ON THE CONSTANT IN THE NONUNIFORM VERSION OF THE BERRY-ESSEEN THEOREM

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Dedicated to Professor Yupaporn Kemprasit on her sixtieth birthday.

In 2001, Chen and Shao gave the nonuniform estimation of the rate of convergence in Berry-Esseen theorem for independent random variables via Stein-Chen-Shao method. The aim of this paper is to obtain a constant in Chen-Shao theorem, where the random variables are not necessarily identically distributed and the existence of their third moments are not assumed. The bound is given in terms of truncated moments and the constant obtained is 21.44 for most values. We use a technique called Stein's method, in particular the Chen-Shao concentration inequality.

1. Introduction and main result

Let $X_1, X_2, \dots, X_n$ be independent and not necessarily identically distributed random variables with zero mean and finite variance. Define $W = X_1 + X_2 + \cdots + X_n$ and assume that $\operatorname{Var}(W) = 1$. Let F_n be the distribution function of W and Φ the standard normal distribution function. It is well known that if the Lindeberg condition,

$$\forall \varepsilon > 0, \quad \sum_{i=1}^{n} EX_i^2 I(|X_i| > \varepsilon) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{1.1}$$

where $I(A)$ is an indicator random variable such that

$$I(A) = \begin{cases} 1 & \text{if } A \text{ is true}, \\ 0 & \text{otherwise}, \end{cases} \tag{1.2}$$

is satisfied, then

$$\forall x \in \mathbb{R}, \quad F_n(x) \longrightarrow \Phi(x) \quad \text{as } n \longrightarrow \infty. \tag{1.3}$$

Furthermore, if $E|X_i|^3 < \infty$, then we have the uniform Berry-Esseen theorem

$$\sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \le C_0 \sum_{i=1}^{n} E|X_i|^3, \tag{1.4}$$

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and the nonuniform Berry-Esseen theorem

$$|F_n(x) - \Phi(x)| \le \frac{C_1}{(1+|x|)^3} \sum_{i=1}^{n} E|X_i|^3, \tag{1.5}$$

where both C_0 and C_1 are absolute constants.

Note that in case X_i's are identically distributed, (1.4) and (1.5) were first obtained by Esseen [4] and Nagaev [8], respectively. Bikjalis [1] generalized Nagaev's result to the case that X_i's are not necessarily identically distributed random variables. Paditz [9, 10] calculated C_1 to be 114.7 and 32 in 1977 and 1989, respectively, and Michel [7] reduced it to 30.84 for the independent and identically distributed case.

In 2001, Chen and Shao gave nonuniform and uniform bounds for independent and not necessarily identically distributed random variables without assuming the existence of third moments. Their result states as follows.

Theorem 1.1 (Chen-Shao theorem). *Let $X_1, X_2, \dots, X_n$ be independent random variables with zero means and $\sum_{i=1}^n EX_i^2 = 1$. Let $W = X_1 + X_2 + \cdots + X_n$ and let F_n be the distribution function of W. Then,*

$$|F(x) - \Phi(x)| \le C \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1+|x|)}{(1+|x|)^2} + \frac{E|X_i|^3 I(|X_i| < 1+|x|)}{(1+|x|)^3} \right\}, \tag{1.6}$$

$$|F(x) - \Phi(x)| \le 4.1 \sum_{i=1}^{n} \left\{ EX_i^2 I(|X_i| \ge 1) + E|X_i|^3 I(|X_i| < 1) \right\}. \tag{1.7}$$

Observe that the constant 4.1 in (1.7) is smaller than 6 as obtained by Feller [5] and it was pointed out by Loh [6] that the truncation at 1 in (1.7) is optimal in the sense that

$$EX^2 I(|X| \ge 1) + E|X|^3 I(|X| < 1) = \inf_A \{EX^2 I(X \in A) + E|X|^3 I(X \in A^C)\}. \tag{1.8}$$

The standard tool used Esseen [4], Nagaev [8], Bikjalis [1], Paditz [9, 10], and Michel [7] is the Fourier-analytic method. But Chen and Shao [3] proved (1.6) and (1.7) by combining truncation with Stein's method [14] and the concentration inequality approach. The concentration inequality approach was originally used by Stein for independent and identically distributed random variables. It was extended by Chen [2] to dependent and nonidentically distributed random variables with arbitrary index sets. In [3], the concentration inequality approach is improved and extended to nonuniform bounds. The improved approach is much more effective than that in [2]. In this paper, we combine the concentration inequality in [3] with the coupling approach to calculate the constant C in (1.6). The followings are our main results.

Theorem 1.2. *Let $X_1, X_2, \dots, X_n$ be independent random variables with zero means and $\sum_{i=1}^n EX_i^2 = 1$. Let $W = X_1 + X_2 + \cdots + X_n$ and let F_n be the distribution function of W. Then*

$$|F_n(x) - \Phi(x)| \le C_0 \sum_{i=1}^{n} \left\{ \frac{EX_i^2 I(|X_i| \ge 1+|x/4|)}{(1+|x/4|)^2} + \frac{E|X_i|^3 I(|X_i| < 1+|x/4|)}{(1+|x/4|)^3} \right\}, \tag{1.9}$$

where

$$C_0 = \begin{cases} 21.44 & \text{if } |x| \leq 3 \text{ or } |x| \geq 14, \\ 32 & \text{if } 3 < |x| \leq 3.99 \text{ or } 7.98 < |x| < 14, \\ 60 & \text{otherwise.} \end{cases} \tag{1.10}$$

Corollary 1.3. *If X_i's in Theorem 1.1 have finite third moment, then*

$$\left| F_n(x) - \Phi(x) \right| \leq \frac{C_1 \sum_{i=1}^{n} E\left|X_i\right|^3}{\left(1+|x/4|\right)^3}, \tag{1.11}$$

where

$$C_1 = \begin{cases} 21.44 & \text{if } |x| \leq 7.98 \text{ or } |x| \geq 14, \\ 32 & \text{if } 7.98 < |x| < 14. \end{cases} \tag{1.12}$$

Observe that the bound in Theorem 1.2 is given in terms of truncated moments. It is worthwhile to note also that truncated moments were considered by Sazonov [13]. In his work, he gave two main methods for deriving speed of convergence results in the central limit theorem (CLT), namely, the Fourier-analytic method and the method of composition which used convolutions directly. These methods are used to derive more results for random vectors. For nonuniform bound in CLT of random vectors, one can see, for examples, Rotar [11, 12].

2. Auxiliary results

In this section, we give auxiliary results in order to prove the main theorem in Section 3. Let $X_1, X_2, \dots, X_n$, W, F_n, and Φ be defined as in Theorem 1.2. In order to use the concentration inequality and the coupling approach, we introduce random variables J, $\tilde{X}_1$, $\tilde{X}_2, \dots, \tilde{X}_n$ defined in the following way. The random variables J, $X_1, X_2, \dots, X_n$, $\tilde{X}_1$, $\tilde{X}_2, \dots, \tilde{X}_n$ are independent, J uniformly distributed over the set $\{1, 2, \dots, n\}$, $(X_i, \tilde{X}_i)$ is a coupling pair, that is, X_i and $\tilde{X}_i$ are the same distributions. For $a > 0$, we also let

$$\begin{gathered} Y_{j,a} = X_j I\left(\left|X_j\right| < 1+a\right), \qquad \tilde{Y}_{j,a} = \tilde{X}_j I\left(\left|\tilde{X}_j\right| < 1+a\right), \\ S_a = \sum_{j=1}^{n} Y_{j,a}, \qquad \tilde{S}_a = S_a - Y_{J,a} + \tilde{Y}_{J,a}, \\ \alpha_a = \sum_{j=1}^{n} E X_j^2 I\left(\left|X_j\right| \geq 1+a\right), \qquad \beta_a = \sum_{j=1}^{n} E\left|X_j\right|^3 I\left(\left|X_j\right| < 1+a\right), \\ \delta_a = \frac{\alpha_a}{(1+a)^2} + \frac{\beta_a}{(1+a)^3}. \end{gathered} \tag{2.1}$$

Observe that $(Y_{j,a}, \tilde{Y}_{j,a})$ is a coupling pair and $(S_a, \tilde{S}_a)$ is an exchangeable pair in the sense that

$$P\left(S_a \in E, \tilde{S}_a \in \tilde{E}\right) = P\left(S_a \in \tilde{E}, \tilde{S}_a \in E\right) \tag{2.2}$$

for arbitrary Borel sets E and $\widetilde{E}$ on $\mathbb{R}$. From the fact that $(a+b)^n \le 2^{n-1}(a^n+b^n)$ for $a,b \ge 0$, we have

$$E\left|Y_{J,a}\right|^3 = \frac{1}{n}\sum_{j=1}^{n} E\left|X_j\right|^3 I(|X_j| < 1+a) = \frac{\beta_a}{n}, \tag{2.3}$$

$$E\left|\tilde{Y}_{J,a} - Y_{J,a}\right|^3 \le \frac{8}{n}\sum_{j=1}^{n} E\left|X_j\right|^3 I(|X_j| < 1+a) = \frac{8\beta_a}{n}. \tag{2.4}$$

In proposition 2.1, we use the coupling approach to bound ES_a^2 and ES_a^4 which are used in the proof of the concentration inequality.

Proposition 2.1. (1) $E^{S_a}\tilde{S}_a = (1-1/n)S_a + (1/n)ES_a$, *where* $E^X Y$ *is the conditional expectation of* Y *with respect to* X.

(2) $ES_a^2 \le 1 + (\alpha_a/(1+a))^2$.

(3) $|ES_a|^3 \le 12\beta_a + 3(\alpha_a/(1+a)) + (\alpha_a/(1+a))^3$.

(4) $ES_a^4 \le 53(1+a)\beta_a + 30\beta_a(\alpha_a/(1+a)) + 6(\alpha_a/(1+a))^2 + (\alpha_a/(1+a))^4 + 6\beta_a + 6\alpha_a + 3$.

(5) *If* $(1+a)^2\alpha_a + (1+a)\beta_a < 1/80$ *and* $a \ge 3$, *then* $ES_a^2 \le 1 + (3.8\times 10^{-8})$ *and* $ES_a^4 \le 3.69$.

(6) *If* $(1+a)^2\alpha_a + (1+a)\beta_a \ge 1/80$ *and* $a \ge 14$, *then* $ES_a^4/a^4 \le 391\delta_a$.

Proof. (1)

$$\begin{aligned}
E^{S_a}\tilde{S}_a &= E^{S_a}\left(S_a - Y_{J,a} + \tilde{Y}_{J,a}\right) \\
&= S_a - E^{S_a}Y_{J,a} + E^{S_a}\tilde{Y}_{J,a} \\
&= S_a - \frac{1}{n}\sum_{j=1}^{n} E^{S_a}Y_{j,a} + \frac{1}{n}\sum_{j=1}^{n} E^{S_a}\tilde{Y}_{j,a} \\
&= S_a - \frac{1}{n}\sum_{j=1}^{n} Y_{j,a} + \frac{1}{n}\sum_{j=1}^{n} EY_{j,a} \\
&= \left(1 - \frac{1}{n}\right)S_a + \frac{1}{n}ES_a.
\end{aligned} \tag{2.5}$$

(2) Let $h : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$h(\tilde{t},t) = \tilde{t}^2 - t^2. \tag{2.6}$$

Since h is antisymmetric in the sense that $h(\tilde{t},t) = -h(t,\tilde{t})$ and $(S_a,\tilde{S}_a)$ is an exchangeable pair, by Stein [15, equation (9), page 10],

$$E(\tilde{S}_a^2 - S_a^2) = Eh(\tilde{S}_a, S_a) = 0. \tag{2.7}$$

From this fact and (1), we have

$$
\begin{aligned}
0 &= E(\tilde{S}_a - S_a)(\tilde{S}_a + S_a) \\
&= 2E(\tilde{S}_a - S_a)S_a + E(\tilde{S}_a - S_a)^2 \\
&= 2E(E^{S_a}\tilde{S}_a - S_a)S_a + E(\tilde{S}_a - S_a)^2 \\
&= -\frac{2}{n}ES_a^2 + E(\tilde{S}_a - S_a)^2 + \frac{2}{n}E^2S_a,
\end{aligned}
\tag{2.8}
$$

which implies that

$$
\begin{aligned}
ES_a^2 &= \frac{n}{2}E(\tilde{Y}_{J,a} - Y_{J,a})^2 + E^2S_a \\
&= \sum_{j=1}^{n}\{EX_j^2I(|X_j| < 1+a) - E^2X_jI(|X_j| < 1+a)\} + E^2S_a \\
&\le \sum_{j=1}^{n}EX_j^2I(|X_j| < 1+a) + E^2S_a \\
&\le 1 + \left(\frac{\alpha_a}{1+a}\right)^2,
\end{aligned}
\tag{2.9}
$$

where we have used the fact that $\sum_{j=1}^{n}EX_j^2 = 1$ and

$$
|ES_a| = \left|\sum_{j=1}^{n}EX_jI(|X_j| < 1+a)\right| = \left|\sum_{j=1}^{n}EX_jI(|X_j| \ge 1+a)\right| \le \frac{\alpha_a}{1+a}
\tag{2.10}
$$

in the last inequality.

(3) By the same argument of (2), with $h(\tilde{t},t) = (\tilde{t}-t)(\tilde{t}^2+t^2)$,

$$
\begin{aligned}
ES_a^3 &= \frac{n}{2}E(\tilde{S}_a - S_a)(\tilde{S}_a^2 - S_a^2) + ES_aES_a^2 \\
&= \frac{n}{2}E(\tilde{S}_a - S_a)^2(\tilde{S}_a + S_a) + ES_aES_a^2 \\
&= \frac{n}{2}E(\tilde{Y}_{J,a} - Y_{J,a})^2[(\tilde{Y}_{J,a} - Y_{J,a}) + 2S_a] + ES_aES_a^2 \\
&= \frac{n}{2}E(\tilde{Y}_{J,a} - Y_{J,a})^3 + nE(\tilde{Y}_{J,a} - Y_{J,a})^2S_a + ES_aES_a^2.
\end{aligned}
\tag{2.11}
$$

Hence,

$$
\begin{aligned}
|ES_a^3| &\le \frac{n}{2}E|\tilde{Y}_{J,a} - Y_{J,a}|^3 + n\left|E(\tilde{Y}_{J,a} - Y_{J,a})^2S_a\right| + ES_aES_a^2 \\
&\le 4\beta_a + n\left|E(\tilde{Y}_{J,a} - Y_{J,a})^2S_a\right| + \left(\frac{\alpha_a}{1+a}\right) + \left(\frac{\alpha_a}{1+a}\right)^3,
\end{aligned}
\tag{2.12}
$$

where we have used (2.4), (2.10), and (2) in the last inequality. Note that

$$\begin{aligned}
\left|E(\tilde{Y}_{J,a}-Y_{J,a})^2S_a\right| &= \left|\frac{1}{n}\sum_{j=1}^{n}E(\tilde{Y}_{j,a}-Y_{j,a})^2\sum_{l=1}^{n}Y_{l,a}\right| \\
&\le \left|\frac{1}{n}\sum_{j=1}^{n}E(\tilde{Y}_{j,a}-Y_{j,a})^2Y_{j,a}\right| + \left|\frac{1}{n}\sum_{j=1}^{n}E(\tilde{Y}_{j,a}-Y_{j,a})^2\sum_{\substack{l=1\\l\neq j}}^{n}EY_{l,a}\right| \\
&\le \frac{1}{n}\sum_{j=1}^{n}E(\tilde{Y}_{j,a}-Y_{j,a})^2|Y_{j,a}| + \frac{1}{n}\sum_{j=1}^{n}E(\tilde{Y}_{j,a}-Y_{j,a})^2|ES_a| \\
&\quad + \frac{1}{n}\sum_{j=1}^{n}E(\tilde{Y}_{j,a}-Y_{j,a})^2E|Y_{j,a}| \\
&\le \frac{8}{n}\sum_{j=1}^{n}E|Y_{j,a}|^3 + \frac{2}{n}\left(\frac{\alpha_a}{1+a}\right) \\
&\le \frac{8\beta_a}{n} + \frac{2}{n}\left(\frac{\alpha_a}{1+a}\right).
\end{aligned} \tag{2.13}$$

Hence, by (2.12) and (2.13), $|ES_a^3| \le 12\beta_a + 3(\alpha_a/(1+a)) + (\alpha_a/(1+a))^3$.

(4) Using the same argument of (2), with $h(\tilde{t},t) = (\tilde{t}-t)(\tilde{t}^3+t^3)$, we have

$$\begin{aligned}
ES_a^4 &= \frac{n}{2}E(\tilde{S}_a-S_a)(\tilde{S}_a^3-S_a^3) + ES_aES_a^3 \\
&= \frac{n}{2}E(\tilde{S}_a-S_a)^2\left[(\tilde{S}_a-S_a)^2+3\tilde{S}_aS_a\right] + ES_aES_a^3 \\
&= \frac{n}{2}E(\tilde{Y}_{J,a}-Y_{J,a})^4 + \frac{3n}{2}E(\tilde{Y}_{J,a}-Y_{J,a})^2\left(S_a^2+(\tilde{Y}_{J,a}-Y_{J,a})S_a\right) + ES_aES_a^3 \\
&\le n(1+a)E|\tilde{Y}_{J,a}-Y_{J,a}|^3 + \frac{3n}{2}E(\tilde{Y}_{J,a}-Y_{J,a})^2S_a^2 \\
&\quad + 3n(1+a)E\left|(\tilde{Y}_{J,a}-Y_{J,a})^2S_a\right| + ES_aES_a^3 \\
&\le 32(1+a)\beta_a + 6\alpha_a + 12\beta_a\left(\frac{\alpha_a}{1+a}\right) + 3\left(\frac{\alpha_a}{1+a}\right)^2 + \left(\frac{\alpha_a}{1+a}\right)^4 \\
&\quad + \frac{3n}{2}E\left(\tilde{Y}_{J,a}-Y_{J,a}\right)^2S_a^2,
\end{aligned} \tag{2.14}$$

where we have used (2.4), (2.10), (2.13), and (3) in the last inequality. From (2.14) and the fact that

$$\begin{aligned}
&E(\tilde{Y}_{J,a}-Y_{J,a})^2S_a^2 \\
&\quad = \frac{1}{n}\sum_{j=1}^{n}E(\tilde{Y}_{j,a}-Y_{j,a})^2E\left(\sum_{\substack{l=1\\l\neq j}}^{n}Y_{l,a}\right)^2 + \frac{2}{n}\sum_{j=1}^{n}E(\tilde{Y}_{j,a}-Y_{j,a})^2Y_{j,a}E\left(\sum_{\substack{l=1\\l\neq j}}^{n}Y_{l,a}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n}\sum_{j=1}^{n}E(\tilde{Y}_{j,a}-Y_{j,a})^2Y_{j,a}^2\\
&\le \frac{2}{n}\sum_{j=1}^{n}EY_{j,a}^2E\left(\sum_{\substack{l=1\\l\neq j}}^{n}Y_{l,a}\right)^2+\frac{8}{n}\sum_{j=1}^{n}E|Y_{j,a}|^3\left|E\left(\sum_{\substack{l=1\\l\neq j}}^{n}Y_{l,a}\right)\right|+\frac{4(1+a)}{n}\sum_{j=1}^{n}E|Y_{j,a}|^3\\
&\le \frac{2}{n}\sum_{j=1}^{n}EY_{j,a}^2ES_a^2-\frac{4}{n}\sum_{j=1}^{n}EY_{j,a}^2ES_aY_{j,a}+\frac{2}{n}\sum_{j=1}^{n}EY_{j,a}^2EY_{j,a}^2\\
&\quad+\frac{8}{n}\sum_{j=1}^{n}E|Y_{j,a}|^3|ES_a|+\frac{8}{n}\sum_{j=1}^{n}E|Y_{j,a}|^3|EY_{j,a}|+\frac{4(1+a)}{n}\beta_a\\
&\le \frac{2}{n}+\frac{2}{n}\left(\frac{\alpha_a}{1+a}\right)^2+\frac{4}{n}\sum_{j=1}^{n}E|Y_{j,a}|^3\sqrt{ES_a^2}+\frac{8\beta_a}{n}\left(\frac{\alpha_a}{1+a}\right)+\frac{14(1+a)\beta_a}{n}\\
&\le \frac{2}{n}+\frac{4\beta_a}{n}+\frac{12\beta_a}{n}\left(\frac{\alpha_a}{1+a}\right)+\frac{2}{n}\left(\frac{\alpha_a}{1+a}\right)^2+\frac{14(1+a)\beta_a}{n},
\end{aligned}
\tag{2.15}
$$

we have

$$
ES_a^4\le 53(1+a)\beta_a+30\beta_a\left(\frac{\alpha_a}{1+a}\right)+6\left(\frac{\alpha_a}{1+a}\right)^2+\left(\frac{\alpha_a}{1+a}\right)^4+6\beta_a+6\alpha_a+3. \tag{2.16}
$$

(5) Follows directly from (2) and (4).

(6)

$$
\begin{aligned}
\frac{ES_a^4}{a^4}&\le \frac{53}{a^3}\left(\frac{1+a}{a}\right)\beta_a+\frac{30\beta_a}{a^4}\left(\frac{\alpha_a}{1+a}\right)+\frac{6}{a^4}\left(\frac{\alpha_a}{1+a}\right)^2\\
&\quad+\frac{1}{a^4}\left(\frac{\alpha_a}{1+a}\right)^4+\frac{6\beta_a}{a^4}+\frac{6\alpha_a}{a^4}+\frac{3}{a^4}\\
&\le \frac{70.697\beta_a}{(1+a)^3}+\frac{0.035\alpha_a}{(1+a)^2}+\frac{3.997}{(1+a)^4}\\
&\le \frac{70.697\beta_a}{(1+a)^3}+\frac{0.035\alpha_a}{(1+a)^2}+319.76\alpha_a\\
&\le 391\delta_a,
\end{aligned}
\tag{2.17}
$$

where we have used the fact that $a\ge 14$, $\alpha_a\le 1$, and $(1+a)/a\le 1.072$ in the second inequality and the fact that $(1+a)^2\alpha_a+(1+a)\beta_a\ge 1/80$ in the last inequality. □

Next, we will prove the concentration inequality.

Proposition 2.2 (concentration inequality). *Let $i\in\{1,2,\dots,n\}$ and $W^{(i)}=W-X_i$. Then for $3\le a<b<\infty$ and $(1+a)^2\alpha_a+(1+a)\beta_a<1/80$,*

$$
P(a\le W^{(i)}\le b)\le \frac{40.98}{(1+a)^3}(b-a)+46.38\delta_a. \tag{2.18}
$$

Proof. Let $S_{i,a} = S_a - Y_{i,a}$. We observe that $W^{(i)} = S_{i,a}$ when $\max_{1\le j\le n, j\ne i} |X_j| < 1+a$. So

$$\begin{aligned} P(a \le W^{(i)} \le b) &\le P(a \le S_{i,a} \le b) + P\left(\max_{\substack{1\le j\le n\\ j\ne i}} |X_j| \ge 1+a\right) \\ &\le P(a \le S_{i,a} \le b) + \frac{\alpha_a}{(1+a)^2}. \end{aligned} \tag{2.19}$$

Let $\gamma = \beta_a/2$ and $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} 0 & \text{for } t < a-\gamma, \\ (1+t+\gamma)^3(t-a+\gamma) & \text{for } a-\gamma \le t \le b+\gamma, \\ (1+t+\gamma)^3(b-a+2\gamma) & \text{for } t > b+\gamma. \end{cases} \tag{2.20}$$

So f is a nondecreasing function satisfying $f'(t) \ge (1+a)^3$ for $a-\gamma < t < b+\gamma$, and $f'(t) \ge 0$ otherwise. Let $M(w,t) = w[I(-w \le t < 0) - I(0 \le t < -w)]$. Hence,

$$\begin{aligned} ES_{i,a}f(S_{i,a}) &= \sum_{\substack{j=1\\ j\ne i}}^{n} EY_{j,a}(f(S_{i,a}) - f(S_{i,a} - Y_{j,a})) \\ &= \sum_{\substack{j=1\\ j\ne i}}^{n} EY_{j,a}\int_{-Y_{j,a}}^{0} f'(S_{i,a}+t)\,dt \\ &= \sum_{\substack{j=1\\ j\ne i}}^{n} EY_{j,a}\left\{\int_{\mathbb{R}} f'(S_{i,a}+t)[I(-Y_{j,a} \le t < 0) - I(0 < t \le -Y_{j,a})]\,dt\right\} \\ &= \sum_{\substack{j=1\\ j\ne i}}^{n} E\left\{\int_{\mathbb{R}} f'(S_{i,a}+t)M(Y_{j,a},t)\,dt\right\} \\ &\ge (1+a)^3 \sum_{\substack{j=1\\ j\ne i}}^{n} E\left\{I(a \le S_{i,a} \le b)\int_{|t|\le\gamma} M(Y_{j,a},t)\,dt\right\} \\ &= (1+a)^3 E\left\{I(a \le S_{i,a} \le b)\sum_{\substack{j=1\\ j\ne i}}^{n} |Y_{j,a}|\min(\gamma, |Y_{j,a}|)\right\} \\ &\ge 0.46(1+a)^3\{P(a \le S_{i,a} \le b) - P(U_i \le 0.46)\}, \end{aligned} \tag{2.21}$$

where $U_i = \sum_{j=1, j\ne i}^{n} |Y_{j,a}|\min(\gamma, |Y_{j,a}|)$ and we have used the fact that

$$I(t_1 \le w \le t_2)y \ge c\left(I(t_1 \le w \le t_2) - \left(1 - \frac{y}{c}\right)I(y \le c)\right) \tag{2.22}$$

for $t_1, t_2, \gamma \geq 0$, $c > 0$ in the last inequality. Hence,

$$P(a \leq S_{i,a} \leq b) \leq \frac{1}{0.46(1+a)^3} ES_{i,a} f(S_{i,a}) + P(U_i \leq 0.46). \tag{2.23}$$

Next, we will bound the two terms on the right-hand side of (2.23). By the same argument as that in Proposition 2.1, we can show that $ES_{i,a}^4 \leq 3.69$ and $ES_{i,a}^2 \leq 1 + (3.8 \times 10^{-8})$. So

$$\begin{aligned} E\left|S_{i,a} f(S_{i,a})\right| &\leq (b - a + 2\gamma) E\left|S_{i,a}\right| \left|S_{i,a} + (1+\gamma)\right|^3 \\ &\leq 4(b - a + \beta_a)\left(ES_{i,a}^4 + |1+\gamma|^3 E\left|S_{i,a}\right|\right) \\ &\leq 4(b - a + \beta_a)\left(ES_{i,a}^4 + \left|1 + \frac{\beta_a}{2}\right|^3 \sqrt{ES_{i,a}^2}\right) \\ &\leq 18.85(b - a + \beta_a). \end{aligned} \tag{2.24}$$

By the facts that $\min(a,b) \geq b - b^2/4a$ for $a, b > 0$,

$$\begin{aligned} &EX_i^2 I(|X_i| \leq 1+a) \leq (\beta_a)^{2/3} < 0.021, \\ &\alpha_a \leq \frac{1}{80(1+a)^2} \leq 7.8 \times 10^{-4} \quad \text{for } a \geq 3, \end{aligned} \tag{2.25}$$

we have

$$\begin{aligned} EU_i &= \sum_{\substack{j=1 \\ j \neq i}}^{n} E\left|Y_{j,a}\right| \min\left(\gamma, \left|Y_{j,a}\right|\right) \\ &\geq \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(EY_{j,a}^2 - \frac{E\left|Y_{j,a}\right|^3}{4\gamma}\right) \\ &\geq \sum_{j=1}^{n} \left\{EX_j^2 I(|X_j| < 1+a) - EX_j^2 I(|X_j| \geq 1+a)\right\} - \frac{\beta}{\gamma} \\ &= \sum_{\substack{j=1 \\ j \neq i}} \left\{EX_j^2 I(|X_j| < 1+a) - E^2 X_j I(|X_j| \geq 1+a)\right\} - 0.5 \\ &= 1 - EX_i^2 I(|X_i| < 1+a) - 2\sum_{j=1}^{n} EX_j^2 I(|X_i| \geq 1+a) - 0.5 \\ &\geq 1 - (\beta_a)^{2/3} - 2\alpha_a - 0.5 \\ &\geq 0.477. \end{aligned} \tag{2.26}$$

Using the same argument as in Proposition 2.1(5), we can show that

$$E\left|U_i - EU_i\right|^4 \leq 3.69\gamma^4 = 0.231\beta_a^4 \leq 4.512 \times 10^{-7} \frac{\beta_a}{(1+a)^3}. \tag{2.27}$$

Hence,

$$\begin{aligned} P(U_i \leq 0.46) &\leq P(EU_i - U_i \geq 0.477 - 0.46) \\ &= P(EU_i - U_i \geq 0.017) \\ &\leq \frac{E|U_i - EU_i|^4}{(0.017)^4} \\ &\leq \frac{5.402\beta_a}{(1+a)^3}. \end{aligned} \tag{2.28}$$

From (2.19), (2.23), (2.24), and (2.28),

$$\begin{aligned} P(a \leq W^{(i)} \leq b) &\leq \frac{40.978}{(1+a)^3}(b - a + \beta_a) + \frac{5.402\beta_a}{(1+a)^3} + \frac{\alpha_a}{(1+a)^2} \\ &\leq \frac{40.98(b-a)}{(1+a)^3} + 46.36\delta_a. \end{aligned} \tag{2.29}$$

□

Proposition 2.3. *For $x \geq 2$,*

$$E|f_x'(W)| \leq \frac{15}{(1+x)^2}, \tag{2.30}$$

where f_x is the unique solution of the Stein equation

$$f'(w) - wf(w) = I(w \leq x) - \Phi(x). \tag{2.31}$$

Proof. From Stein [15, pages 22 and 24], we know that

$$\begin{gathered} 0 < f_x'(w) < 1 - \Phi(x) \quad \text{for } w \leq 0, \\ 0 < f_x'(x) \leq 1 - \Phi(x)\left[1 + \sqrt{2\pi} w e^{(1/2)w^2}\Phi(x)\right] \quad \text{for } 0 < w \leq x, \\ |f_x'(w)| \leq 1 \quad \forall w \in \mathbb{R}. \end{gathered} \tag{2.32}$$

Hence,

$$\begin{aligned} E|f_x'(W)| &= E|f_x'(W)|I(W \leq 0) + E|f_x'(W)|I\left(0 < W \leq \frac{4x}{5}\right) \\ &\quad + E|f_x'(W)|I\left(W > \frac{4x}{5}\right) \\ &\leq (1-\Phi(x))P(W \leq 0) + (1-\Phi(x))E(1 + \sqrt{2\pi} W e^{W^2/2})I\left(0 < W \leq \frac{4x}{5}\right) \\ &\quad + \frac{E(1+W^2)}{(1+(4x/5))^2} \\ &\leq (1-\Phi(x)) + (1-\Phi(x))\left(1 + \frac{4\sqrt{2\pi}}{5} x e^{8x^2/25}\right) + \frac{2}{(1+(4x/5))^2}. \end{aligned} \tag{2.33}$$

Since

$$1-\Phi(x)\le \frac{e^{-(1/2)x^2}}{\sqrt{2\pi}x} \quad \text{for } x\ge 0, \tag{2.34}$$

(see Stein [15, equation (25), page 23]) and $e^{x^2/2}>x$ for $x\ge 2$, we have

$$(1-\Phi(x))(1+x)^2 \le \frac{1}{\sqrt{2\pi}x^2}(1+x)^2=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}+1\right)^2\le 0.9, \tag{2.35}$$

which implies that

$$1-\Phi(x)\le \frac{0.9}{(1+x)^2}. \tag{2.36}$$

From (2.34) and the fact that $e^{9x^2/50}>9x^2/50$, we derive

$$\begin{aligned}\sqrt{2\pi}(1-\Phi(x))(1+x)^2xe^{8x^2/25} &\le e^{-9x^2/50}(1+x)^2\\ &\le \frac{50}{9}\left(\frac{1}{x}+1\right)^2\\ &\le 12.5,\end{aligned} \tag{2.37}$$

that is,

$$\frac{4\sqrt{2\pi}}{5}(1-\Phi(x))xe^{8x^2/25}\le \frac{10}{(1+x)^2}. \tag{2.38}$$

From (2.33), (2.36), (2.38), and the fact that $(1+x)/(1+4x/5)\le 5/4$, we have proved the proposition. □

Proposition 2.4. *Let* $x\ge 14$ *and* $g:\mathbb{R}\to\mathbb{R}$ *defined by* $g(w)=(wf_x(w))'$. *If* $(1+x)^2\alpha_x+(1+x)\beta_x<1/80$, *then for* $|u|\le 1+x/4$,

$$Eg(W^{(i)}+u)\le \frac{4.60}{(1+x/4)^3}+5.13\delta_{x/4}(1+x). \tag{2.39}$$

Proof. From Chen and Shao [3, pages 248–249], we know that

$$g(x-1)=\left[\sqrt{2\pi}(1+(x-1)^2)e^{(x-1)^2/2}\Phi(x-1)+(x-1)\right](1-\Phi(x)), \tag{2.40}$$

g is increasing for $0\le w<x$, and

$$Eg(W^{(i)}+u)\le \frac{2}{1+x^3}+2(1-\Phi(x))+g(x-1)+Eg(W^{(i)}+u)I(x-1<W^{(i)}+u<x). \tag{2.41}$$

For $x\ge 14$, elementary calculation yields

$$\frac{(1+x)^3}{1+x^3}\le 1.23 \tag{2.42}$$

and $e^{x^2/2} \geq (1/3!)(x^2/2)^3 \geq 800x^2$. Using similar argument as that in deriving (2.36), we have

$$1-\Phi(x) \leq \frac{0.0006}{(1+x)^3}. \tag{2.43}$$

By (2.40) and (2.34),

$$\begin{aligned} g(x-1) &\leq \sqrt{2\pi}\left(1+(x-1)^2\right)e^{(x-1)^2/2}\left(1-\Phi(x)\right)+x\left(1-\Phi(x)\right) \\ &\leq \left(x-2+\frac{2}{x}\right)e^{-x+1/2}+\frac{1}{\sqrt{2\pi}}e^{-(1/2)x^2} \\ &\leq \frac{0.056}{(1+x)^3}, \end{aligned} \tag{2.44}$$

where we have used the fact that $f(x)=\{(x-2+2/x)e^{-x+1/2}+(1/\sqrt{2\pi})e^{-(1/2)x^2}\}(1+x)^3$ is decreasing on $[14,\infty)$ in the last inequality. So, by (2.41), (2.42), (2.43), and (2.44),

$$\begin{aligned} Eg\left(W^{(i)}+u\right) &\leq \frac{2.517}{(1+x)^3}+Eg\left(W^{(i)}+u\right)I\left(x-1<W^{(i)}+u<x\right) \\ &= \frac{2.517}{(1+x)^3}+\int_{x-1}^{x}-g(w)dP\left(w<W^{(i)}+u<x\right) \\ &= \frac{2.517}{(1+x)^3}+g(x-1)P\left(x-1<W^{(i)}+u<x\right) \\ &\quad+\int_{x-1}^{x}g'(w)P\left(w<W^{(i)}+u<x\right)dw \\ &\leq \frac{2.573}{(1+x)^3}+\int_{x-1}^{x}g'(w)\left[\frac{40.98}{(1+w-u)^3}(x-w)+46.38\delta_{w-u}\right]dw, \end{aligned} \tag{2.45}$$

where the last inequality follows from Proposition 2.2, (2.44), and the fact that

$$w-u \geq (x-1)-\left(1+\frac{x}{4}\right) \geq \frac{3x}{5} \geq 3 \quad \text{for } |u|<1+\frac{x}{4}. \tag{2.46}$$

Since δ_x is decreasing in x, g is nonnegative and increasing on $[0,x)$, and $|g(x)| \leq 1+|x|$, (2.45) can then be bounded by

$$\begin{aligned} Eg\left(W^{(i)}+u\right) &\leq \frac{2.573}{(1+x)^3}+\frac{40.98}{(1+3x/5)^3}\int_{x-1}^{x}g'(w)(x-w)dw+46.38\delta_{3x/5}g(x) \\ &\leq \frac{2.573}{(1+x)^3}+\frac{40.98}{(1+3x/5)^3}\int_{x-1}^{x}(x-w)dg(w)+46.38(1+x)\delta_{3x/5} \\ &\leq \frac{2.573}{(1+x)^3}+\frac{40.98}{(1+3x/5)^3}\int_{x-1}^{x}g(w)dw+46.38(1+x)\delta_{3x/5} \\ &= \frac{2.573}{(1+x)^3}+\frac{40.98}{(1+3x/5)^3}\left(xf_x(x)\right)+46.38(1+x)\delta_{3x/5} \\ &\leq \frac{4.60}{(1+x/4)^3}+5.13\delta_{x/4}(1+x), \end{aligned} \tag{2.47}$$

where we have applied the fact that $|xf_x(x)| \le 1$, $(1+x/4)/(1+x) \le 0.30$, $(1+x/4)/(1+3x/5) \le 0.48$, and $\delta_{3x/5} \le (1+x/4)^3/(1+3x/5)^3\delta_{x/4} \le 0.111\delta_{x/4}$ for $x \ge 14$ in the last inequality. □

We are now ready to prove our main results.

3. Proof of main results

3.1. Proof of Theorem 1.2. It suffices to consider only $x \ge 0$ as we can simply apply the result to $-W$ when $x < 0$.

Case 3.1 ($0 \le x \le 3$). By (1.7) and (1.8),

$$\begin{aligned}
21.44&\sum_{i=1}^{n}\left[\frac{EX_i^2 I(|X_i| \ge 1+x/4)}{(1+x/4)^2}+\frac{E|X_i|^3 I(|X_i| < 1+x/4)}{(1+x/4)^3}\right]\\
&\ge 4.1\sum_{i=1}^{n}\left[EX_i^2 I\left(|X_i| \ge 1+\frac{x}{4}\right)+E|X_i|^3 I\left(|X_i| < 1+\frac{x}{4}\right)\right]\\
&\ge 4.1\sum_{i=1}^{n}\left[EX_i^2 I(|X_i| \ge 1)+E|X_i|^3 I(|X_i| < 1)\right]\\
&\ge |F(x)-\Phi(x)|.
\end{aligned} \tag{3.1}$$

Case 3.2 ($x \ge 14$).

Subcase 3.3 ($(1+x)^2\alpha_x+(1+x)\beta_x \ge 1/80$). Using similar argument as that in showing (2.36), we see that $1-\Phi(x) \le 0.009/(1+x)^4$. This inequality, together with inequality $(1+x)^2\alpha_x+(1+x)\beta_x \ge 1/80$, gives $1-\Phi(x) \le 0.738\delta_x$, which in turn implies that

$$\begin{aligned}
|F(x)-\Phi(x)| &\le P(W \ge x)+(1-\Phi(x))\\
&\le P(W \ge x)+0.738\delta_x.
\end{aligned} \tag{3.2}$$

Using the same argument as in (2.19), we have

$$\begin{aligned}
P(W \ge x) &\le P(S_x \ge x)+\frac{\alpha_x}{(1+x)^2}\\
&\le \frac{ES_x^4}{x^4}+\frac{\alpha_x}{(1+x)^2} \quad \text{(which by Proposition 2.1(6))}\\
&\le 391\delta_a+\frac{\alpha_x}{(1+x)^2}\\
&\le 392\delta_x.
\end{aligned} \tag{3.3}$$

From (3.2), (3.3), and the fact that $\delta_x \le ((1+x/4)/(1+x))^3\delta_{x/4} \le 0.05\delta_{x/4}$, we have

$$|F(x)-\Phi(x)| \le 20.26\delta_{x/4}. \tag{3.4}$$

Subcase 3.4 ($(1+x)^2\alpha_x + (1+x)\beta_x < 1/80$). Let $K_{i,x/4}(t) = EY_{i,x/4}\{I(0 < t \le Y_{i,x/4}) - I(Y_{i,x/4} \le t < 0)\}$. From Chen and Shao [3, pages 250–251], we set

$$F(x) - \Phi(x) = R_1 + R_2 + R_3 + R_4, \tag{3.5}$$

where

$$\begin{aligned} R_1 &= \sum_{i=1}^{n} E\Big\{I\Big(|X_i| < 1 + \frac{x}{4}\Big)\int_{|t|\le 1+x/4}\big(f_x'(W^{(i)}+X_i) - f_x'(W^{(i)}+t)\big)K_{i,x/4}(t)dt\Big\},\\ R_2 &= \sum_{i=1}^{n} E\Big\{I\Big(|X_i| \ge 1 + \frac{x}{4}\Big)\int_{|t|\le 1+x/4}\big(f_x'(W^{(i)}+X_i) - f_x'(W^{(i)}+t)\big)K_{ix/4}(t)dt\Big\},\\ R_3 &= \alpha_{x/4}Ef_x'(W),\\ R_4 &= -\sum_{i=1}^{n} E\Big\{X_iI\Big(|X_i| \ge 1 + \frac{x}{4}\Big)\big(f(W) - f(W^{(i)})\big)\Big\}, \end{aligned} \tag{3.6}$$

and observe that $|R_1| \le R_{11} + R_{12}$, where

$$\begin{aligned} R_{11} &= \sum_{i=1}^{n} \Big|E\Big\{I\Big(|X_i| < 1 + \frac{x}{4}\Big)\int_{|t|\le 1+x/4}K_{i,x/4}(t)\int_t^{X_i}Eg(W^{(i)}+u)\,du\,dt\Big\}\Big|,\\ R_{12} &= \sum_{i=1}^{n} E\Big\{I\Big(|X_i| < 1 + \frac{x}{4}\Big)\int_{|t|\le 1+x/4}P\big(x - \max(t,X_i) \le W^{(i)} \le x\\ &\qquad - \min(t,X_i) \mid X_i\big)K_{i,x/4}(t)dt\Big\}. \end{aligned} \tag{3.7}$$

By the fact that $|f_x'(s) - f_x'(t)| \le 1$ for all $x,s,t \in \mathbb{R}$ (see Chen and Shao [3, page 246]), we have

$$\begin{aligned} |R_2| &\le \sum_{i=1}^{n} P\Big(|X_i| \ge 1 + \frac{x}{4}\Big)\\ &\le \sum_{i=1}^{n} \frac{EX_i^2I(|X_i| \ge 1 + x/4)}{(1+x/4)^2}\\ &= \frac{\alpha_{x/4}}{(1+x/4)^2} \end{aligned} \tag{3.8}$$

and by Proposition 2.3,

$$|R_3| = \alpha_{x/4}|Ef_x'(W)| \le \frac{15\alpha_{x/4}}{(1+x)^2} \le \frac{15\alpha_{x/4}}{(1+x/4)^2}. \tag{3.9}$$

Since $0 \le f_x(w) \le \min(\sqrt{2\pi}/4, 1/|x|)$ for all $x > 0$ (see Chen and Shao [3, equation (4.6), page 246]),

$$\begin{aligned}
|R_4| &\le \sum_{i=1}^{n} \frac{E|X_i|I(|X_i| \ge 1+x/4)}{x} \\
&\le \sum_{i=1}^{n} \frac{EX_i^2 I(|X_i| \ge 1+x/4)}{(1+x/4)x} \\
&\le \frac{\alpha_{x/4}}{(1+x/4)x} \\
&\le \frac{\alpha_{x/4}}{(1+x/4)^2}.
\end{aligned} \tag{3.10}$$

Hence,

$$|R_2 + R_3 + R_4| \le \frac{17\alpha_{x/4}}{(1+x/4)^2}. \tag{3.11}$$

By Proposition 2.4, we have

$$\begin{aligned}
R_{11} &\le 2\left[\frac{4.60}{(1+x/4)^3} + 5.13(1+x)\delta_{x/4}\right]\sum_{i=1}^{n} E|Y_{i,x/4}|^3 \\
&\le \frac{9.2\beta_{x/4}}{(1+x/4)^3} + 10.26(1+x)\delta_{x/4}\beta_{x/4} \\
&\le \frac{9.2\beta_{x/4}}{(1+x/4)^3} + 0.128\delta_{x/4},
\end{aligned} \tag{3.12}$$

where we have used the fact that $(1+x)\beta_{x/4} \le (1+x)\beta_x < 1/80$ in the last inequality.

By Proposition 2.2 and the fact that

$$x - \max(t, X_i) \ge x - \left(1 + \frac{x}{4}\right) = \frac{3x}{4} - 1 \ge \frac{2x}{3} \quad \text{for } |t| \le 1 + \frac{x}{4}, \tag{3.13}$$

we have

$$\begin{aligned}
|R_{12}| &\le \sum_{i=1}^{n} E\left\{I\left(|X_i| \le 1+\frac{x}{4}\right)\int_{|t|\le 1+x/4}\left(\frac{40.98}{(1+2x/3)^3}(|t|+|X_i|) + 46.38\delta_{2x/3}\right)K_{i,x/4}(t)\,dt\right\} \\
&\le \frac{81.96}{(1+2x/3)^3}\beta_{x/4} + 46.38\delta_{2x/3} \\
&\le \frac{6.982}{(1+x/4)^3}\beta_{x/4} + 3.942\delta_{x/4},
\end{aligned} \tag{3.14}$$

where we have used the fact that $(1+x/4)/(1+2x/3) \le 0.44$ and $\delta_{2x/3} \le (1+x/4)^3/(1+2x/3)^3\delta_{x/4} \le 0.085\delta_{x/4}$ for $x \ge 14$ in the last inequality.

Hence, by (3.5), (3.11), (3.12), and (3.14),

$$|F(x)-\Phi(x)| \le |R_1+R_2+R_3+R_4| \le 20.26\delta_{x/4}. \tag{3.15}$$

For the case when $3<x<14$, we can use the same arguments as in Case 3.1 and Case 3.2.

3.2. Proof of Corollary 1.3. From van Beek [16], we know that

$$|F(x)-\Phi(x)| \le 0.7975\sum_{i=1}^{n} E|X_i|^3. \tag{3.16}$$

Using the same technique in proving Case 3.1 of Theorem 1.2, we see that

$$|F(x)-\Phi(x)| \le 21.44\sum_{i=1}^{n} E|X_i|^3 \tag{3.17}$$

for $0\le x\le 7.98$. For $x>7.98$, we use Theorem 1.1 and the fact that

$$\frac{EX_i^2 I(|X_i| \ge 1+|x/4|)}{(1+|x/4|)^2}+\frac{E|X_i|^3 I(|X_i|<1+|x/4|)}{(1+|x/4|)^3} \le \frac{E|X_i|^3}{(1+|x/4|)^3}. \tag{3.18}$$

4. Remark

In the proof of Theorem 1.1, we assume that $x\ge a$, where $a=14$. It is not necessary to assume that $a=14$. If $a>14$ is used, the constant will be sharper.

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