A FIXED POINT THEOREM FOR A PAIR OF MAPS SATISFYING A GENERAL CONTRACTIVE CONDITION OF INTEGRAL TYPE

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We give a general condition which enables one to easily establish fixed point theorems for a pair of maps satisfying a contractive inequality of integral type.

Branciari [1] obtained a fixed point result for a single mapping satisfying an analogue of Banach's contraction principle for an integral-type inequality. The second author [3] proved two fixed point theorems involving more general contractive conditions. In this paper, we establish a general principle, which makes it possible to prove many fixed point theorems for a pair of maps of integral type.

Define $\Phi = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}\}$ such that φ is nonnegative, Lebesgue integrable, and satisfies

$$\int_0^{\epsilon} \varphi(t)dt > 0 \quad \text{for each } \epsilon > 0. \tag{1}$$

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy that

(i) ψ is nonnegative and nondecreasing on $\mathbb{R}^+$,

(ii) $\psi(t) < t$ for each $t > 0$,

(iii) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each fixed $t > 0$.

Define $\Psi = \{\psi : \psi$ satisfies (i)–(iii)$\}$.

Lemma 1. *Let S and T be self-maps of a metric space (X,d). Suppose that there exists a sequence $\{x_n\} \subset X$ with $x_0 \in X$, $x_{2n+1} := Sx_{2n}$, $x_{2n+2} := Tx_{2n+1}$, such that $\overline{\{x_n\}}$ is complete and there exists a $k \in [0,1)$ such that*

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \le \psi\left(\int_0^{d(x,y)} \varphi(t)dt\right) \tag{2}$$

for each distinct $x, y \in \overline{\{x_n\}}$ satisfying either $x = Ty$ or $y = Sx$, where $\varphi \in \Phi$, $\psi \in \Psi$.

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Then, either
(a) *S or T has a fixed point in $\{x_n\}$ or*
(b) *$\{x_n\}$ converges to some point $p \in X$ and*

$$\int_0^{d(x_n,p)} \varphi(t)dt \leq \sum_{i=n}^{\infty} \psi^i(d) \quad \textit{for } n > 0, \tag{3}$$

where

$$d := \int_0^{d(x_0,x_1)} \varphi(t)dt. \tag{4}$$

Proof. Suppose that $x_{2n+1} = x_{2n}$ for some n. Then $x_{2n} = x_{2n+1} = Sx_{2n}$, and x_{2n} is a fixed point of S. Similarly, if $x_{2n+2} = x_{2n+1}$ for some n, then x_{2n+1} is a fixed point of T.

Now assume that $x_n \neq x_{n+1}$ for each n. With $x = x_{2n}$, $y = x_{2n+1}$, (2) becomes

$$\int_0^{d(x_{2n+1},x_{2n+2})} \varphi(t)dt \leq \psi\left(\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)dt\right). \tag{5}$$

Substituting $x = x_{2n}$, $y = x_{2n-1}$, (2) becomes

$$\int_0^{d(x_{2n+1},x_{2n})} \varphi(t)dt \leq \psi\left(\int_0^{d(x_{2n},x_{2n-1})} \varphi(t)dt\right). \tag{6}$$

Therefore, for each $n \geq 0$,

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq \psi\left(\int_0^{d(x_{n-1},x_n)} \varphi(t)dt\right) \leq \cdots \leq \psi^n(d). \tag{7}$$

Let $m,n \in \mathbb{N}$, $m > n$. Then, using the triangular inequality,

$$d(x_n,x_m) \leq \sum_{i=n}^{m-1} d(x_i,x_{i+1}). \tag{8}$$

It can be shown by induction that

$$\int_0^{d(x_n,x_m)} \varphi(t)dt \leq \sum_{i=n}^{m-1} \int_0^{d(x_i,x_{i+1})} \varphi(t)dt. \tag{9}$$

Using (7) and (9),

$$\int_0^{d(x_n,x_m)} \varphi(t)dt \leq \sum_{i=n}^{\infty} \psi^i(d) \leq \sum_{i=n}^{\infty} \psi^i(d). \tag{10}$$

Taking the limit of (10) as $m,n \to \infty$ and using condition (iii) for ψ, it follows that $\{x_n\}$ is Cauchy, hence convergent, since X is complete. Call the limit p. Taking the limit of (10) as $m \to \infty$ yields (3). □

Theorem 2. *Let (X,d) be a complete metric space, and let S, T be self-maps of X such that for each distinct $x,y \in X$,*

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \le \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right), \tag{11}$$

where $k \in [0,1)$, $\varphi \in \Phi$, $\psi \in \Psi$, and

$$M(x,y) := \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{[d(x,Ty)+d(y,Sx)]}{2}\right\}. \tag{12}$$

Then S and T have a unique common fixed point.

Proof. We will first show that any fixed point of S is also a fixed point of T, and conversely.

Let $p = Sp$. Then

$$M(p,p) = \max\left\{0, 0, d(p,Tp), \frac{d(p,Tp)}{2}\right\} = d(p,Tp), \tag{13}$$

and (11) becomes

$$\int_0^{d(p,Tp)} \varphi(t)dt \le \psi\left(\int_0^{d(p,Tp)} \varphi(t)dt\right), \tag{14}$$

which, from (1), implies that $p = Tp$.

Similarly, $p = Tp$ implies that $p = Sp$.

We will now show that S and T satisfy (2).

$$M(x,Sx) = \max\left\{d(x,Sx), d(x,Sx), d(Sx,TSx), \frac{[d(x,TSx)+0]}{2}\right\}. \tag{15}$$

From the triangular inequality,

$$\frac{d(x,TSx)}{2} \le \frac{[d(x,Sx)+d(Sx,TSx)]}{2} \le \max\{d(x,Sx), d(Sx,TSx)\}. \tag{16}$$

Thus, (11) becomes

$$\int_0^{d(Sx,TSx)} \varphi(t)dt \le k\int_0^{d(Sx,TSx)} \varphi(t)dt, \tag{17}$$

a contradiction to (1).

Therefore, for all $x \in X$, $M(x,Sx) = d(x,Sx)$, and (2) is satisfied. If condition (a) of Lemma 1 is true, then S or T has a fixed point. But it has already been shown that any fixed point of S is also a fixed point of T, and conversely. Thus S and T have a common fixed point.

Suppose that conclusion (b) of Lemma 1 is true. Then, from (3),

$$\int_0^{d(Sx_{2n},Tp)} \varphi(t)dt \le \psi\left(\int_0^{d(x_{2n},p)} \varphi(t)dt\right), \tag{18}$$

which implies, since X is complete, that $\lim d(Sx_{2n},Tp) = 0$.

Therefore,

$$d(p,Tp) \leq d(p,Sx_{2n}) + d(Sx_{2n},Tp) \longrightarrow 0, \tag{19}$$

and p is a fixed point of T, hence a fixed point of S. Condition (11) clearly implies uniqueness of the fixed point. □

Every contractive condition of integral type automatically includes a corresponding contractive condition not involving integrals, by setting $\varphi(t) \equiv 1$ over $\mathbb{R}^+$.

There are many contractive conditions of integral type which satisfy (2). Included among these are the analogues of the many contractive conditions involving rational expressions and/or products of distances. We conclude this paper with one such example.

Corollary 3. *Let (X,d) be a complete metric space, S and T self-maps of X such that, for each distinct $x, y \in X$,*

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \leq k\int_0^{n(x,y)} \varphi(t)dt, \tag{20}$$

where $\varphi \in \Phi$, $k \in [0,1)$, and

$$n(x,y) := \max\left\{\frac{d(y,Ty)[1+d(x,Sx)]}{1+d(x,y)}, d(x,y)\right\}. \tag{21}$$

Then S and T have a unique common fixed point.

Proof.

$$n(x,Sx) = \max\{d(Sx,TSx), d(x,Sx)\}. \tag{22}$$

As in the proof of Theorem 2, it is easy to show that any fixed point of S is also a fixed point of T, and conversely.

If $n(x,Sx) = d(Sx,TSx)$, then an argument similar to that of Theorem 2 leads to a contradiction. Therefore $n(x,Sx) = d(x,Sx)$, and either S or T has a common fixed point, or (3) is satisfied. In the latter case, with $\lim x_n = p$, $n(p,p) = 0$, so that, from (20), p is a fixed point of S, hence of T. Uniqueness of p is easily established.

Corollary 3 is also a consequence of Lemma 1.

We now provide an example, kindly supplied by one of the referees, to show that Lemma 1 is more general than [2, Theorem 3.1].

Example 4. Let $X := \{1/n : n \in \mathbb{N} \cup \{0\}\}$ with the Euclidean metric and S, T are self-maps of X defined by

$$S\left(\frac{1}{n}\right) = \begin{cases} \dfrac{1}{n+1} & \text{if } n \text{ is odd}, \\ \dfrac{1}{n+2} & \text{if } n \text{ is even}, \\ 0 & \text{if } n = \infty, \end{cases} \qquad T\left(\frac{1}{n}\right) = \begin{cases} \dfrac{1}{n+1} & \text{if } n \text{ is even}, \\ \dfrac{1}{n+2} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n = \infty. \end{cases} \tag{23}$$

For each n, define $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$. With $x_0 = 1$, let $O(1)$ denote the orbit of $x_0 = 1$; that is, $O(1) = \{1, 1/2, 1/3, \ldots\}$ and $\overline{O(1)} = O(1) \cup \{0\} = X$. For $x, y \in O(1)$, $y = 1/m$, m even and $x = 1/n = Ty = 1/(m+1)$, $Sx = 1/(m+2)$, so that

$$\begin{aligned} d(Sx, Ty) &= \left| \frac{1}{m+1} - \frac{1}{m+1} \right| = \frac{1}{m+1} - \frac{1}{m+2} = \frac{1}{(m+1)(m+2)}, \\ d(x,y) &= \left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{1}{m+1} - \frac{1}{n} \right| = \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}. \end{aligned} \tag{24}$$

Thus

$$\frac{d(Sx, Ty)}{d(x,y)} = \frac{m}{m+2} \leq 1. \tag{25}$$

Also

$$\sup_{n \in \mathbb{N}} \frac{d(Sx, Ty)}{d(x,y)} = 1, \tag{26}$$

so that there is no number $c \in [0, 1)$ such that $d(Sx, Ty) \leq cd(x, y)$ for $x, y \in O(1)$ and $x = Ty$. Therefore, [2, Theorem 3.1] cannot be used. On the other hand, the hypotheses of Lemma 1 are satisfied. To see this, it will be shown that condition (2) is satisfied for some $\varphi \in \Phi$.

We will first show that for any $x = 1/n$, $y = 1/m \in O(1)$ satisfying either $x = Ty$ or $y = Sx$,

$$d(Sx, Ty) \leq \left| \frac{1}{n+1} - \frac{1}{m+1} \right|. \tag{27}$$

There are four cases.

Case 1. $y = 1/m$, m even, $x = 1/n = Ty = 1/(m+1)$, and $Sx = 1/(m+2)$. Then

$$d(Sx, Ty) = \left| \frac{1}{m+2} - \frac{1}{m+1} \right| = \left| \frac{1}{n+1} - \frac{1}{m+1} \right|. \tag{28}$$

Case 2. $y = 1/m$, m odd, $x = 1/n = Ty = 1/(m+2)$, and $Sx = 1/(m+3)$. Then

$$\begin{aligned} d(Sx, Ty) &= \left| \frac{1}{m+3} - \frac{1}{m+2} \right| = \frac{1}{m+2} - \frac{1}{m+3} \\ &\leq \frac{1}{m+1} - \frac{1}{m+3} = \left| \frac{1}{n+1} - \frac{1}{m+1} \right|. \end{aligned} \tag{29}$$

Case 3. $x = 1/n$, n even, $y = 1/m = Sx = 1/(n+2)$, and $Ty = 1/(n+3)$. Then

$$\begin{aligned} d(Sx, Ty) &= \left| \frac{1}{n+2} - \frac{1}{n+3} \right| = \frac{1}{n+2} - \frac{1}{n+3} \\ &\leq \frac{1}{n+1} - \frac{1}{n+3} = \left| \frac{1}{n+1} - \frac{1}{n+3} \right|. \end{aligned} \tag{30}$$

Case 4. $x = 1/n$, n odd, $y = 1/m = Sx = 1/(n+1)$, and $Ty = 1/(n+2)$. Then

$$d(Sx, Ty) = \left| \frac{1}{n+1} - \frac{1}{n+2} \right| = \left| \frac{1}{n+1} - \frac{1}{m+1} \right|. \tag{31}$$

Thus in all cases, (20) is satisfied.

Define φ by $\varphi(t) = t^{1/2-2}[1 - \log t]$ for $t > 0$ and $\varphi(0) = 0$. Then, for any $\tau > 0$,

$$\int_0^{\tau} \varphi(t)dt = \tau^{1/\tau}, \tag{32}$$

and $\varphi \in \Phi$.

Using [1, Example 3.6],

$$\begin{aligned} \int_0^{d(Sx,Ty)} \varphi(t)dt &\le d(Sx, Ty)^{1/d(Sx,Ty)} \\ &\le \left| \frac{1}{n+1} - \frac{1}{m+1} \right|^{1/|(1/n+1)-(1/m+1)|} \\ &\le \frac{1}{2} \left| \frac{1}{n} - \frac{1}{m} \right|^{1/|(1/n)-(1/m)|} = d(x, y)^{1/d(x,y)} \end{aligned} \tag{33}$$

for each x, y as in Lemma 1, and condition (2) is satisfied with $\psi(t) = t/2$. □

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