ON SEMIABELIAN GROUPS

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We establish some structural results on the semiabelian groups, that is, groups generated by their cyclic normal subgroups. Such groups play a significant role in the theory of supersoluble groups. A description of two-generated semiabelian groups and torsion-free semiabelian groups is obtained.

Recall that a group is called semiabelian if it is generated by its normal cyclic subgroups [6]. The class of semiabelian groups is a very natural generalization of the well-known class of Dedekind groups (the groups in which all cyclic subgroups are normal). In the paper [6] Venzke showed that these groups could play a major role in the theory of supersoluble finite groups. Based on the notion of semiabelian group, he developed a theory of finite supersoluble group similar to the theory of finite nilpotent group. He also proved that finite semiabelian groups are nilpotent of class 2, closed under homomorphic images and direct products, and are not closed under subgroups or normal subgroups. In this paper we study semiabelian (finite and infinite) groups under some restrictions. In particular, the torsion-free semiablian groups and two-generated semiabelian groups are described.

Lemma 1. (i) *Any semiabelian group can be presented as a product of normal infinite and normal cyclic p-subgroup.*

(ii) *Any factor group of a direct product of semiabelian groups is semiabelian.*

(iii) *There are semiabelian groups with non-semiabelian normal subgroups.*

Proof. The first and the second assertions are obvious (see also [6, page 573]). In [6, page 573] one can find an appropriate example of a semiabelian p-group for odd prime p, which contains a non-semiabelian normal subgroup. $\square$

Lemma 2. *A derived subgroup $[G,G]$ of a semiabelian group G is periodic and central. It coincides with the derived subgroup $[H,H]$ of a periodic subgroup H decomposing into a product of G-normal cyclic subgroups. Moreover, $G = HZ$, where $Z \leq Z(G)$ is a subgroup generated by infinite normal in G cyclic subgroups.*

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Proof. By Lemma 1, $G = \prod_{i\in I}\langle g_i\rangle$ (here and below $\prod_{i\in I} K_i$ is a product of subgroups K_i, $i \in I$), where $\langle g_i\rangle \trianglelefteq G$ for all $i \in I$. Denote by $G_i = C_G(g_i)$ the centralizer of element g_i in G, $i \in I$. Then $G_i \trianglelefteq G$, and $G/G_i \leq \mathrm{Aut}(\langle g_i\rangle)$ is abelian as a group of automorphism of a cyclic group. Hence, $[G,G] \leq G_i$. It is clear that $\bigcap_{i\in I} G_i = Z(G)$ and $[G,G] \leq Z(G)$.

It is obvious that in a group with the property $[G,G] \leq Z(G)$, for any two elements $a,b \in G$ the following equation holds: $[a,b]^n = [a,b^n]$ (see [5, Exercise 5.1.4]). This fact will help us to show that $[G,G]$ is a periodic group. In turn, $[G,G]$ is generated by commutators $[g_i,g_j]$. Assume that one of the elements g_i or g_j (let say g_i) has an infinite order. If $|g_j| = n$, then $[g_i,g_j]^n = [g_i,g_j^n] = 1$. So $|[g_i,g_j]| < \infty$. Therefore, we need to consider the case when $|g_j| = |g_i| = \infty$. If $\langle g_i\rangle \cap \langle g_j\rangle = 1$, then $[g_i,g_j] = 1$. If $\langle g_i\rangle \cap \langle g_j\rangle \neq 1$, then $\langle g_i\rangle \cap \langle g_j\rangle = \langle g_j^n\rangle \leq Z(\langle g_i,g_j\rangle)$. In this case, the commutator identity $[a,b]^n = [a,b^n]$ implies that $1 = [g_i,g_j^n] = [g_i,g_j]^n$. Hence, $[G,G]$ is generated by elements of finite order. Since it is central, $[G,G]$ is periodic.

Denote by Z the product of all $\langle g_i\rangle$ of infinite order. Since $[G,G]$ is periodic, $[G,G] \cap \langle g_i\rangle = 1$ for any $\langle g_i\rangle$ of infinite order. Therefore, $\langle g_i\rangle \leq Z(G)$, and hence $Z \leq Z(G)$. Let H be the product of all $\langle g_j\rangle$ such that $\langle g_j\rangle \nleq Z(G)$. Then such $\langle g_j\rangle$ have finite orders, H is periodic, and $G = HZ$. Since $Z \leq Z(G)$, $[G,G] = [H,H]$. □

The following corollary has been proven above.

Corollary 3. *If a group G decomposes into a product of its normal cyclic subgroups $\langle g_i\rangle$, then every infinite factor $\langle g_i\rangle$ is central.*

Corollary 4. *A finitely generated semiabelian group G decomposes as a direct product of a finite number of factors, each of which is either a finitely generated abelian group or a finite semiabelian Sylow p-subgroup of G.*

Proof. Let G be a finitely generated semiabelian group. By Lemma 2, $G = HZ$, where $Z \leq Z(G)$, H is periodic, and $[H,H] = [G,G] \leq Z(G)$. Since G is a nilpotent finitely generated group, its periodic part $t(G)$ is also finitely generated (see, e.g., [5, 5.2.17]). Therefore, H is finite. Let $C = t(G) \cap Z$. It is obvious that $G = t(G)Z$, and $C = t(Z)$. Since G is finitely generated, the factor group $G/t(G)$ is also finitely generated, and therefore by Lemmas 1 and 2 it is abelian. Then $Z/t(Z)$ is a finitely generated abelian group. By a Kulikov theorem (see, e.g., [3, page 174]), $Z = C \times D$, where D is a finitely generated abelian group. Since D is a central subgroup, $G = t(G) \times D$. Obviously, $t(G) = HC$. Lemma 2 implies that H decomposes into a product of $G-$invariant cyclic subgroups. Since $C \leq Z(G)$, it also has this property. Therefore, $t(G)$ also decomposes into a product of $G-$normal cyclic subgroups. In [6, Theorem 2.1] Venzke proved that a semiabelian finite group is nilpotent of class 2, and each Sylow p-subgroup of such a group is also semiabelian. □

Lemma 2 implies the following theorem.

Theorem 5. *Semiabelian torsion free groups are abelian.*

Corollary 6. *A nonperiodic semiabelian two-generated group decomposes into a direct product of two of its cyclic subgroups.*

Proof. Let G be a nonperiodic two-generated semiabelian group. By Corollary 4, $G = t(G) \times D$, where D an abelian finitely generated group. Since G is two-generated nonperiodic, D is at most two-generated, and $D \neq 1$. If rank of D is 2, then $t(G) = 1$, and everything is proved. If D is a cyclic, then $t(G)$ is also cyclic, and our corollary is proved. □

A group G is called the almost direct product of groups G_i, $i \in I$, if it satisfies the following conditions:

(1) for any G_i group G contains a normal subgroup $G_i^* \cong G_i$;

(2) $G = \prod_{i \in I} G_i^*$;

(3) for any $i \in I$ $G_i^* \cap \prod_{i,j,\in I, j \neq i} G_j^* < G_i^*$.

We say that G_i^* is an almost direct factor of the corresponding almost direct decomposition of G, and the cardinality $|I|$ is the cardinality of this almost direct product.

An almost direct product is a wide generalization of direct and central products (see, e.g., [2]). However, not every group being a product of its, let say, cyclic subgroups decomposes into an almost direct product of its cyclic subgroups. For example, every Dedekind group is a product of its cyclic subgroups, but not every abelian group (a quasicyclic group, an additive group of rational numbers) is an almost direct product of its cyclic subgroups.

Lemma 7. *Let G be a semiabelian p-group of bounded exponent. Then $G/[G,G]$ is a direct product of cardinality σ of its cyclic p-subgroups, and G is an almost direct product of cardinality σ of its normal cyclic subgroups.*

Proof. By Lemma 1 G is a product of its normal subgroups $\langle g_i \rangle$, $i \in I$. Since the exponent of G is bounded, then $G/[G,G]$ is a direct product of its cyclic subgroups $\langle [G,G]a_l \rangle$, $l \in L$, such that $|\langle [G,G]a_l \rangle| > 1$. Let $\Phi/[G,G]$ denote the Frattini subgroup $\Phi(G/[G,G])$ of the group $G/[G,G]$. Then $\Phi/[G,G]$ is a direct product of subgroups $\langle [G,G]a_l^p \rangle$, $l \in L$. It is clear that $\Phi \trianglelefteq G$, and G/Φ is a direct product of subgroups $\langle \Phi a_l \rangle$, $|\Phi a_l| = p$, $l \in L$. Hence, G/Φ is an elementary abelian p-group decomposed into a direct product of cardinality $|L|$ of cyclic subgroups of prime order p. Clearly, G/Φ is generated by cosets Φg_i, $i \in I$, such that $|\Phi g_i| \leq p$. Then I contains a subset I_1 such that for any $i \in I_1$ $|\Phi g_i| = p$, and G/Φ is a direct product of such factors $\langle \Phi g_i \rangle$. By the theorem about the invariants of a decomposition of abelian p-groups into a direct product of cyclic subgroups (see, e.g., [3, page 147]) $|L| = |I_i|$, and without loss of generality, we can assume that $L = I_1$. Let $N \leq G$ be a product of $\langle g_i \rangle$, $i \in L$. Clearly, $N \trianglelefteq G$. Put $X = [G,G]N$. Obviously, $X \trianglelefteq G$. Since G/Φ is a direct product of subgroups $\langle \Phi g_i \rangle$, $i \in I$, we can conclude that $G = \Phi N = \Phi X$. Put $Y = \Phi \cap X$. Then $[G,G] \leq Y \trianglelefteq G$, $G/Y = \Phi/Y \times X/Y$. Since $[G,G] \leq Y$ and $\Phi([G,G]) = \Phi(G/[G,G])$, we can write

$$G/Y \cong (G/[G,G])/(Y/[G,G]) = (\Phi/[G,G])/(Y/[G,G]) \times (X/[G,G])/(Y/[G,G]). \quad (1)$$

Since $G/[G,G]$ is a direct product of cyclic groups, and $\exp(G/[G,G])$ is bounded, it follows that the Frattini subgroup of any homomorphic image of $G/[G,G]$ coincides with the homomorphic image of the Frattini subgroup of the group, that is, $\Phi/Y = \Phi(G/Y)$. Since the exponent of G/Y is bounded, in the case when $|\Phi/Y| > 1$, there is a maximal subgroup in G/Y, which does not include Φ/Y, a contradiction. Hence, $\Phi/Y = 1$, $\Phi \leq X$, and $G = X = [G,G]N$. By Lemma 2, $[G,G]$ is abelian, and therefore $N = G$. Then G is a product of $\langle g_i \rangle$, $i \in L$. So G/Φ decomposes into a direct product of $\langle \Phi g_i \rangle$, $|\Phi g_i| = p$.

Let $i \neq j$, and let G_i be the product of all $\langle g_j \rangle$, $j \in L$. Clearly, $G_i \trianglelefteq G$, and $|G : G_i| < \infty$. Put $G_i^* = \langle g_i^p \rangle G_i$. Then $|G : G_i^*| \leq p$. It follows that $[G,G] \leq G_i^*$, $\Phi \leq G_i^*$, and $G_i^* \trianglelefteq G$. Consider the group G/Φ. It is a direct product of $\langle \Phi g_i \rangle$, $i \in L$, and G_i^*/Φ is a direct product of all $\langle \Phi g_j \rangle$, $j \in L$, $i \neq j$. It follows that $G_i^* \cap \langle g_i \rangle = \langle g_i \rangle$, and $\langle g_i \rangle \cap G_i \leq \langle g_i \rangle$. □

Using the results above, we obtain the following theorem describing the two-generated semiabelain groups.

Theorem 8. *Two-generated semiabelian groups are exhausted by the following types of groups:*

(1) $G = \langle a \rangle \times \langle b \rangle$;

(2) $G = \langle a, b \rangle$, *G is a quaternion group*, $|a| = |b| = 4$, $[a,b] = a^2 = b^2$;

(3) $G = \langle a \rangle \leftthreetimes \langle b \rangle$, $|a| = p^\alpha$, $|b| = p^\beta$, $\langle [a,b] \rangle = \langle a^{p^k} \rangle$, $\alpha \geq k \geq \alpha - k > 0$, $p^k > 2$, $k \geq \beta \geq \alpha - k$;

(4) *G is a finite nilpotent nonabelian non-p-group, every Sylow p-subgroup of which is a group of one of the types 1–3 above or cyclic.*

Proof. Necessity. Observe that if G is a nonperiodic group, then by Corollary 6, G is a direct product of two cyclic groups; that is, G is a group of the type 1. By Lemma 2, we can assume that G is a periodic two-generated nilpotent group; therefore G is finite, and G is a direct product of its Sylow p-subgroups. If G is abelian, then it is a group of the type 1. If G is nonabelian, then it includes a nonabelian Sylow p-subgroup P whose minimal number of generators is 2. Further we consider two cases: $G = P$, and G is a non-p nonabelian group. In the first case G is a metacyclic (in the meaning of [5, page 290]) two-generated p-group. Since P is nonabelian, and $[G,G] \leq \Phi(G)$ (the Frattini subgroup of G), $G/[G,G]$ is a direct product of two cyclic subgroups. By Lemma 7, G is an almost direct product of two cyclic subgroups $\langle g_1 \rangle$ and $\langle g_2 \rangle$. Assume that $|g_1| \geq |g_2|$, and denote $\langle g_1 \rangle = \langle a \rangle$. Since $G = \langle g_1, g_2 \rangle$, $\langle g_1 \rangle \lhd G$ and $\langle g_2 \rangle \lhd G$, we can observe that $[G,G] \leq \langle g_1 \rangle \cap \langle g_2 \rangle \leq Z(G)$. G is nonabelian, and therefore $a^{p^\alpha} = 1$ some $\alpha \geq 2$.

If $|a| = 4$ and $|g_2| = 2$, then $G = \langle a, g_2 \rangle$ is an abelian group, which is impossible. So $|g_2| > 2$, that is, $|g_1| = |g_2| = 4$, and G belongs to the type 2.

If $|a| > 4$, $[\langle a \rangle, \langle g_2 \rangle] = \langle a^{p^k} \rangle$, $\alpha > k$, $p^k > 2$. Since G is a nonabelian nilpotent group with a central derived subgroup, and $G = \langle a \rangle \langle g_2 \rangle$, $|a| \geq |g_2|$, $|a| > 4$, $[\langle a \rangle, \langle g_2 \rangle] = \langle a^{p^k} \rangle$, $\alpha > k$, $p^k > 2$, using the detailed classification of metacyclic p-groups given in [4, Theorem 1.2.2], we obtain that $G = \langle a \rangle \leftthreetimes \langle b \rangle$, $|a| = p^\alpha$, $|b| = p^\beta$, $\langle [a,b] \rangle = \langle a^{p^k} \rangle$, $k \geq \alpha - k > 0$, $\alpha \geq \beta \geq \alpha - k$. Obviously, we can write $g_2 = a^i b^j$ for some whole numbers i, j, and since $G = \langle a \rangle \langle g_2 \rangle$, $(j, p) = 1$. Since $\langle g_2 \rangle \geq [G,G] = \langle a^{p^k} \rangle \leq Z(G)$, it is follows that there is a positive whole number δ such that $\langle g_2^{p^\delta} \rangle = \langle a^{p^k} \rangle$. Using [4, Proposition 1.1.10 (2)] (see also the formula obtained inductively in the proof of [1, Theorem 12.5.4]), we can write

$$g_2^{p^\delta} = (a^i b^j)^{p^\delta} = a^{ip^\delta} b^{jp^\delta} [a,b]^{-(1/2)ijp^\delta(p^\delta - 1)} \tag{2}$$

and $\langle g_2^{p^\delta}\rangle \le \langle a^{p^k}\rangle$. It is clear that $b^{jp^\delta}=1$, $[a,b]=a^{lp^k}$, $(l,p)=1$. Therefore,

$$[a,b]^{-(1/2)ijp^\delta(p^\delta-1)} = a^{-(1/2)lijp^{\delta+k}(p^\delta-1)}. \tag{3}$$

Hence,

$$g_2^{p^\delta} = a^{ip^\delta}a^{-(1/2)lijp^{\delta+k}(p^\delta-1)}. \tag{4}$$

Since $p^k>2$, $-(1/2)lijp^{\delta+k}(p^\delta-1)=tp^{\delta+1}$ for some whole number t. Then $a^{ip^\delta}a^{tp^{\delta+1}}$ $=a^{p^\delta(i+tp)}$. Since $g_2^{p^\delta}\le\langle a^{p^k}\rangle$ and $\langle g_2^{p^\delta}\rangle=\langle a^{p^\delta(i+tp)}\rangle$, we can conclude that $\delta\le k$. Since $\delta\ge\beta$, $k\ge\beta$, and G is a group from the type 3.

In the second case when G is not a p-group, it is a direct product of its Sylow p_i-subgroups, each of which by Corollary 4 is a semiabelian two-generated group or a cyclic group. By Lemma 7 each of these Sylow subgroups is a product of no more than two normal in G cyclic subgroups.

Sufficiency. For the groups of types 1 and 2, the sufficiency is obvious.

For the type 3, $G=\langle a\rangle\rightthreetimes\langle b\rangle$, $|a|=p^\alpha$, $|b|=p^\beta$, $\langle[a,b]\rangle=\langle a^{p^k}\rangle$, $\alpha\ge k\ge\alpha-k>0$, $p^k>2$, $k\ge\beta\ge\alpha-k$, [4, Theorem 1.2.2] implies that for such metacyclic groups $[G,G]\le Z(G)$. Using the formula from the proof of [1, Theorem 12.5.4] (see, also, [4, Proposition 1.1.10 (2)]), we can obtain for $g=ab$,

$$g^{p^\beta}=a^{p^\beta}b^{p^\beta}[a,b]^{-(1/2)p^\beta(p^\beta-1)}=a^{p^\beta}(a^{lp^k})^{-(1/2)p^\beta(p^\beta-1)}=a^{p^\beta}a^{-(1/2)lp^{\beta+k}(p^\beta-1)}, \tag{5}$$

where $(l,p)=1$. Since $p^k>2$, we can find a number t, such that $-(1/2)lp^{\beta+k}(p^\beta-1)=tp^{\beta+1}$, and therefore $g^{p^\beta}=a^{p^\beta(1+tp)}$, where $p\nmid 1+tp$. By the conditions, $\beta\le k$ $\langle g^{p^\beta}\rangle\ge\langle a^{p^k}\rangle=[G,G]$. Therefore, $\langle g\rangle\lhd G$, and $G=\langle a,b\rangle=\langle a,ab\rangle=\langle a,g\rangle$. Hence, $G=\langle a\rangle\langle g\rangle$ is a semiabelian two-generated group.

For the groups of type 4, the sufficiency directly follows from the sufficiency of the types 1–3. □

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