ON GENERALIZED ω-CLOSED SETS

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The class of ω-closed subsets of a space (X,τ) was defined to introduce ω-closed functions. The aim of this paper is to introduce and study the class of $g\omega$-closed sets. This class of sets is finer than g-closed sets and ω-closed sets. We study the fundamental properties of this class of sets. In the space (X,τ_ω), the concepts closed set, g-closed set, and $g\omega$-closed set coincide. Further, we introduce and study $g\omega$-continuous and $g\omega$-irresolute functions.

1. Introduction

Throughout this work, a space will always mean a topological space on which no separation axiom is assumed unless explicitly stated. Let (X,τ) be a space and let A be a subset of X. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is called ω-closed [10] if it contains all its condensation points. The complement of an ω-closed set is called ω-open. It is well known that a subset W of a space (X,τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open subsets of a space (X,τ), denoted by τ_ω, forms a topology on X finer than τ.

In 1970, Levine [13] introduced the notion of generalized closed sets. He defined a subset A of a space (X,τ) to be generalized and closed (briefly g-closed) if $\mathrm{cl}_\tau(A) \subseteq U$ whenever $U \in \tau$ and $A \subseteq U$.

Generalized semiclosed [4] (resp., α-generalized closed [14], θ-generalized closed [8], generalized semi-preclosed [6], δ-generalized closed [7]) sets are defined by replacing the closure operator in Levine's original definition by the semiclosure (resp., α-closure, θ-closure, semi-preclosure, δ-closure) operator.

In Section 2 of the present work, we follow a similar line to introduce generalized ω-closed sets by utilizing the ω-closure operator. We study g-closed sets and $g\omega$-closed sets in the spaces (X,τ) and (X,τ_ω). In particular, we show that a subset A of a space (X,τ) is closed in (X,τ_ω) if and only if it is g-closed in (X,τ_ω) if and only if it is $g\omega$-closed in (X,τ_ω).

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In Section 3, we introduce $g\omega$-continuity and $g\omega$-irresoluteness by using $g\omega$-closed sets and study some of their fundamental properties.

Now we begin to recall some known notions, definitions, and results which will be used in the work.

Let (X,τ) be a space and let A be a subset of X. The closure of A, the interior of A, and the relative topology on A will be denoted by $\mathrm{cl}_\tau(A)$, $\mathrm{int}_\tau(A)$, and τ_A, respectively. The ω-interior (ω-closure) of a subset A of a space (X,τ) is the interior (closure) of A in the space (X,τ_ω), and is denoted by $\mathrm{int}_{\tau_\omega}(A)(\mathrm{cl}_{\tau_\omega}(A))$.

Definition 1.1. A space (X,τ) is called

(a) locally countable [3] if each point $x \in X$ has a countable open neighborhood;
(b) anti-locally countable [1] if each nonempty open set is uncountable;
(c) $T_{1/2}$-space [13] if every g-closed set is closed (equivalently if every singleton is open or closed, see [8]).

Definition 1.2. A function $f : (X,\tau) \to (Y,\sigma)$ is called

(a) g-continuous [5] if $f^{-1}(V)$ is g-closed in (X,τ) for every closed set V of (Y,σ);
(b) g-irresolute [5] if $f^{-1}(V)$ is g-closed in (X,τ) for every g-closed set V of (Y,σ);
(c) ω-continuous [11] if $f^{-1}(V)$ is ω-open in (X,τ) for every open set V of (Y,σ);
(d) ω-irresolute [2] if $f^{-1}(V)$ is ω-open in (X,τ) for every ω-open set V of (Y,σ);
(e) α-continuous [15] if $f^{-1}(V)$ is α-set in (X,τ) for every open set V of (Y,σ).

Lemma 1.3 [3]. *Let A be a subset of a space (X,τ). Then,*

(a) $(\tau_\omega)_\omega = \tau_\omega$;
(b) $(\tau_A)_\omega = (\tau_\omega)_A$.

2. Generalized ω-closed sets

Definition 2.1. A subset A of a space (X,τ) is called generalized ω-closed (briefly, $g\omega$-closed) if $\mathrm{cl}_{\tau_\omega}(A) \subseteq U$ whenever $U \in \tau$ and $A \subseteq U$.

We denote the family of all generalized ω-closed (generalized closed) subsets of a space (X,τ) by $G\omega C(X,\tau)(GC(X,\tau))$.

It is clear that if (X,τ) is a countable space, then $G\omega C(X,\tau) = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X.

Proposition 2.2. *Every g-closed set is $g\omega$-closed.*

The proof follows immediately from the definitions and the fact that τ_ω is finer than τ for any space (X,τ). However, the converse is not true in general as the following example shows.

Example 2.3. Let $X = \{a,b,c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{a,b\}\}$ and let $A = \{a\}$. Then $A \in G\omega C(X,\tau)$. But $A \notin GC(X,\tau)$ since $A \subseteq A \in \tau$ and $\mathrm{cl}_\tau(A) = X \not\subseteq A$.

Lemma 2.4. *Let (A,τ_A) be an anti-locally countable subspace of a space (X,τ). Then* $\mathrm{cl}_\tau(A) = \mathrm{cl}_{\tau_\omega}(A)$.

Proof. We need to prove that $\mathrm{cl}_\tau(A) \subseteq \mathrm{cl}_{\tau_\omega}(A)$. Suppose that there exists $x \in \mathrm{cl}_\tau(A) - \mathrm{cl}_{\tau_\omega}(A)$. Then, $x \notin \mathrm{cl}_{\tau_\omega}(A)$, and so there exists $W_x \in \tau_\omega$ such that $x \in W_x$ and $W_x \cap A = \varnothing$.

Now choose $V_x \in \tau$ such that $x \in V_x$ and $V_x - W_x = C_x$ is countable. Then $\emptyset \neq V_x \cap A \subseteq A \cap (W_x \cup C_x) = (A \cap W_x) \cup (A \cap C_x) = A \cap C_x \subseteq V_x \cap A$. Thus $V_x \cap A = A \cap C_x \in \tau_A$ (i.e., $V_x \cap A$ is a nonempty countable open set in (A, τ_A)), which is a contradiction and the result follows. □

Corollary 2.5. *Let (A, τ_A) be an anti-locally countable subspace of a space (X, τ). Then $A \in GC(X, \tau)$ if and only if $A \in G\omega C(X, \tau)$.*

Theorem 2.6. *Let (X, τ) be any space and $A \subseteq X$. Then the following are equivalent.*

(a) *A is ω-closed in (X, τ) (equivalently A is closed in (X, τ_ω)).*

(b) $A \in GC(X, \tau_\omega)$.

(c) $A \in G\omega C(X, \tau_\omega)$.

Proof. (a)⇒(b). It follows from the fact that every closed set is g-closed.

(b)⇒(c). It is obvious by using Proposition 2.2.

(c)⇒(a). We show that $\mathrm{cl}_{\tau_\omega}(A) \subseteq A$. Suppose that $x_0 \notin A$. Then $U = X - \{x_0\}$ is an ω-open set containing A. Since $A \in G\omega C(X, \tau_\omega)$, $\mathrm{cl}_{(\tau_\omega)_\omega}(A) = \mathrm{cl}_{\tau_\omega}(A) \subseteq U$ (Lemma 1.3(a)), and thus $x_0 \notin \mathrm{cl}_{\tau_\omega}(A)$. Therefore, $\mathrm{cl}_{\tau_\omega}(A) = A$, that is, A is ω-closed in (X, τ).

In the same way, it can be shown that a subset A of a space (X, τ) is closed if and only if $\mathrm{cl}_\tau(A) \subseteq U$ whenever $U \in \tau_\omega$ and $A \subseteq U$.

Proposition 2.7. *If $A \in GC(X, \tau_\omega)$, then $A \in G\omega C(X, \tau)$ but not conversely.*

The proof is obvious. □

Example 2.8. Let $X = \mathbb{R}$ be the set of all real numbers with the topology $\tau = \{\phi, X, \{1\}\}$ and put $A = \mathbb{R} - Q$. Then A is an ω-open subset of (X, τ) such that $\mathrm{cl}_{\tau_\omega}(A) = \mathbb{R} - \{1\} \nsubseteq A$ (i.e., $A \notin GC(X, \tau_\omega)$). However, $A \in G\omega C(X, \tau)$ since the only open set in (X, τ) containing A is X.

In Example 2.8, $A \in GC(X, \tau) - GC(X, \tau_\omega)$. In the following, we give an example of a space (X, τ) and a subset A of X such that $A \in GC(X, \tau_\omega) - GC(X, \tau)$. In other words, for a space (X, τ), the collections $GC(X, \tau)$ and $GC(X, \tau_\omega)$ are independent from each other.

Example 2.9. Consider $X = \mathbb{R}$ with the usual topology τ_u. Put $A = (0,1) \cap Q$. Then $\mathrm{cl}_{(\tau_u)_\omega}(A) = A$ (A is countable), and so $A \in GC(\mathbb{R}, (\tau_u)_\omega)$. On the other hand, $A \notin GC(\mathbb{R}, \tau_u)$ since $U = (0,1)$ is open in $(\mathbb{R}, \tau_u)$ such that $A \subseteq U$ and $\mathrm{cl}_{\tau_u}(A) = [0,1] \nsubseteq U$.

Note that in Example 2.9, $(\mathbb{R}, \tau_u)$ is anti-locally countable and $A = (0,1) \bigcap Q \in G\omega C(\mathbb{R}, \tau_u) - GC(\mathbb{R}, \tau_u)$. Thus the condition that (A, τ_A) is anti-locally countable in Corollary 2.5 cannot be replaced by the condition that (X, τ) is anti-locally countable.

Proposition 2.10. *Let A be a $g\omega$-closed subset of a space (X, τ) and $B \subseteq X$. Then the following hold.*

(a) *$\mathrm{cl}_{\tau_\omega}(A) - A$ contains no nonempty closed set.*

(b) *If $A \subseteq B \subseteq \mathrm{cl}_{\tau_\omega}(A)$, then $B \in G\omega C(X, \tau)$.*

Proof. (a) Suppose by contrary that $\mathrm{cl}_{\tau_\omega}(A) - A$ contains a nonempty closed set C. Then $A \subseteq X - C$ and $X - C$ is open in (X, τ). Thus, $\mathrm{cl}_{\tau_\omega}(A) \subseteq X - C$ or equivalently, $C \subseteq X - \mathrm{cl}_{\tau_\omega}(A)$. Therefore, $C \subseteq (X - \mathrm{cl}_{\tau_\omega}(A)) \cap (\mathrm{cl}_{\tau_\omega}(A) - A) = \emptyset$.

(b) Let $U \in \tau$ and $B \subseteq U$. Then $A \subseteq B \subseteq U$. Since $A \in G\omega C(X,\tau)$, $\mathrm{cl}_{\tau_\omega}(B) \subseteq \mathrm{cl}_{\tau_\omega}(\mathrm{cl}_{\tau_\omega}(A)) = \mathrm{cl}_{\tau_\omega}(A) \subseteq U$, and the result follows. □

Theorem 2.11. *If (X,τ) is a $T_{1/2}$-space, then every $g\omega$-closed set in (X,τ) is g-closed in (X,τ_ω).*

Proof. Let A be a $g\omega$-closed subset of (X,τ). By Theorem 2.6, we show that A is ω-closed in (X,τ). Suppose, to the contrary, that there exists $x \in \mathrm{cl}_{\tau_\omega}(A) - A$. Then, by Proposition 2.10(a), $\{x\}$ is not closed. Since (X,τ) is a $T_{1/2}$-space, $\{x\}$ is open in (X,τ), and thus it is ω-open. Therefore, $\{x\} \cap A \neq \emptyset$, a contradiction. □

In the space X from Example 2.3, every $g\omega$-closed set is ω-closed while (X,τ) is not a $T_{1/2}$-space. Thus, the converse of Theorem 2.11 is not true in general.

Theorem 2.12. *Let (X,τ) be an anti-locally countable space. Then (X,τ) is a T_1-space if and only if every $g\omega$-closed set is ω-closed.*

Proof. We need to show the sufficiency part only. Let $x \in X$ and suppose that $\{x\}$ is not closed. Then $A = X - \{x\}$ is not open, and thus A is $g\omega$-closed (the only open set containing A is X). Therefore, by assumption, A is ω-closed, and thus $\{x\}$ is ω-open. So there exists $U \in \tau$ such that $x \in U$ and $U - \{x\}$ is countable. It follows that U is a nonempty countable open subset of (X,τ), a contradiction. □

Proposition 2.13. *If $\mathcal{A} = \{A_\alpha : \alpha \in I\}$ is a locally finite collection of $g\omega$-closed sets of a space (X,τ), then $A = \bigcup_{\alpha\in I} A_\alpha$ is $g\omega$-closed (in particular, a finite union of $g\omega$-closed sets is $g\omega$-closed).*

Proof. Let U be an open subset of (X,τ) such that $A \subseteq U$. Since $A_\alpha \in G\omega C(X,\tau)$ and $A_\alpha \subseteq U$ for each $\alpha \in I$, $\mathrm{cl}_{\tau_\omega}(A_\alpha) \subseteq U$. As τ_ω is a topology on X finer than τ, $\mathcal{A}$ is locally finite in (X,τ_ω). Therefore, $\mathrm{cl}_{\tau_\omega}(A) = \mathrm{cl}_{\tau_\omega}(\bigcup_{\alpha\in I} A_\alpha) = \bigcup_{\alpha\in I} \mathrm{cl}_{\tau_\omega}(A_\alpha) \subseteq U$. Thus, A is $g\omega$-closed in (X,τ). □

The following two examples show that a countable union of $g\omega$-closed sets and a finite intersection of $g\omega$-closed sets need not be $g\omega$-closed.

Example 2.14. (a) Consider $X = \mathbb{R}$ with the usual topology τ_u. For each $n \in \mathbb{N}$, put $A_n = [1/n, 1]$ and $A = \bigcup_{n\in\mathbb{N}}^{\infty} A_n$. Then A is a countable union of $g\omega$-closed sets but A is not $g\omega$-closed since $U = (0,2) \in \tau_u$, $A \subseteq U$ and $\mathrm{cl}_{\tau_\omega}(A) = [0,1] \nsubseteq U$.

(b) Let X be an uncountable set and let A be a subset of X such that A and $X - A$ are uncountable. Let $\tau = \{\emptyset, A, X\}$. Choose $x_0, x_1 \notin A$ and $x_0 \neq x_1$. Then $A_0 = A \cup \{x_0\}$ and $A_1 = A \cup \{x_1\}$ are two $g\omega$-closed subsets of (X,τ). But $A_0 \bigcap A_1 = A$ is not $g\omega$-closed since $A \subseteq A \in \tau$ and $\mathrm{cl}_{\tau_\omega}(A) \neq A$.

Proposition 2.15. *If $A \in G\omega C(X,\tau)$ and B is closed in (X,τ), then $A \cap B \in G\omega C(X,\tau)$.*

Proof. Let U be an open set in (X,τ) such that $A \cap B \subseteq U$. Put $W = X - B$. Then $A \subseteq U \cup W \in \tau$. Since $A \in G\omega C(X,\tau)$, $\mathrm{cl}_{\tau_\omega}(A) \subseteq U \cup W$. Now, $\mathrm{cl}_{\tau_\omega}(A \cap B) \subseteq \mathrm{cl}_{\tau_\omega}(A) \bigcap \mathrm{cl}_{\tau_\omega}(B) \subseteq \mathrm{cl}_{\tau_\omega}(A) \cap \mathrm{cl}_{\tau}(B) = \mathrm{cl}_{\tau_\omega}(A) \cap B \subseteq (U \cup W) \cap B \subseteq U$. □

In [11], Hdeib shows that if A is an ω-open subset of a space (X,τ) and B is an ω-open subset of a space (Y,σ), then $A \times B$ need not be ω-open in $(X \times Y, \tau \times \sigma)$, that is,

$\tau_\omega \times \sigma_\omega \nsubseteq (\tau \times \sigma)_\omega$. To prove that the other inclusion always holds, we need the following lemma.

Lemma 2.16. (a) *If A is an ω-open subset of a space (X,τ), then $A - C$ is ω-open for every countable subset C of X.*

(b) *The open image of an ω-open set is ω-open.*

Proof. Part (a) is clear. To prove part (b), let $f : (X,\tau) \to (Y,\sigma)$ be an open function and let W be an ω-open subset of (X,τ). Let $y \in f(W)$. There exists $x \in W$ such that $y = f(x)$. Choose $U \in \tau$ such that $x \in U$ and $U - W = C$ is countable. Since f is open, $f(U)$ is open in (Y,σ) such that $y = f(x) \in f(U)$ and $f(U) - f(W) \subseteq f(U - W) = f(C)$ is countable. Therefore, $f(W)$ is ω-open in (Y,σ). □

Theorem 2.17. *Let (X,τ) and (Y,σ) be two topological spaces. Then $(\tau \times \sigma)_\omega \subseteq \tau_\omega \times \sigma_\omega$.*

Proof. Let $W \in (\tau \times \sigma)_\omega$ and $(x,y) \in W$. There exist $U \in \tau$ and $V \in \sigma$ such that $(x,y) \in U \times V$ and $U \times V - W = C$ is countable. Put $W_1 = (U \cap p_X(W)) - (p_X(C) - \{x\})$ and $W_2 = (V \cap p_Y(W)) - (p_Y(C) - \{y\})$, where $p_X : (X \times Y, \tau \times \sigma) \to (X,\tau)$ and $p_Y : (X \times Y, \tau \times \sigma) \to (Y,\sigma)$ are the natural projections. Then $W_1 \in \tau_\omega$, $W_2 \in \sigma_\omega$ (Lemma 2.16) and $(x,y) \in W_1 \times W_2 \subseteq W$. Thus $W \in \tau_\omega \times \sigma_\omega$. □

Definition 2.18. A subset A of a space (X,τ) is called generalized ω-open (briefly, $g\omega$-open) if its complement $X - A$ is $g\omega$-closed in (X,τ).

It is clear that a subset A of a space (X,τ) is $g\omega$-open if and only if $F \subseteq \mathrm{int}_{\tau_\omega}(A)$, whenever $F \subseteq A$ and F is closed in (X,τ).

Theorem 2.19. *If $A \times B$ is a $g\omega$-open subset of $(X \times Y, \tau \times \sigma)$, then A is $g\omega$-open in (X,τ) and B is $g\omega$-open in (Y,σ).*

Proof. Let F_A be a closed subset of (X,τ) and let F_B be a closed subset of (Y,σ) such that $F_A \subseteq A$ and $F_B \subseteq B$. Then $F_A \times F_B$ is closed in $(X \times Y, \tau \times \sigma)$ such that $F_A \times F_B \subseteq A \times B$. By assumption, $A \times B$ is $g\omega$-open in $(X \times Y, \tau \times \sigma)$, and so $F_A \times F_B \subseteq \mathrm{int}_{(\tau\times\sigma)_\omega}(A \times B) \subseteq \mathrm{int}_{\tau_\omega}(A) \times \mathrm{int}_{\sigma_\omega}(B)$ by using Theorem 2.17. Therefore, $F_A \subseteq \mathrm{int}_{\tau_\omega}(A)$ and $F_B \subseteq \mathrm{int}_{\sigma_\omega}(A)$, and the result follows.

The converse of the above theorem need not be true in general. □

Example 2.20. Let $X = Y = \mathbb{R}$ with the usual topology τ_u. Let $A = \mathbb{R} - Q$ and $B = (0,3)$. Then A and B are ω-open subsets of $(\mathbb{R},\tau_u)$, while $A \times B$ is not $g\omega$-open in $(\mathbb{R} \times \mathbb{R}, \tau_u \times \tau_u)$, since $\mathrm{int}_{(\tau_u\times\tau_u)_\omega} A \times B = \varnothing$ and $\{\sqrt{2}\} \times [1,2]$ is a closed set in $(\mathbb{R} \times \mathbb{R}, \tau_u \times \tau_u)$ contained in $A \times B$.

Theorem 2.21. *Let (Y,τ_Y) be a subspace of a space (X,τ) and $A \subseteq Y$. Then the following hold.*

(a) *If $A \in G\omega C(X,\tau)$, then $A \in G\omega C(Y,\tau_Y)$.*

(b) *If $A \in G\omega C(Y,\tau_Y)$ and Y is ω-closed in (X,τ), then $A \in G\omega C(X,\tau)$.*

Proof. (a) Let V be an open set of (Y,τ_Y) such that $A \subseteq V$. By using Lemma 1.3(b), there exists an open set $U \in \tau$ such that $V = Y \cap U$. Since $A \in G\omega C(X,\tau)$ and $A \subseteq U$, $\mathrm{cl}_{\tau_\omega}(A) \subseteq U$. Now, $\mathrm{cl}_{(\tau_Y)_\omega}(A) = \mathrm{cl}_{(\tau_\omega)_Y}(A) = \mathrm{cl}_{\tau_\omega}(A) \cap Y \subseteq Y \cap U = V$. Therefore, $A \in G\omega C(Y,\tau_Y)$.

(b) Let $A \subseteq U$, where $U \in \tau$. Then $A \subseteq Y \cap U \in \tau_Y$. Since $A \in G\omega C(Y, \tau_Y)$, $\mathrm{cl}_{(\tau_Y)_\omega}(A) = \mathrm{cl}_{(\tau_\omega)_Y}(A) = \mathrm{cl}_{\tau_\omega}(A) \cap Y \subseteq Y \cap U$. Finally, $\mathrm{cl}_{\tau_\omega}(A) = \mathrm{cl}_{\tau_\omega}(A \cap Y) \subseteq \mathrm{cl}_{\tau_\omega}(A) \cap \mathrm{cl}_{\tau_\omega}(Y) = (Y$ is ω-closed) $\mathrm{cl}_{\tau_\omega}(A) \cap Y \subseteq Y \cap U \subseteq U$. Thus $A \in G\omega C(X, \tau)$. □

If we choose $A = Y$ in Example 2.14(b), then $A \in G\omega C(Y, \tau_Y) - G\omega C(X, \tau)$. Therefore, the condition that Y is ω-closed in Theorem 2.21 (b) cannot be dropped.

3. $g\omega$-continuous functions

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called
(a) $g\omega$-continuous if $f^{-1}(C) \in G\omega C(X, \tau)$ for every closed subset C of (Y, σ);
(b) $g\omega$-irresolute if $f^{-1}(A) \in G\omega C(X, \tau)$ for every $A \in G\omega C(Y, \sigma)$.

It follows from the definitions that a function $f : (X, \tau) \to (Y, \sigma)$ is $g\omega$-continuous ($g\omega$-irresolute) if and only if $f^{-1}(V)$ is $g\omega$-open in (X, τ) for every open ($g\omega$-open) subset V of (Y, σ).

Proposition 3.2. *Every g-continuous function and ω-continuous function is $g\omega$-continuous.*

The proof follows from the definitions and Propositions 2.2 and 2.7.

Example 3.3. (a) Let X be an uncountable set and let A be a proper uncountable subset of X. Let $f : (X, \tau_{\mathrm{indis}}) \to (X, \tau_{\mathrm{dis}})$ be the identity function. Then f is $g\omega$-continuous ($GC(X, \tau_{\mathrm{indis}}) = \mathcal{P}(X)$). However, f is not ω-continuous since A is closed in (X, τ_{dis}) and $A = f^{-1}(A)$ is not ω-closed in $(X, \tau_{\mathrm{indis}})$.
(b) Let (X, τ) be as in Example 2.3. Then, the identity function $f : (X, \tau) \to (X, \tau_{\mathrm{dis}})$ is $g\omega$-continuous but not g-continuous.

Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then a function $f^\omega_\omega : (X, \tau_\omega) \to (Y, \sigma_\omega)$ (resp., $f_\omega : (X, \tau_\omega) \to (Y, \sigma)$, $f^\omega : (X, \tau) \to (Y, \sigma_\omega)$) associated with f is defined as follows: $f^\omega_\omega(x) = f(x)$ (resp., $f_\omega(x) = f(x)$, $f^\omega(x) = f(x)$) for each $x \in X$.

Theorem 3.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following are equivalent.*
(a) *f^ω_ω is continuous.*
(b) *f^ω_ω is g-continuous.*
(c) *f^ω_ω is ω-continuous.*
(d) *f^ω_ω is $g\omega$-continuous.*
(e) *f^ω_ω is $g\omega$-irresolute.*
(f) *f^ω_ω is ω-irresolute.*
(g) *f^ω_ω is g-irresolute.*

The proof follows from Theorem 2.6.

The following result follows immediately from the definitions, Theorem 2.6, and Propositions 2.2 and 2.7.

Theorem 3.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following hold.*
(a) *f_ω is $g\omega$-continuous if and only if it is g-continuous.*
(b) *If f_ω is $g\omega$-irresolute, then f is g-irresolute.*

(c) *If f_ω is g-continuous, then f is $g\omega$-continuous.*
(d) *If f^ω is $g\omega$-continuous, then f is $g\omega$-continuous.*

In Example 3.3(a), f is both $g\omega$-continuous and g-irresolute. However, f_ω is neither g-continuous nor $g\omega$-irresolute ($(\tau_{\text{indis}})_\omega = \tau_{\text{coc}}$ is the cocountable topology). Therefore, the converses of parts (b) and (c) of Theorem 3.5 are not true in general. Example 3.8 shows that also the converse of part (d) is not true.

Proposition 3.6. *Every $g\omega$-irresolute function is $g\omega$-continuous but not conversely.*

The proof follows immediately from the definitions. For the converse, see Example 3.8.

Theorem 3.7. *If $f:(X,\tau)\to(Y,\sigma)$ is closed and f^ω is $g\omega$-continuous, then f is $g\omega$-irresolute.*

Proof. Assume that A is a $g\omega$-open subset of (Y,σ) and that $F\subseteq f^{-1}(A)$, where F is closed in (X,τ). Then, $f(F)$ is closed in (Y,σ) such that $f(F)\subseteq A$. Since A is $g\omega$-open in (Y,σ), $f(F)\subseteq \text{int}_{\sigma_\omega}(A)$, and thus $F\subseteq f^{-1}(\text{int}_{\sigma_\omega}(A))$. Since f^ω is $g\omega$-continuous and $\text{int}_{\sigma_\omega}(A)$ is open in (Y,σ_ω), $f^{-1}(\text{int}_{\sigma_\omega}(A))$ is $g\omega$-open in (X,τ). Therefore, $F\subseteq \text{int}_{\tau_\omega}(f^{-1}(\text{int}_{\sigma_\omega}(A)))\subseteq \text{int}_{\tau_\omega}(f^{-1}(A))$. This means that $f^{-1}(A)$ is $g\omega$-open in (X,τ), and thus f is $g\omega$-irresolute. □

The following example shows that the condition that f^ω is $g\omega$-continuous in Theorem 3.7 cannot be weakened to f being $g\omega$-continuous.

Example 3.8. Let (X,τ) and $A\subset X$ be as in Example 2.14(b). Let $Y=\mathbb{R}$ with the topology $\sigma=\{U\subseteq\mathbb{R}:1\in U\}\bigcup\{\emptyset\}$. Define $f:(X,\tau)\to(Y,\sigma)$ as follows:

$$f(x)=\begin{cases}0, & x\in X-A,\\ 1, & x\in A.\end{cases}\tag{3.1}$$

Then f is closed, open, and $g\omega$-continuous. To show that f is $g\omega$-continuous, let $U\in\sigma$ and let F be any closed set in (X,τ) such that $F\subseteq f^{-1}(U)$. Then $f^{-1}(U)$ must be X, and hence $f^{-1}(U)$ is $g\omega$-open in (X,τ). But neither f is $g\omega$-irresolute nor f^ω is $g\omega$-continuous since $\{0\}$ is ω-open, hence $g\omega$-open in (Y,σ), while $f^{-1}(\{0\})=X-A$ is not $g\omega$-open in (X,τ) ($X-A$ is closed but not ω-open in (X,τ)).

Proposition 3.9. *If $f:(X,\tau)\to(Y,\sigma)$ is $g\omega$-continuous, then for each $x\in X$ and each open set V in (Y,σ) with $f(x)\in V$, there exists a $g\omega$-open set U in (X,τ) such that $x\in U$ and $f(U)\subseteq V$.*

Proof. Let $x\in X$ and let V be any open set in (Y,σ) containing $f(x)$. Put $U=f^{-1}(V)$. Then, by assumption, U is a $g\omega$-open set in (X,τ) such that $x\in U$ and $f(U)\subseteq V$, and the result follows. □

The converse of the above proposition is not true in general as the following example shows.

Example 3.10. Let (X,τ) and $A \subset X$ be as in Example 2.14(b) and let $Y = \{0,1\}$ with the topology $\sigma = \{\emptyset, \{0\}, Y\}$. Define $f : (X,\tau) \to (Y,\sigma)$ as follows:

$$f(x) = \begin{cases} 0, & x \in X - A, \\ 1, & x \in A. \end{cases} \tag{3.2}$$

Then f is not $g\omega$-continuous since $X - A = f^{-1}(\{0\})$ is closed but not ω-open in (X,τ). On the other hand, f satisfies the property stated in Proposition 3.9 because $\{x\}$ is $g\omega$-open in (X,τ) for each $x \in X$.

Recall that a function $f : (X,\tau) \to (Y,\sigma)$ is called θ-continuous [9] (resp., almost continuous [16], weakly continuous [12]) if for each $x \in X$ and each open set V in (Y,σ) containing $f(x)$, there exists an open set U in (X,τ) such that $x \in U$ and $f(\mathrm{cl}_\tau(U)) \subseteq \mathrm{cl}_\sigma(V)$ (resp., $f(U) \subseteq \mathrm{int}_\sigma(\mathrm{cl}_\sigma(V))$, $f(U) \subseteq \mathrm{cl}_\sigma(V)$).

The following result was obtained in [15].

Theorem 3.11. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function from a space (X,τ) into a regular space (Y,σ). Then the following are equivalent.*

(a) *f is continuous.*
(b) *f is θ-continuous.*
(c) *f is almost continuous.*
(d) *f is α-continuous.*
(e) *f is weakly continuous.*

Theorem 3.12. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function from an anti-locally countable space (X,τ) onto a regular space (Y,σ). Then the following are equivalent.*

(a) *f is continuous.*
(b) *f is ω-continuous.*
(c) *For each $x \in X$ and each open set V in (Y,σ) with $f(x) \in V$, there exists an ω-open set U in (X,τ) such that $x \in U$ and $f(U) \subseteq \mathrm{int}_\sigma(\mathrm{cl}_\sigma(V))$.*
(d) *For each $x \in X$ and each open set V in (Y,σ) with $f(x) \in V$, there exists an ω-open set U in (X,τ) such that $x \in U$ and $f(U) \subseteq \mathrm{int}_{\sigma_\omega}(\mathrm{cl}_\sigma(V))$.*
(e) *For each $x \in X$ and each open set V in (Y,σ) with $f(x) \in V$, there exists an ω-open set U in (X,τ) such that $x \in U$ and $f(U) \subseteq \mathrm{cl}_\sigma(V)$.*

Proof. In general, the implications (a)⇒(b)⇒(c)⇒(d)⇒(e) follow from the definitions and the fact that the topology σ_ω is finer than σ.

(e)⇒(a). We show that f is continuous at each $x \in X$. Let $x \in X$ and let V be any open set in (Y,σ) such that $f(x) \in V$. By regularity of (Y,σ), choose two open sets W and H in (Y,σ) such that $f(x) \in H \subseteq \mathrm{cl}_\sigma(H) \subseteq W \subseteq \mathrm{cl}_\sigma(W) \subseteq V$. By assumption, there exists an ω-open set U in (X,τ) such that $x \in U$ and $f(U) \subseteq \mathrm{cl}_\sigma(H)$. Now, choose an open set G in (X,τ) such that $x \in G$ and $G - U$ is countable. We claim that $f(G) \subseteq \mathrm{cl}_\sigma(W)$. If not, choose $t \in f(G) - \mathrm{cl}_\sigma(W)$. Therefore, $t = f(g)$ for some $g \in G$. Now, $t \in Y - \mathrm{cl}_\sigma(W)$ which is an open set in (Y,σ), and so there exist $U_1 \in \tau_\omega$ and an open set $G_1 \in \tau$ such that $g \in U_1 \cap G_1$, $f(U_1) \subseteq \mathrm{cl}_\sigma(Y - \mathrm{cl}_\sigma(W))$, and $G_1 - U_1$ is countable. Finally, since $f(U) \bigcap f(U_1) \subseteq \mathrm{cl}_\sigma(H) \cap \mathrm{cl}_\sigma(Y - \mathrm{cl}_\sigma(W)) \subseteq W \cap \mathrm{cl}_\sigma(Y - \mathrm{cl}_\sigma(W)) = \emptyset$, $U \cap U_1 = \emptyset$, and so $g \in G \cap G_1 \subseteq (G - U) \cup (G_1 - U_1)$, that is, $G \cap G_1$ is a nonempty countable

open set in (X,τ), which contradicts the fact that X is anti-locally countable. Thus $f(G) \subseteq \mathrm{cl}_\sigma(W) \subseteq V$, and hence f is continuous at x. □

The following two examples show that the conditions that X is anti-locally countable and Y is regular in Theorem 3.12 are essential.

Example 3.13. (a) Let (Y,σ) be as in Example 3.10. Then the function $f:(\mathbb{R},\tau_u)\to(Y,\sigma)$ defined by

$$f(x)=\begin{cases}0, & x\in\mathbb{R}-Q,\\ 1, & x\in Q\end{cases} \tag{3.3}$$

is ω-continuous but not continuous. Here, $(\mathbb{R},\tau_u)$ is anti-locally countable and (Y,σ) is not regular.

(b) Let (Y,σ) be as in (a), $Z=\{0,1\}$ with the discrete topology τ_{dis} and let $f:(Y,\sigma)\to(Z,\tau_{\mathrm{dis}})$ be the identity function. Clearly, (Y,σ) is not anti-locally countable, (Z,τ_{dis}) is regular, and f is ω-continuous but not continuous.

Corollary 3.14. *Let $f:(X,\tau)\to(Y,\sigma)$ be a function from an anti-locally countable $T_{1/2}$-space (X,τ) onto a regular space (Y,σ). Then f is continuous if and only if it is $g\omega$-continuous.*

The proof follows from Theorems 2.11 and 3.12.

Example 3.3(a) shows that the assumption that (X,τ) is a $T_{1/2}$-space in the above corollary cannot be dropped.

Theorem 3.15. *Let $f:(X,\tau)\to(Y,\sigma)$ be a $g\omega$-continuous function and let A be a closed subset of (X,τ). Then, the restriction $f|_A:(A,\tau_A)\to(Y,\sigma)$ is $g\omega$-continuous.*

Proof. Let F be a closed subset of (Y,σ). Then $(f|_A)^{-1}(F)=f^{-1}(F)\cap A$. Since f is $g\omega$-continuous, $f^{-1}(F)\in G\omega C(X,\tau)$ and so, by Proposition 2.15, $f^{-1}(F)\cap A\in G\omega C(X,\tau)$. Therefore, by Theorem 2.21(a), $(f|_A)^{-1}(F)\in G\omega C(A,\tau_A)$ and the result follows. □

Theorem 3.16. *Let (X,τ) be a topological space such that $X=A\cup B$, where A, B are both ω-closed in (X,τ). Let $f:(X,\tau)\to(Y,\sigma)$ be given such that the restricions $f|_A$ and $f|_B$ are both $g\omega$-continuous. Then f is $g\omega$-continuous.*

Proof. Let F be a closed subset of (Y,σ). Then, $f^{-1}(F)=(f|_A)^{-1}(F)\cup(f|_B)^{-1}(F)$. Since $(f|_A)^{-1}(F)\in G\omega C(A,\tau_A)$ and A is ω-closed in (X,τ), by Theorem 2.21(b), $(f|_A)^{-1}(F)\in G\omega C(X,\tau)$. Similarly, $(f|_B)^{-1}(F)\in G\omega C(X,\tau)$. By Proposition 2.13, $f^{-1}(F)\in G\omega C(X,\tau)$. Thus f is $g\omega$-continuous. □

Theorem 3.17. *Let (X,τ) and (Y,σ) be topological spaces, where (Y,σ) is locally countable. Then the projection $p_X:(X\times Y,\tau\times\sigma)\to(X,\tau)$ is $g\omega$-irresolute.*

Proof. Let A be a $g\omega$-open subset of (X,τ) and let F be a closed subset of $(X\times Y,\tau\times\sigma)$ such that $F\subseteq p_X^{-1}(A)=A\times Y$. For each $(x,y)\in F$, the closed set $\mathrm{cl}_\tau\{x\}$ is contained in A. By assumption, $\mathrm{cl}_\tau\{x\}\subseteq\mathrm{int}_{\tau_\omega}(A)$. Therefore, $(x,y)\in\mathrm{cl}_\tau\{x\}\times\{y\}\subseteq\mathrm{int}_{\tau_\omega}(A)\times Y$. Now,

we show that $\text{int}_{\tau_\omega}(A)\times Y \subseteq \text{int}_{(\tau\times\sigma)_\omega}(A\times Y)$. Let $(s,t)\in \text{int}_{\tau_\omega}(A)\times Y$. Choose $U\in\tau$, $W\in\tau_\omega$, and a countable open subset V of (Y,σ) such that $(s,t)\in(U\cap W)\times V$, $s\in W\subseteq A$, and $U-W$ is countable. Since $U\times V - W\times Y = (U-W)\times V$ is countable, $W\times Y\in(\tau\times\sigma)_\omega$ and $(s,t)\in W\times Y\subseteq A\times Y$. Therefore, $(s,t)\in\text{int}_{(\tau\times\sigma)_\omega}(A\times Y)$, and hence the result follows. It follows that $(x,y)\in\text{int}_{(\tau\times\sigma)_\omega}(A\times Y)$ for each $(x,y)\in F$, which means that $F\subseteq\text{int}_{(\tau\times\sigma)_\omega}(A\times Y)$. Therefore, $p_X^{-1}(A)=A\times Y$ is $g\omega$-open in $(X\times Y,\tau\times\sigma)$, and hence p_X is $g\omega$-irresolute. □

To show that the condition (Y,σ) being locally countable in Theorem 3.17 is essential, we consider the following example.

Example 3.18. Consider the projection $p:(\mathbb{R}\times\mathbb{R},\tau_u\times\tau_u)\to(\mathbb{R},\tau_u)$ and let $A=\mathbb{R}-Q$. Then A is ω-open (and hence $g\omega$-open) in $(\mathbb{R},\tau_u)$ while $p^{-1}(A)=(\mathbb{R}-Q)\times\mathbb{R}$ is not $g\omega$-open in $(\mathbb{R}\times\mathbb{R},\tau_u\times\tau_u)$ (see Example 2.20). Thus p is not $g\omega$-irresolute.

The proof of the following theorem is left to the reader.

Theorem 3.19. *Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\gamma)$ be two functions. Then the following hold.*

(a) *$g\circ f$ is $g\omega$-continuous if g is continuous and f is $g\omega$-continuous.*
(b) *$g\circ f$ is $g\omega$-irresolute if f and g are $g\omega$-irresolute.*
(c) *$g\circ f$ is $g\omega$-continuous if g is $g\omega$-continuous and f is $g\omega$-irresolute.*
(d) *Let (Y,σ) be a $T_{1/2}$-space. Then, $g\circ f$ is $g\omega$-continuous if g is $g\omega$-continuous and f^ω is $g\omega$-continuous.*

The following example shows that the composition of two $g\omega$-continuous functions need not be $g\omega$-continuous.

Example 3.20. Let (Y,σ) and f be as in Example 3.13, let (Y,γ) be the set $\{0,1\}$ with the topology $\gamma=\{\emptyset,\{1\},Y\}$ and let $g:(Y,\sigma)\to(Y,\gamma)$ be the identity function. Then f and g are both $g\omega$-continuous but $g\circ f$ is not $g\omega$-continuous.

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