NOETHERIAN AND ARTINIAN ORDERED GROUPOIDS—SEMIGROUPS

NIOVI KEHAYOPULU AND MICHAEL TSINGELIS

Chain conditions, finiteness conditions, growth conditions, and other forms of finiteness, Noetherian rings and Artinian rings have been systematically studied for commutative rings and algebras since 1959. In pursuit of the deeper results of ideal theory in ordered groupoids (semigroups), it is necessary to study special classes of ordered groupoids (semigroups). Noetherian ordered groupoids (semigroups) which are about to be introduced are particularly versatile. These satisfy a certain finiteness condition, namely, that every ideal of the ordered groupoid (semigroup) is finitely generated. Our purpose is to introduce the concepts of Noetherian and Artinian ordered groupoids. An ordered groupoid is said to be Noetherian if every ideal of it is finitely generated. In this paper, we prove that an equivalent formulation of the Noetherian requirement is that the ideals of the ordered groupoid satisfy the so-called ascending chain condition. From this idea, we are led in a natural way to consider a number of results relevant to ordered groupoids with descending chain condition for ideals. We moreover prove that an ordered groupoid is Noetherian if and only if it satisfies the maximum condition for ideals and it is Artinian if and only if it satisfies the minimum condition for ideals. In addition, we prove that there is a homomorphism π of an ordered groupoid (semigroup) S having an ideal I onto the Rees quotient ordered groupoid (semigroup) S/I. As a consequence, if S is an ordered groupoid and I an ideal of S such that both I and the quotient groupoid S/I are Noetherian (Artinian), then so is S. Finally, we give conditions under which the proper prime ideals of commutative Artinian ordered semigroups are maximal ideals.

1. Introduction and prerequisites

Notherian and Artinian rings have been extensively studied since 1959. For Noetherian and Artinian rings we refer, for example, in [1]. In pursuit of the deeper results of ideal theory in ordered groupoids (semigroups), it is necessary to study special kinds of ordered groupoids (semigroups). Noetherian ordered groupoids are particularly versatile since they satisfy a certain finiteness condition for ideals. The fact that, when dealing with Noetherian ordered groupoids, we can restrict our attention to finitely generated ideals is of great advantage. We call them Noetherian ordered groupoids, in honor of Emmy

Noether who first initiated their study for rings. An equivalent definition of the Noetherian ordered groupoids is that they satisfy the ascending chain condition for ideals. It is natural then to study ordered groupoids which satisfy the descending chain condition. These are the Artinian ordered groupoids (after Emil Artin). The aim of this paper is to introduce the concepts of Noetherian and Artinian ordered groupoids. We call an ordered groupoid S Noetherian if every ideal of S is finitely generated. We call an ordered groupoid Artinian if it satisfies the descending chain condition for ideals. We prove the following: An ordered groupoid is Noetherian if and only if it satisfies the ascending chain condition for ideals, equivalently, if it satisfies the maximum condition for ideals. An ordered groupoid is Artinian if and only if it satisfies the minimum condition for ideals. Any homomorphism mapping of a Noetherian (resp., Artinian) ordered groupoid onto an ordered groupoid is Noetherian (resp., Artinian). We prove that there is a homomorphism π of an ordered groupoid (resp., ordered semigroup) S having an ideal I onto the Rees quotient ordered groupoid (resp., ordered semigroup) S/I. For an ideal I of an ordered groupoid S, we define $S/I := S\backslash I \cup \{0\}$, where 0 is the zero of S/I. If S is an ordered groupoid and I an ideal of S, then for each ideal A of S containing I, we have $\pi(A) = A/I$, and the set A/I is an ideal of S/I. As a consequence, if S is an ordered groupoid and I an ideal of S such that both I and the quotient groupoid S/I are Noetherian (resp., Artinian), then so is S. We finally find conditions under which the proper prime ideals of commutative Artinian ordered semigroups are maximal. We prove that if S is a commutative, Artinian ordered semigroup having an element b such that the ideal generated by b is S, then each proper prime ideal of S, having the property "if $a \in S\backslash P$, $z \in S^1$, $n \in \mathbb{N}$, and $ba^n \le za^{n+1}$, then $b \le za$", is a maximal ideal of S. This leads to some conditions under which in some Artinian ordered semigroups, the proper prime ideals are maximal ideals. A basic one is that in commutative, cancellative, Artinian ordered semigroups having an identity element, each proper prime ideal is a maximal ideal. Our results, with the appropriate modifications, hold for groupoids (semigroups) without order, as well.

Let S be an ordered groupoid. A nonempty subset A of S is called an ideal of S if (1) $AS \subseteq A$, $SA \subseteq A$; (2) if $a \in A$ and $S \ni b \le a$, then $b \in A$ [2]. For $H \subseteq S$, we denote $(H] := \{t \in S \mid t \le h$ for some $h \in H\}$. For a nonempty subset A of S, we denote by $I(A)$ the ideal of S generated by A. We have $I(A) = (A \cup AS \cup SA \cup SAS]$. For $A = \{a_1, a_2, \ldots, a_n\}$, we write $I(a_1, a_2, \ldots, a_n)$ instead of $I(\{a_1, a_2, \ldots, a_n\})$. An ideal P of S is called proper if $P \ne S$. An ideal P of S is called prime if for $a, b \in S$ such that $ab \in P$, we have $a \in P$ or $b \in P$. Equivalently, if $A, B \subseteq S$, $AB \subseteq P$, this implies that $A \subseteq P$ or $B \subseteq P$. An ordered groupoid $(S, \cdot, \le)$ is called right (resp., left) cancellative if for each $a, b, c \in S$ such that $ac \le bc$ (resp., $ca \le cb$), we have $a \le b$. It is called cancellative if it is both right and left cancellative [3]. For commutative ordered groupoids, we just use the term "cancellative". An identity of an ordered groupoid S is an element of S, denoted by e, such that $ea = ae = a$ for every $a \in S$.

2. Main results

Definition 2.1. An ordered groupoid S satisfies the *ascending chain condition* for ideals if, for any sequence of ideals $I_1, I_2, \ldots, I_i, \ldots$ of S such that

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_i \ldots, \tag{2.1}$$

there exists an element $n \in \mathbb{N}$ such that $I_m = I_n$ for each $m \in \mathbb{N}$, $m \geq n$. $\mathbb{N} = \{1, 2, \ldots\}$ always denotes the set of natural numbers.

Definition 2.2. An ordered groupoid S satisfies the *maximum condition* for ideals if each nonempty set of ideals $\mathcal{A}$ of S, partially ordered by inclusion, has a maximal element. That is, for each nonempty set $\mathcal{A}$ of ideals of S, there is an element $M \in \mathcal{A}$ such that there is no element $T \in \mathcal{A}$ such that $T \supset M$. Equivalently, if $T \in \mathcal{A}$ such that $T \supseteq M$, then $T = M$.

Definition 2.3. An ideal T of an ordered groupoid S is called *finitely generated* if there are elements $a_1, a_2, \ldots, a_n$ in S such that $T = I(a_1, a_2, \ldots, a_n)$.

Definition 2.4. An ordered groupoid S is called *Noetherian* if every ideal of S is finitely generated.

THEOREM 2.5. *Let S be an ordered groupoid. The following are equivalent:*
  (i) *S is Noetherian;*
 (ii) *S satisfies the ascending chain condition for ideals;*
(iii) *S satisfies the maximum condition for ideals.*

Proof. (i)$\Rightarrow$(ii) Let $\{T_i \mid i \in \mathbb{N}\}$ be a sequence of ideals of S such that

$$T_1 \subseteq T_2 \subseteq \cdots \subseteq T_i \ldots, \tag{2.2}$$

and let $T := \bigcup_{i \in \mathbb{N}} T_i$. One can easily see that T is an ideal of S. Then, by (i), there exist $a_1, a_2, \ldots, a_t \in S$ such that $T = I(a_1, a_2, \ldots, a_t)$. Clearly, $a_1, a_2, \ldots, a_t \in T$. Let $i_k \in \mathbb{N}$ such that $a_k \in T_{i_k}$ for each $k = 1, 2, \ldots, t$. We put $n := \max\{i_1, i_2, \ldots, i_t\}$.

Since $i_k \leq n$ for all $k = 1, 2, \ldots, t$, we have $T_{i_k} \subseteq T_n$ for all $k = 1, 2, \ldots, t$. Hence

$$a_k \in T_n \quad \forall k = 1, 2, \ldots, t, \ldots. \tag{2.3}$$

Since T_n is an ideal of S, by (2.3), we have $I(a_1, a_2, \ldots, a_t) \subseteq T_n$. Hence we have $T = I(a_1, a_2, \ldots, a_t) \subseteq T_n \subseteq T$. Then $T_n = T$. We have $T_m = T_n$ for all $\mathbb{N} \ni m \geq n$. Indeed, let $\mathbb{N} \ni m \geq n$. Then we have $T = T_n \subseteq T_m \subseteq T$, so $T_m = T_n$.

(ii)$\Rightarrow$(iii) Let $\mathcal{A}$ be a nonempty set of ideals of S. Suppose there is no maximal ideal of $\mathcal{A}$. Then for each $J \in \mathcal{A}$, there exists

$$I \in \mathcal{A} \text{ such that } J \subset I. \tag{2.4}$$

Indeed, let $J \in \mathcal{A}$. Suppose there is no $I \in \mathcal{A}$ such that $J \subset I$. Then J is a maximal element of $\mathcal{A}$, which is impossible.

Let now $I_1 \in \mathcal{A}$ ($\mathcal{A} \neq \varnothing$). By (2.4), there exists $I_2 \in \mathcal{A}$ such that $I_1 \subset I_2$. Since $I_2 \in \mathcal{A}$, by (2.4), there exists $I_3 \in \mathcal{A}$ such that $I_2 \subset I_3$. Continuing this way, we get a sequence of ideals $I_1, I_2, \ldots, I_i, \ldots$ of S such that

$$I_1 \subset I_2 \subset \cdots \subset I_i \cdots. \tag{2.5}$$

Then, clearly, $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_i \cdots$. Since S satisfies the ascending chain condition for ideals, there is an element $n \in \mathbb{N}$ such that $I_m = I_n$ for each $m \in \mathbb{N}$, $m \geq n$. Then we have $I_{n+1} = I_n$, which is impossible.

(iii)⇒(i) Let T be an ideal of S. Let

$$\mathscr{A} := \{K \subseteq S \mid \exists n \in \mathbb{N},\ a_1, a_2, \ldots, a_n \in T \text{ such that } K = I(a_1, a_2, \ldots, a_n)\}. \qquad (2.6)$$

Let $a \in T$ ($T \neq \varnothing$). Since $I(a) \in \mathscr{A}$, $\mathscr{A}$ is a nonempty set of ideals of S. By (iii), there exists an element $M \in \mathscr{A}$ which is maximal in $\mathscr{A}$. Since $M \in \mathscr{A}$, there exist $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in T$ such that $M = I(a_1, a_2, \ldots, a_n)$. Then $T = I(a_1, a_2, \ldots, a_n)$. In fact, since $a_1, a_2, \ldots, a_n \in T$, we have $I(a_1, a_2, \ldots, a_n) \subseteq T$. Let $b \in T$ such that $b \notin I(a_1, a_2, \ldots, a_n)$. Since

$$M = I(a_1, a_2, \ldots, a_n) \subseteq I(a_1, a_2, \ldots, a_n, b) \in \mathscr{A} \qquad (2.7)$$

and M is maximal in $\mathscr{A}$, we have $I(a_1, a_2, \ldots, a_n, b) = I(a_1, a_2, \ldots, a_n)$. Then $b \in I(a_1, a_2, \ldots, a_n)$, which is impossible. $\qquad \square$

Definition 2.6. An ordered groupoid S satisfies the *descending chain condition* for ideals if, for any sequence of ideals $I_1, I_2, \ldots, I_i, \ldots$ of S such that

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_i \cdots, \qquad (2.8)$$

there exists an element $n \in \mathbb{N}$ such that $I_m = I_n$ for each $m \in \mathbb{N}$, $m \geq n$.

Definition 2.7. An ordered groupoid S satisfies the *minimum condition* for ideals if each nonempty set of ideals $\mathscr{A}$ of S, partially ordered by inclusion, has a minimal element. That is, for each nonempty set $\mathscr{A}$ of ideals of S, there is an element $M \in \mathscr{A}$ such that there is no element $T \in \mathscr{A}$ such that $T \subset M$. Equivalently, if $T \in \mathscr{A}$ such that $T \subseteq M$, then $T = M$.

Definition 2.8. An ordered groupoid S is called *Artinian* if S satisfies the descending chain condition for ideals.

THEOREM 2.9. *An ordered groupoid S is Artinian if and only if it satisfies the minimum conditions for ideals.*

Proof. "If" part. This is the dual of (ii)⇒(iii) of Theorem 2.5.

"Only if" part. Let $\{T_i \mid i \in \mathbb{N}\}$ be a sequence of ideals of S such that

$$T_1 \supseteq T_2 \supseteq \cdots \supseteq T_i \cdots. \qquad (2.9)$$

We put $\mathscr{A} = \{T_i \mid i \in \mathbb{N}\}$. Since $\mathscr{A}$ is a nonempty set of ideals of S, by hypothesis, there is an element T_n in $\mathscr{A}$ which is minimal element of $\mathscr{A}$. We have $T_m = T_n$ for every $\mathbb{N} \ni m \geq n$. Indeed, let $\mathbb{N} \ni m \geq n$. Since $T_n \supseteq T_m \in \mathscr{A}$ and T_n is a minimal element of $\mathscr{A}$, we have $T_n = T_m$. $\qquad \square$

A mapping $f : S \to T$ of an ordered groupoid S into an ordered groupoid T is called homomorphism if (1) $f(xy) = f(x)f(y)$ for all $x, y \in S$; and (2) if $x \leq y$ implies that $f(x) \leq f(y)$.

LEMMA 2.10. *If S, T are two ordered groupoids, $f : S \to T$ a homomorphism and onto mapping, and I an ideal of T, then $f^{-1}(I)$ is an ideal of S.*

The proof of the lemma is easy.

PROPOSITION 2.11. *Let S be a Noetherian (resp., Artinian) ordered groupoid, T an ordered groupoid, and $f : S \to T$ a homomorphism and onto mapping. Then T is Noetherian (resp., Artinian).*

Proof. Let S be Noetherian and $\{J_i \mid i \in I\}$ a family of ideals of S such that

$$J_1 \subseteq J_2 \subseteq \cdots \subseteq J_i \cdots . \tag{2.10}$$

Let $I_i := f^{-1}(J_i)$, $i \in \mathbb{N}$.

By Lemma 2.10, I_i is an ideal of S for every $i \in \mathbb{N}$. Moreover, by (2.10), we have

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_i \cdots . \tag{2.11}$$

Since S is Noetherian, there exists $n \in \mathbb{N}$ such that $I_m = I_n$ for each $m \in \mathbb{N}$, $m \geq n$. Then $J_m = J_n$ for every $m \in \mathbb{N}$, $m \geq n$. In fact, let $m \in \mathbb{N}$, $m \geq n$. Since $I_m = f^{-1}(J_m)$ and f is onto, we have

$$f(I_m) = f\big(f^{-1}(J_m)\big) = J_m. \tag{2.12}$$

Similarly, $f(I_n) = J_n$. Since $I_m = I_n$, we have $f(I_m) = f(I_n)$. Then $J_m = J_n$.   □

LEMMA 2.12 (cf. [4, Lemma 2]). *Let $(S, \cdot, \leq_S)$ be an ordered groupoid, I an ideal of S. Let $S/I := S\backslash I \cup \{0\}$, where 0 is an arbitrary element of I. Define an operation "$*$" and an order "$\leq$" on S/I as follows:*

$$* : S/I \times S/I \longrightarrow S/I \mid (x, y) \longrightarrow x * y,$$

$$x * y := \begin{cases} xy & \text{if } xy \in S\backslash I, \\ 0 & \text{if } xy \in I \end{cases} \tag{2.13}$$

$$\leq := \big(\leq_S \cap \big[(S\backslash I) \times (S\backslash I)\big]\big) \cup \{(0, x) \mid x \in S/I\}.$$

Then $(S/I, *, \leq)$ is an ordered groupoid (called the Rees quotient of S by I) and 0 is its zero. In particular, if the multiplication on S is associative, then S/I is an ordered semigroup.

LEMMA 2.13. *Let $(S, \cdot, \leq_S)$ be an ordered groupoid, I an ideal of S. Then the mapping*

$$\pi : (S, \cdot, \leq_S) \longrightarrow (S/I, *, \leq) \mid a \longrightarrow \begin{cases} a & \text{if } a \in S\backslash I, \\ 0 & \text{if } a \in I \end{cases} \tag{2.14}$$

is a homomorphism and onto mapping.

Proof. The mapping π is clearly well defined. Let $a, b \in S$. Then $\pi(ab) = \pi(a) * \pi(b)$. In fact, we have the following.

(1) Let $ab \in S\backslash I$. Then if $a \in I$, then $ab \in IS \subseteq I$. If $b \in I$, then $ab \in SI \subseteq I$, which is impossible. Thus we have $a, b \in S\backslash I$. Since $a, b, ab \in S\backslash I$, we have $\pi(ab) := ab$, $\pi(a) := a$, $\pi(b) := b$. Since $ab \in S\backslash I$, we have $a * b := ab$. Hence we have

$$\pi(ab) = ab = a * b = \pi(a) * \pi(b). \tag{2.15}$$

(2) Let $ab \in I$. Then $\pi(ab) := 0$. Then
   (a) if $a, b \in I$, then $\pi(a) := 0$, $\pi(b) := 0$, and $\pi(ab) = 0 = 0 * 0 = \pi(a) * \pi(b)$;
   (b) if $a \in I$, $b \in S \backslash I$, then $\pi(a) := 0$, $\pi(b) := b$, and $\pi(ab) = 0 = 0 * b = \pi(a) * \pi(b)$ (since $0b \in IS \subseteq I$);
   (c) if $a \in S \backslash I$, $b \in I$, then $\pi(a) := a$, $\pi(b) := 0$, and $\pi(ab) = 0 = a * 0 = \pi(a) * \pi(b)$;
   (d) let $a, b \in S \backslash I$. Then $\pi(a) := a$, $\pi(b) := b$. Since $ab \in I$, we have $a * b := 0$ and $\pi(ab) := 0$. Thus, we have

$$\pi(ab) = 0 = a * b = \pi(a) * \pi(b). \tag{2.16}$$

Let $a, b \in S$, $a \leq_S b$. Then $\pi(a) \leq \pi(b)$. Indeed, we have the following.
   (1) If $a \in I$, then $\pi(a) := 0$. Since $b \in S$, we have $\pi(b) \in S/I$. Then

$$(\pi(a), \pi(b)) = (0, \pi(b)) \in \{(0, x) \mid x \in S/I\} \subseteq \leq . \tag{2.17}$$

   (2) Let $a \in S \backslash I$. Then $\pi(a) := a$.
If $b \in I$, then since $S \ni a \leq_S b \in I$ and I is an ideal of S, we have $a \in I$, which is impossible. Thus $b \in S \backslash I$ and $\pi(b) := b$. Then we have

$$(\pi(a), \pi(b)) = (a, b) \in \leq_S \cap \big[(S \backslash I) \times (S \backslash I)\big] \subseteq \leq . \tag{2.18}$$

The mapping π is clearly an onto mapping. This is because the set I is nonempty.   □

Notation 2.14. Let S be an ordered groupoid and I an ideal of S. If A is an ideal of S such that $I \subseteq A$, we denote $A/I := A \backslash I \cup \{0\}$, where 0 is the zero of S/I.

LEMMA 2.15. *Let $(S, \cdot, \leq_S)$ be an ordered groupoid and I an ideal of S. Let A be an ideal of S such that $I \subseteq A$. Then the set A/I is an ideal of S/I.*

Proof. (A) We have $\pi(A) = A/I$. In fact, let $x \in \pi(A)$. Then $x = \pi(y)$ for some $y \in A$. Then the following hold.
   (1) If $y \in I$, then $x = \pi(y) := 0 \in A/I$.
   (2) If $y \in A \backslash I$ then, since $A \backslash I \subseteq S \backslash I$, we have

$$x = \pi(y) := y \in A \backslash I \subseteq A/I. \tag{2.19}$$

Let $x \in A/I$. Then $x \in \pi(A)$. Indeed, we have the following.
   (1) If $x \in A \backslash I$, then $x \in S \backslash I$, and $\pi(x) := x$. Since $x \in A$, we have $\pi(x) \in \pi(A)$, thus we have $x \in \pi(A)$.
   (2) Let $x = 0$. Take an element $y \in I$ ($I \neq \varnothing$). Then $\pi(y) := 0$. Since $y \in I \subseteq A$, we have $\pi(y) \in \pi(A)$, thus we have $x \in \pi(A)$.
   (B) The set $\pi(A)$ is an ideal of S/I. Indeed, $\varnothing \neq \pi(A) \subseteq S/I$ (since $A \neq \varnothing$).
Let $a \in S/I$, $b \in \pi(A)$. Let $a = \pi(x)$ for some $x \in S$ (π is onto) and $b = \pi(y)$ for some $y \in A$. Since π is a homomorphism and $xy \in SA \subseteq A$, we have

$$a * b = \pi(x) * \pi(y) = \pi(xy) \in \pi(A). \tag{2.20}$$

Similarly, we get $\pi(A) * S/I \subseteq \pi(A)$.

Let $S/I \ni a \leq b \in \pi(A)$. Then $a \in \pi(A)$. Indeed, let $a = \pi(x)$ for some $x \in S$ (π is onto) and $b = \pi(y)$ for some $y \in A$. Then the following hold.

(1) If $x \in I$, then $x \in A$, hence $a = \pi(x) \in \pi(A)$.

(2) Let $x \in S \setminus I$. Then $a = \pi(x) := x \in S \setminus I$, so $a \neq 0$ (since $0 \in I$).

Since $(a,b) \in \leq = (\leq_S \cap [(S \setminus I) \times (S \setminus I)]) \cup \{(0,x) \mid x \in S/I\}$ and $a \neq 0$, we have

$$
\begin{aligned}
a &\leq_S b, \\
a,b &\in S \setminus I.
\end{aligned}
\tag{2.21}
$$

If $y \in I$, then $S \setminus I \ni b = \pi(y) := 0$, which is impossible. Thus we have $y \in S \setminus I$. Then $b = \pi(y) := y \in A$. Since $S \ni a \leq_S b \in A$ and A is an ideal of S, we have $a \in A$, so $\pi(a) \in \pi(A)$. Since $a \in S \setminus I$, we have $\pi(a) := a$. Hence we have $a \in \pi(A)$. $\qquad\square$

Remark 2.16. If S is a set and T_1, T_2, I are subsets of S such that $T_1 \subseteq T_2$, $T_2 \cap I \subseteq T_1 \cap I$ and $T_2 \setminus I \subseteq T_1 \setminus I$, then $T_1 = T_2$. Indeed, let $a \in T_2$. If $a \in I$, then $a \in T_2 \cap I \subseteq T_1 \cap I \subseteq T_1$. If $a \notin I$, then

$$
a \in T_2 \setminus I \subseteq T_1 \setminus I \subseteq T_1.
\tag{2.22}
$$

Clearly, if $T_1 \subseteq T_2$, then $T_2 \cap I \subseteq T_1 \cap I$ is equivalent to $T_1 \cap I = T_2 \cap I$ and $T_2 \setminus I \subseteq T_1 \setminus I$ is equivalent to $T_1 \setminus I = T_2 \setminus I$.

Remark 2.17. If $(S, \cdot, \leq_S)$ is an ordered groupoid, then each nonempty subset A of S with the multiplication " $\circ$ " and the order " $\leq_A$ " on A defined by

$$
\begin{aligned}
\circ &: A \times A \longrightarrow A \mid (a,b) \longrightarrow a.b \\
\leq_A &:= \leq_S \cap (A \times A)
\end{aligned}
\tag{2.23}
$$

is an ordered groupoid (a subgroupoid of S). In particular, if the multiplication on S is associative, then A is an ordered semigroup.

Theorem 2.18. *Let S be an ordered groupoid and I an ideal of S. If both I and S/I are Noetherian, then so is S.*

Proof. Let $\{T_i \mid i \in \mathbb{N}\}$ be a sequence of ideals of S such that

$$
T_1 \subseteq T_2 \subseteq \cdots \subseteq T_i \subseteq \cdots \ (\implies \exists n \in \mathbb{N} \text{ such that } T_m = T_n \ \forall m \in \mathbb{N}, \ m \geq n \, ?) \tag{2.24}
$$

Since T_k, I are ideals of S, $T_k \cap I$ is an ideal of S for all $k \in \mathbb{N}$. We put $I_k := T_k \cap I$ ($k \in \mathbb{N}$), and we have

$$
I_1 \subseteq I_2 \subseteq \cdots \subseteq I_i \subseteq \cdots .
\tag{2.25}
$$

Since I_k is an ideal of S and $I_k \subseteq I$, I_k is an ideal of I for each $k \in \mathbb{N}$. Since I is Noetherian, there exists $p \in \mathbb{N}$ such that

$$
I_p = I_m \quad \forall \mathbb{N} \ni m \geq p.
\tag{2.26}
$$

Since T_k, I are ideals of S, $T_k \cup I$ is an ideal of S for all $k \in \mathbb{N}$.

We put $M_k := T_k \cup I \; (k \in \mathbb{N})$, and we have

$$
\begin{aligned}
& M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq \cdots \\
& \quad \Longrightarrow M_1 \backslash I \subseteq M_2 \backslash I \subseteq \cdots \subseteq M_i \backslash I \subseteq \cdots \\
& \quad \Longrightarrow (M_1 \backslash I) \cup \{0\} \subseteq (M_2 \backslash I) \cup \{0\} \subseteq \cdots \subseteq (M_i \backslash I) \cup \{0\} \qquad (2.27) \\
& \quad \subseteq \cdots \quad \text{(where 0 is the zero of } S/I) \\
& \quad \Longrightarrow M_1/I \subseteq M_2/I \subseteq \cdots \subseteq M_i/I \subseteq \cdots .
\end{aligned}
$$

Since M_k is an ideal of S and $I \subseteq M_k$, by Lemma 2.15, the set M_k/I is an ideal of S/I for each $k \in \mathbb{N}$. Since S/I is Noetherian, there exists $q \in \mathbb{N}$ such that

$$
M_q/I = M_m/I \quad \forall \mathbb{N} \ni m \geq q. \qquad (2.28)
$$

We put $n := \max\{p, q\}$.

Let $m \in \mathbb{N}$, $m \geq n$. Then $T_m = T_n$. Indeed, since $m \geq n$, we have

$$
T_n \subseteq T_m. \qquad (2.29)
$$

Since $m \geq n \geq p$, by (2.26), we have $I_p = I_m = I_n$, then

$$
T_m \cap I = T_n \cap I. \qquad (2.30)
$$

Since $m \geq n \geq q$, by (2.28), we have $M_q/I = M_m/I = M_n/I$, then

$$
(M_m \backslash I) \cup \{0\} = (M_n \backslash I) \cup \{0\}. \qquad (2.31)
$$

Since $M_m := T_m \cup I \subseteq S$, we have $M_m \backslash I \subseteq S \backslash I$. Since $0 \notin S \backslash I$, $0 \notin M_m \backslash I$.

Similarly, $0 \notin M_n \backslash I$. By (2.31), we have $M_m \backslash I = M_n \backslash I$. Then, since $M_m \backslash I = (T_m \cup I) \backslash I = T_m \backslash I$ and $M_n \backslash I = T_n \backslash I$, we have

$$
T_m \backslash I = T_n \backslash I. \qquad (2.32)
$$

By (2.29), (2.30), (2.32), and Remark 2.16, we have $T_n = T_m$. $\qquad \square$

In a similar way, we prove the following.

THEOREM 2.19. *Let S be an ordered groupoid and I an ideal of S. If both I and S/I are Artinian, then so is S.*

In the following, we consider ordered semigroups.

Definition 2.20. An ideal M of an ordered semigroup S is called maximal if the following hold.

(1) $M \neq S$.

(2) If T is an ideal of S such that $T \supset M$, then $T = S$.

For convenience, we denote $S^1 := S \cup \{1\}$, where 1 is a symbol denoting that $a1 = 1a = a$ for all $a \in S$. We have

$$
\begin{aligned}
I(a) = (a \cup Sa \cup aS \cup SaS] &= (a] \cup (Sa] \cup (aS] \cup (SaS] \\
&= \{t \in S \mid \ t \le a \text{ or } t \le za;\ z \in S \text{ or } t \le ah;\ h \in S \text{ or } t \le xay;\ x, y \in S\}.
\end{aligned} \tag{2.33}
$$

Using this notation, for an element $t \in I(a)$, we can write $t \le xay$ for some $x, y \in S^1$.

Also, the relation $ba^n \le za^{n+1}$; $z \in S^1$ in Theorem 2.21, means that $ba^n \le a^{n+1}$ or $ba^n \le za^{n+1}$ for some $z \in S$.

THEOREM 2.21. *Let S be a commutative, Artinian ordered semigroup. Let $b \in S$ such that $I(b) = S$ and P a proper prime ideal of S having the property:*

$$
\text{If } a \in S \backslash P,\ z \in S^1,\ n \in \mathbb{N},\ ba^n \le za^{n+1}, \text{ then } b \le za. \tag{2.34}
$$

Then P is a maximal ideal of S.

Proof. By hypothesis, $P \ne S$. Let now T be an ideal of S such that $T \supset P$. Then $T = S$. Indeed, let $a \in T$, $a \notin P$. Clearly, we have

$$
I(a) \supseteq I(a^2) \supseteq I(a^3) \supseteq \cdots \supseteq I(a^n) \supseteq \cdots \tag{2.35}
$$

where $I(a^k)$ is an ideal of S for every $k \in \mathbb{N}$. Since S is Artinian, there exists $n \in \mathbb{N}$ such that $I(a^m) = I(a^n)$ for every $\mathbb{N} \ni m \ge n$. So $I(a^{n+1}) = I(a^n)$. Since

$$
ba^n \in Sa^n \subseteq I(a^n) = I(a^{n+1}), \tag{2.36}
$$

there exist $x, y \in S^1$ such that $ba^n \le xa^{n+1}y = xya^{n+1}$. Since $x, y \in S^1$, we have $xy \in S^1$. Since $a \in S \backslash P$, $xy \in S^1$, $n \in \mathbb{N}$, and $ba^n \le (xy)a^{n+1}$, by hypothesis, we have

$$
b \le (xy)a \in Sa \subseteq I(a). \tag{2.37}
$$

Since $a \in T$ and T is an ideal of S, we have $I(a) \subseteq T$. Since $S \ni b \le (xy)a \in T$, we have $b \in T$. Then $I(b) \subseteq T$. By hypothesis, $I(b) = S$. So we have $T = S$. □

COROLLARY 2.22. *Let S be a commutative, Artinian ordered semigroup with an identity element " e". Then each proper prime ideal P of S having the property "if $a \in S \backslash P$, $z \in S^1$, $n \in \mathbb{N}$, $a^n \le za^{n+1}$, then $e \le za$", is a maximal ideal of S.*

Proof. We have

$$
I(e) = (e \cup Se \cup eS \cup SeS] = (e \cup S \cup S^2] = (e \cup S] = S. \tag{2.38}
$$

The proof is an immediate consequence of Theorem 2.21. □

THEOREM 2.23. *Let S be a commutative, Artinian ordered semigroup having an element b such that $I(b) = S$. Then each proper prime ideal P of S for which the ordered semigroup $S\backslash P$ is cancellative is a maximal ideal of S.*

Proof. Let P be a proper prime ideal of S and let $S\backslash P$ be cancellative. It is enough to prove that condition (2.34) of Theorem 2.21 is satisfied.

Let $a \in S\backslash P$, $z \in S^1$, $n \in \mathbb{N}$, and $ba^n \leq za^{n+1}$. Then $b \leq za$. In fact, we have the following.

   (A) $b \notin P$. Since $b \in P$ implies that $S = I(b) \subseteq P$, then $P = S$, which is impossible.
   (B) $a^n \notin P$. Indeed, if $a^n \in P$ then, since P is prime and $a \notin P$, we have $a^{n-1} \in P$. Continuing this way, we get that $a \in P$, which is impossible.
   (C) $za \notin P$. Indeed, since $z \in S^1$ and $a \in S$, we have $za \in S$. Let $za \in P$. Then $S \ni ba^n \leq za^{n+1} = (za)a^n \in PS \subseteq P$. Then $ba^n \in P$. Since P is prime, we have $b \in P$ or $a^n \in P$, which is impossible.

Since $ba^n \leq (za)a^n$; $b, a^n, za \in S\backslash P$ and $S\backslash P$ is cancellative, we get that $b \leq za$. $\square$

COROLLARY 2.24. *Let S be a commutative, cancellative, Artinian ordered semigroup having an element b such that $I(b) = S$. Then each proper prime ideal P of S is a maximal ideal of S.*

COROLLARY 2.25. *Let S be a commutative, Artinian ordered semigroup having an identity element e. Then each proper prime ideal P of S for which the ordered semigroup $S\backslash P$ is cancellative is a maximal ideal of S.*

COROLLARY 2.26. *Let S be a commutative, cancellative, Artinian ordered semigroup having an identity element e. Then each proper prime ideal P of S is a maximal ideal of S.*

THEOREM 2.27. *Let S be a commutative, Artinian ordered semigroup, b an element of S such that $I(b) = S$, and P a proper prime ideal of S having the property:*

$$If \ a, c \in S\backslash P \ such \ that \ bc \leq ac, \ then \ b \leq a. \tag{2.39}$$

Then P is a maximal ideal of S.

Proof. It is enough to prove that condition (2.34) of Theorem 2.21 is satisfied. Let $a \in S\backslash P$, $z \in S^1$, $n \in \mathbb{N}$, and $ba^n \leq za^{n+1}$. Then $b \leq za$. In fact, as in Theorem 2.23, we prove that $b \notin P$, $a^n \notin P$, $za \notin P$. On the other hand,

$$ba^n \leq (za)a^n; \quad za, a^n \in S\backslash P. \tag{2.40}$$

By hypothesis, we have $b \leq za$. $\square$

COROLLARY 2.28. *Let S be a commutative, Artinian ordered semigroup having an identity element e. Then each proper prime ideal P of S having the property "if $a, c \in S\backslash P$ such that $c \leq ac$, then $e \leq a$", is a maximal ideal of S.*

Acknowledgments

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Niovi Kehayopulu: University of Athens, Department of Mathematics, 15784 Panepistimiopolis, Greece
E-mail address: nkehayop@cc.uoa.gr

Michael Tsingelis: University of Athens, Department of Mathematics, 15784 Panepistimiopolis, Greece