

RIEMANN BOUNDARY VALUE PROBLEM FOR HYPERANALYTIC FUNCTIONS

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Received 23 February 2005

We deal with Riemann boundary value problem for hyperanalytic functions. Furthermore, necessary and sufficient conditions for solvability of the problem are derived. At the end the explicit form of general solution for singular integral equations with a hypercomplex Cauchy kernel in the Douglis sense is established.

1. Introduction

The theory of Riemann boundary value problem for analytic functions of one complex variable and singular integral equations that are equivalent to it has been extensively studied in the literature. For classical books on this topic see [7, 12, 13] and for an actual overview of them the reader is directed to the monograph by Estrada and Kanwal [6], and the references therein.

In the more recent times several generalizations and extensions of the theory are treated and have led to numerous important results not only for nonsmoothly bounded domain, which differs with the former, but for general assumptions on the data of the problem, such as generalized Hölder coefficients or special subspaces of this space and the desired boundary behavior condition for the solution. During the last decades, the Riemann boundary value problem was studied for generalized analytic functions, as well as for many other linear and nonlinear elliptic systems in the plane [1, 2, 8, 15, 16, 17].

Let γ be a rectifiable positively oriented Jordan closed curve with diameter d which is the boundary of a bounded simply connected domain Ω_+ in the complex plane \mathbb{C} and let $\Omega_- := \mathbb{C} \setminus (\Omega_+ \cup \gamma)$.

In the Douglis commuting function algebra sense, a continuously differentiable null solution to the Douglis differential operator provides us with the class of hyperanalytic functions. Let $\mathcal{A}(\Omega_\pm)$ be the spaces of all continuous functions on $\overline{\Omega_\pm} := \Omega_\pm \cup \gamma$ and hyperanalytic in Ω_\pm .

The classical Riemann boundary value problem for analytic functions consists in finding a function $\Phi(z)$ analytic in $\mathbb{C} \setminus \gamma$, such that Φ has a finite order at infinity, and satisfies a prescribed jump condition across the curve γ . The basic boundary condition takes

the form

$$\Phi^+(t) - G(t)\Phi^-(t) = g(t), \quad t \in \gamma, \quad (1.1)$$

where G, g are given continuous functions on γ , and $\Phi^+(t)$ and $\Phi^-(t)$ represent the limit values, in a suitable sense, of the desired function Φ at a point t as this point is approached from Ω_+ and from Ω_- , respectively.

The Riemann boundary value problem for analytic functions as well as for hyperanalytic functions in the case where the unknown functions are continuous up to the boundary or their continuity is violated only at a finite number of points are well studied and described in the fundamental monographs [7, 8, 12, 13]. The Riemann problems mentioned above are usually called a continuous or piecewise continuous boundary value problem, respectively.

The present paper is concerned with hyperanalytic Riemann boundary value problem (where instead of analyticity one requires the hyperanalyticity of Φ) in the continuous case (the solutions including their boundary values on γ are continuous). The purpose of the paper is to present an explicit form of the general solution of the problem.

The motivation comes on the one hand from the study of the hyperanalytic Riemann boundary value problem with continuous coefficients [10] and on the other from the necessary and sufficient solvability condition, which will be imposed on the layer function of the Cauchy type integral so that this integral provides the solution of the basic jump problem.

The main result is moreover applied to describe the general solution of a singular integral equation with a hypercomplex Cauchy kernel.

2. Preliminaries

For the sake of completeness we recall some basic notions and results in Douglis analysis, that is, a Douglis-algebra-valued function theory which is a generalization of classical complex analysis in the plane. For more details concerning this function theory and its application, we refer the reader to [1, 8, 16, 17].

Let \mathbb{D} be the Douglis algebra generated by the elements i and e . The multiplication in \mathbb{D} is governed by the rules

$$i^2 = -1, \quad ie = ei, \quad e^r = 0, \quad e^0 = 1, \quad (2.1)$$

where r is a positive integer.

An arbitrary element $a \in \mathbb{D}$ may be written as a hypercomplex number of the form

$$a = \sum_{k=0}^{r-1} a_k e^k, \quad (2.2)$$

where each a_k is a complex number, a_0 is called the complex part of a , and $A := \sum_{k=1}^{r-1} a_k e^k$ is the nilpotent part.

The conjugation $a \rightarrow \bar{a}$ in \mathbb{D} is defined as

$$\bar{a} := \sum_{k=0}^{r-1} \bar{a}_k e^k. \tag{2.3}$$

In \mathbb{D} the algebraic norm of a is defined by

$$|a| := \sum_{k=0}^{r-1} |a_k|. \tag{2.4}$$

It is easily seen that

$$|ab| \leq |a||b|, \quad |a+b| \leq |a| + |b|, \tag{2.5}$$

for any hypercomplex numbers a and b . The multiplicative inverse a^{-1} of a with complex part $a_0 \neq 0$ is given by

$$a^{-1} \text{ or } \frac{1}{a} = \frac{1}{a_0} \sum_{k=0}^{r-1} (-1)^k \left(\frac{A}{a_0}\right)^k. \tag{2.6}$$

Conversely, if $a_0 = 0$, then a does not have a multiplicative inverse and a is called nilpotent.

In what follows, we will consider functions \mathbb{D} -valued, which are defined in some subset $\Omega \subset \mathbb{C}$.

We say that $f = \sum_{k=0}^{r-1} f_k e^k$, where f_k are complex-valued functions, belongs to some classical class of functions on Ω if each of its components f_k belongs to that class.

The Douglis operator $\partial_{\bar{z}}^q$ is given by

$$\partial_{\bar{z}}^q := \partial_{\bar{z}} + q(z)\partial_z, \quad z = x + iy \tag{2.7}$$

here $q(z)$ is a known nilpotent hypercomplex function and

$$\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y), \quad \partial_z := \frac{1}{2}(\partial_x - i\partial_y). \tag{2.8}$$

Definition 2.1. A continuously differentiable hypercomplex function f is hyperanalytic in Ω if $\partial_{\bar{z}}^q f = 0$ in Ω .

The basic example of a hyperanalytic function is the generating solution of the Douglis operator given by

$$W(z) = z + \sum_{k=1}^{r-1} W_k(z)e^k, \tag{2.9}$$

namely, $\partial_{\bar{z}}^q W(z) = 0$ and its nilpotent part $\sum_{k=1}^{r-1} W_k e^k$ possess bounded and continuous derivatives up to order two in \mathbb{C} .

Since in the paper we have employed the letter t to denote a generic point in the curve γ , we have decided to denote the generating solution of the Douglis operator with $W(z)$ instead of the standard notation $t(z)$.

The hypercomplex Cauchy kernel, that is, the fundamental solution of the Douglis operator, is given by

$$2e_z(\zeta) := \frac{1}{\pi} \frac{\partial_\zeta W(\zeta)}{W(\zeta) - W(z)}, \quad \zeta \neq z. \tag{2.10}$$

For $f, g \in \mathcal{C}^1(\Omega_+) \cap \mathcal{C}(\overline{\Omega_+})$ Green's identity can be formulated within the framework of hypercomplex function theory in the following way:

$$\int_\gamma \partial_\zeta W(\zeta) f(\zeta) n_q(\zeta) g(\zeta) ds = 2 \int_{\Omega_+} \partial_\zeta W(\zeta) \left(f \partial_{\bar{\zeta}}^q g + g \partial_{\bar{\zeta}}^q f \right) d\Omega_+, \tag{2.11}$$

where $n_q(\zeta) := n(\zeta) + \bar{n}(\zeta)q(\zeta)$, $n(\zeta)$ denotes the exterior unit normal vector to γ at the point ζ , and ds is an arc length differential.

Green's identity leads to the Cauchy-Pompeiu integral representation formula for smooth functions

$$f(z) = \int_\gamma e_z(\zeta) n_q(\zeta) f(\zeta) ds - 2 \int_{\Omega_+} e_z(\zeta) \partial_{\bar{\zeta}}^q f(\zeta) d\Omega_+, \quad z \in \Omega_+, \tag{2.12}$$

while for hyperanalytic functions coincides with Cauchy's formula.

$$f(z) = \int_\gamma e_z(\zeta) n_q(\zeta) f(\zeta) ds, \quad z \in \Omega_+. \tag{2.13}$$

Definition 2.2. Suppose F is a hyperanalytic function outside of an open ball B_R with radius $R > 0$ and center at the origin and let γ_0 be any rectifiable positively oriented Jordan closed curve such that γ_0 lies in $\mathbb{C} \setminus \overline{B_R}$ and surrounds $\overline{B_R}$. The hypercomplex number

$$\text{Res}_{\zeta=\infty} [F(\zeta)] := -\frac{1}{2\pi} \int_{\gamma_0} W_\zeta(\zeta) n_q(\zeta) F(\zeta) ds \tag{2.14}$$

is called the residue of F at infinity.

Note that, according to (2.11), the integral in the right-hand side does not depend on the choice of the curve γ_0 .

THEOREM 2.3. *Let $F \in \mathcal{A}(\Omega_-)$, then*

$$\int_\gamma e_z(\zeta) n_q(\zeta) F(\zeta) ds = \begin{cases} -\text{Res}_{\zeta=\infty} \left[\frac{F(\zeta)}{W(\zeta) - W(z)} \right], & z \in \Omega_+ \\ -F(z) - \text{Res}_{\zeta=\infty} \left[\frac{F(\zeta)}{W(\zeta) - W(z)} \right], & z \in \Omega_-. \end{cases} \tag{2.15}$$

Proof. Suppose $z \in \Omega_-$ and let $R > 0$ such that $\Omega_+ \subset B_R$ and $z \in B_R$. From Cauchy’s formula we get

$$\begin{aligned} \int_{\gamma} e_z(\zeta)n_q(\zeta)F(\zeta)ds &= -F(z) + \int_{\partial B_R} e_z(\zeta)n_q(\zeta)F(\zeta)ds \\ &= -F(z) - \text{Res}_{\zeta=\infty} \left[\frac{F(\zeta)}{W(\zeta) - W(z)} \right]. \end{aligned} \tag{2.16}$$

The case $z \in \Omega_+$ is similar. □

Notation c will be used for constants which may vary from one occurrence to the next; in general these constants only depend on q .

LEMMA 2.4. Let $\gamma_\epsilon(t) := \{\zeta \in \gamma : |\zeta - t| \leq \epsilon\}$, for $t \in \gamma$ and let $\epsilon \in (0, d]$.

(i) If $F \in \mathcal{A}(\Omega_+)$, then

$$\left| \int_{\gamma \setminus \gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta)(F(\zeta) - F(t))ds \right| \leq c \max_{z \in \Omega_+, |z-t|=\epsilon} |F(z) - F(t)|. \tag{2.17}$$

(ii) If $F \in \mathcal{A}(\Omega_-)$, then

$$\begin{aligned} \left| \int_{\gamma \setminus \gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta)(F(\zeta) - F(t))ds + F(t) + \text{Res}_{\zeta=\infty} \left[\frac{F(\zeta)}{W(\zeta) - W(t)} \right] \right| \\ \leq c \max_{z \in \Omega_-, |z-t|=\epsilon} |F(z) - F(t)|. \end{aligned} \tag{2.18}$$

Proof. The first assertion was already proved in [10]. We consider the second one. Suppose $B_\epsilon(t)$ is the open ball with center t and radius ϵ , and let

$$Q := \Omega_+ \cup B_\epsilon(t). \tag{2.19}$$

Since $\gamma \setminus \gamma_\epsilon(t) = \partial Q \setminus (\partial Q \cap \partial B_\epsilon(t))$, we have

$$\begin{aligned} \int_{\gamma \setminus \gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta)(F(\zeta) - F(t))ds \\ = \int_{\partial Q} e_t(\zeta)n_q(\zeta)(F(\zeta) - F(t))ds - \int_{\partial Q \cap \partial B_\epsilon(t)} e_t(\zeta)n_q(\zeta)(F(\zeta) - F(t))ds \\ = \int_{\partial Q} e_t(\zeta)n_q(\zeta)F(\zeta)ds - F(t) - \int_{\partial Q \cap \partial B_\epsilon(t)} e_t(\zeta)n_q(\zeta)(F(\zeta) - F(t))ds. \end{aligned} \tag{2.20}$$

In view of the previous theorem,

$$\begin{aligned} \left| \int_{\gamma \setminus \gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta)(F(\zeta) - F(t))ds + F(t) + \text{Res}_{\zeta=\infty} \left[\frac{F(\zeta)}{W(\zeta) - W(t)} \right] \right| \\ = \left| \int_{\partial Q \cap \partial B_\epsilon(t)} e_t(\zeta)n_q(\zeta)(F(\zeta) - F(t))ds \right| \leq c \max_{z \in \Omega_-, |z-t|=\epsilon} |F(z) - F(t)|. \end{aligned} \tag{2.21}$$

□

For each $f \in \mathcal{C}(\gamma)$ the Cauchy-type integral is given by

$$(\mathbf{C}_\gamma f)(z) := \int_\gamma e_z(\zeta)n_q(\zeta)f(\zeta)ds, \quad z \notin \gamma. \tag{2.22}$$

Clearly $\mathbf{C}_\gamma f$ is hyperanalytic in $\mathbb{C} \setminus \gamma$.

Thus the singular Cauchy integral operator (Hilbert transform) on γ reads as

$$(\mathbf{S}_\gamma f)(t) := 2 \int_\gamma e_t(\zeta)n_q(\zeta)(f(\zeta) - f(t))ds + f(t), \quad t \in \gamma, \tag{2.23}$$

where the integral which defines $\mathbf{S}_\gamma f$ has to be taken in the sense of Cauchy’s principal value, and the function f is such that the integrals

$$\int_{\gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta)(f(\zeta) - f(t))ds \tag{2.24}$$

converge uniformly to zero for $t \in \gamma$ as $\epsilon \rightarrow 0$.

The space of all continuous functions on γ which satisfy the above condition will be denoted by $\mathcal{S}(\gamma)$. Taking into account Lemma 2.4, we get $\mathcal{A}(\Omega_\pm) \subset \mathcal{S}(\gamma)$.

From now on we always assume γ to be a regular closed curve, that is, the quotient of the length of γ inside any circle to the radius of the circle is less than some fixed constant.

In a previous paper [3] we already studied the problem of establishing necessary and sufficient condition for the Cauchy-type integral to be continuously extended onto γ . The following theorem was proved in [3].

THEOREM 2.5. *Let $f \in \mathcal{C}(\gamma)$, then $\mathbf{C}_\gamma f$ has continuous limit values on γ if and only if $f \in \mathcal{S}(\gamma)$. Moreover,*

$$\lim_{\Omega_\pm \ni z \rightarrow t} (\mathbf{C}_\gamma f)(z) = \frac{1}{2}((\mathbf{S}_\gamma f)(t) \pm f(t)). \tag{2.25}$$

Therefore, for any $f \in \mathcal{S}(\gamma)$ the functions

$$(\mathbf{C}_\gamma^\pm f)(z) := \begin{cases} (\mathbf{C}_\gamma f)(z), & z \in \Omega_\pm, \\ \frac{1}{2}((\mathbf{S}_\gamma f)(z) \pm f(z)), & z \in \gamma, \end{cases} \tag{2.26}$$

are continuous on $\overline{\Omega_\pm}$.

Let $\mathcal{S}^\pm(\gamma)$ be the spaces of all continuous functions f on γ which have hyperanalytic extensions f^\pm to Ω_\pm and $f^-(\infty) = 0$.

By Theorem 2.5 we get the splitting of the space $\mathcal{S}(\gamma)$ as

$$\mathcal{S}(\gamma) = \mathcal{S}^+(\gamma) \oplus \mathcal{S}^-(\gamma), \tag{2.27}$$

and the corresponding splitting of the functions considered $f = f^+ + f^-$, where $f^\pm \in \mathcal{S}^\pm(\gamma)$.

Now, for $f \in \mathcal{C}(\gamma)$ we consider the modulus of continuity for f as

$$\omega_f(\xi) := \xi \sup_{\rho \geq \xi} \rho^{-1} \max_{t_1, t_2 \in \gamma, |t_1 - t_2| \leq \rho} |f(t_1) - f(t_2)|, \quad \xi > 0. \tag{2.28}$$

Let

$$I_0(\gamma) := \left\{ f \in \mathcal{C}(\gamma) : \int_0^d \frac{\omega_f(\xi)}{\xi} d\xi < +\infty \right\}. \tag{2.29}$$

For a majorant φ , that is, φ is a positive real function on $(0, d]$ such that $\varphi(\xi)$ does not decrease, $\varphi(\xi)/\xi$ does not increase, and $\varphi(\xi) \rightarrow 0$ as $\xi \rightarrow 0$, we now introduce the generalized Hölder continuous functions space

$$\mathcal{H}_\varphi(\gamma) := \left\{ f \in \mathcal{C}(\gamma) : \omega_f(\xi) \leq c\varphi(\xi), \xi \in (0, d] \right\} \tag{2.30}$$

and its subspace

$$\mathcal{L}_\varphi(\gamma) := \left\{ f \in \mathcal{H}_\varphi(\gamma) : \Theta_f(\xi) \leq c\varphi(\xi), \xi \in (0, d] \right\}, \tag{2.31}$$

where

$$\Theta_f(\xi) := \xi \sup_{\rho \geq \xi} \rho^{-1} \sup_{\epsilon \in (0, \rho], t \in \gamma} \left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) (f(\zeta) - f(t)) ds \right|. \tag{2.32}$$

One can define a norm in $\mathcal{L}_\varphi(\gamma)$ by

$$\|f\|_{\mathcal{L}_\varphi} := \max_{t \in \gamma} |f(t)| + \sup_{\xi \in (0, d]} \frac{\omega_f(\xi)}{\varphi(\xi)} + \sup_{\xi \in (0, d]} \frac{\Theta_f(\xi)}{\varphi(\xi)}. \tag{2.33}$$

In the sequel we will make use of the following two technical lemmas, from which other results will follow.

LEMMA 2.6. *Let $f \in \mathcal{F}(\gamma)$, then for $t \in \gamma$ and $\epsilon \in (0, d]$,*

$$\sup_{z \in \overline{\Omega}_\pm, |z-t|=\epsilon} |(C_\gamma^\pm f)(z) - (C_\gamma^\pm f)(t)| \leq c \left(\omega_f(\epsilon) + \Theta_f(\epsilon) + \epsilon \int_\epsilon^d \frac{\omega_f(\xi)}{\xi^2} d\xi \right), \tag{2.34}$$

is true.

Proof. It is only necessary to distinguish the following two cases.

Case 1. $z \in \Omega_{\pm}$. Let $t_z \in \gamma$ such that $|z - t_z| = \text{dist}(z, \gamma) =: \nu$. Then for $|z - t| = \epsilon$,

$$\begin{aligned}
 & (\mathbf{C}_{\gamma} f)(z) - (\mathbf{C}_{\gamma}^{\pm} f)(t) \\
 &= \int_{\gamma_{\nu}(t_z)} e_z(\zeta) n_q(\zeta) (f(\zeta) - f(t_z)) ds \\
 &+ \int_{\gamma \setminus \gamma_{\nu}(t_z)} (e_z(\zeta) - e_{t_z}(\zeta)) n_q(\zeta) (f(\zeta) - f(t_z)) ds \\
 &- \int_{\gamma_{\nu}(t_z)} e_{t_z}(\zeta) n_q(\zeta) (f(\zeta) - f(t_z)) ds + \frac{1}{2} ((\mathbf{S}_{\gamma} f)(t_z) - (\mathbf{S}_{\gamma} f)(t)) \\
 &\pm \frac{1}{2} (f(t_z) - f(t)) =: \sum_{k=1}^5 J_k.
 \end{aligned} \tag{2.35}$$

Because $|z - t_z| \leq |\zeta - z|$ and $|\zeta - t_z| \leq |\zeta - z| + |z - t_z| \leq 2|\zeta - z|$,

$$\begin{aligned}
 |J_1| + |J_2| &\leq c \left(\int_{\gamma_{\nu}(t_z)} \frac{\omega_f(|\zeta - t_z|)}{|\zeta - z|} ds + |z - t_z| \int_{\gamma \setminus \gamma_{\nu}(t_z)} \frac{\omega_f(|\zeta - t_z|)}{|\zeta - z| |\zeta - t_z|} ds \right) \\
 &\leq c \left(\omega_f(\nu) + |z - t_z| \int_{\gamma \setminus \gamma_{\nu}(t_z)} \frac{\omega_f(|\zeta - t_z|)}{|\zeta - t_z|^2} ds \right) \\
 &\leq c \left(\omega_f(\nu) + \nu \int_{\gamma} \frac{\omega_f(\xi)}{\xi^2} d\xi \right).
 \end{aligned} \tag{2.36}$$

First, we remark that the second term in the last inequality is an almost-increasing function in ν . There is no loss of generality in assuming that $\nu \leq \epsilon \leq d/2$, then

$$\begin{aligned}
 & 3\epsilon \int_{\epsilon}^d \frac{\omega_f(\xi)}{\xi^2} d\xi - \nu \int_{\nu}^d \frac{\omega_f(\xi)}{\xi^2} d\xi \\
 &= (\epsilon - \nu) \int_{\epsilon}^d \frac{\omega_f(\xi)}{\xi^2} d\xi + \left(2\epsilon \int_{\epsilon}^d \frac{\omega_f(\xi)}{\xi^2} d\xi - \nu \int_{\nu}^{\epsilon} \frac{\omega_f(\xi)}{\xi^2} d\xi \right).
 \end{aligned} \tag{2.37}$$

Further, it is clear that

$$2\epsilon \int_{\epsilon}^d \frac{\omega_f(\xi)}{\xi^2} d\xi \geq 2\epsilon \omega_f(\epsilon) \int_{\epsilon}^d \frac{d\xi}{\xi^2} = 2\epsilon \omega_f(\epsilon) \frac{d - \epsilon}{\epsilon d} = 2\omega_f(\epsilon) \frac{d - \epsilon}{d} \geq \omega_f(\epsilon). \tag{2.38}$$

On the other hand,

$$\nu \int_{\nu}^{\epsilon} \frac{\omega_f(\xi)}{\xi^2} d\xi \leq \nu \omega_f(\epsilon) \int_{\nu}^{\epsilon} \frac{d\xi}{\xi^2} = \nu \omega_f(\epsilon) \frac{\epsilon - \nu}{\nu \epsilon} = \omega_f(\epsilon) \frac{\epsilon - \nu}{\epsilon} \leq \omega_f(\epsilon), \tag{2.39}$$

and hence

$$3\epsilon \int_{\epsilon}^d \frac{\omega_f(\xi)}{\xi^2} d\xi - \nu \int_{\nu}^d \frac{\omega_f(\xi)}{\xi^2} d\xi \geq 0. \quad (2.40)$$

It then follows that

$$|J_1| + |J_2| \leq c \left(\omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\xi)}{\xi^2} d\xi \right). \quad (2.41)$$

For J_3 we obtain

$$|J_3| \leq \Theta_f(\nu) \leq \Theta_f(\epsilon). \quad (2.42)$$

Since $|t_z - t| \leq |t_z - z| + |z - t| \leq 2|z - t| = 2\epsilon$, then

$$|J_4| \leq \omega_{S_{\gamma}f}(\epsilon), \quad |J_5| \leq \omega_f(\epsilon). \quad (2.43)$$

On the other hand, starting from the reasonings in [3, Theorem 2] and using the definition of Θ_f , we get

$$|J_4| \leq \omega_{S_{\gamma}f}(\epsilon) \leq c \left(\omega_f(\epsilon) + \Theta_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\xi)}{\xi^2} d\xi \right). \quad (2.44)$$

Therefore

$$|(\mathbf{C}_{\gamma}f)(z) - (\mathbf{C}_{\gamma}^{\pm}f)(t)| \leq c \left(\omega_f(\epsilon) + \Theta_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\xi)}{\xi^2} d\xi \right). \quad (2.45)$$

Case 2. $z \in \gamma$. If $|z - t| = \epsilon$, then

$$(\mathbf{C}_{\gamma}^{\pm}f)(z) - (\mathbf{C}_{\gamma}^{\pm}f)(t) = \frac{1}{2}(\mathbf{S}_{\gamma}f(z) - \mathbf{S}_{\gamma}f(t)) \pm \frac{1}{2}(f(z) - f(t)). \quad (2.46)$$

For that

$$|(\mathbf{C}_{\gamma}^{\pm}f)(z) - (\mathbf{C}_{\gamma}^{\pm}f)(t)| \leq \omega_{S_{\gamma}f}(\epsilon) + \omega_f(\epsilon), \quad (2.47)$$

hence

$$|(\mathbf{C}_{\gamma}^{\pm}f)(z) - (\mathbf{C}_{\gamma}^{\pm}f)(t)| \leq c \left(\omega_f(\epsilon) + \Theta_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\xi)}{\xi^2} d\xi \right). \quad (2.48)$$

The statement of the lemma follows now from the above-considered cases. \square

LEMMA 2.7. *Suppose $f \in \mathcal{G}(\gamma)$ and let $\epsilon \in (0, d]$. Then*

$$\Theta_{S_\gamma f}(\epsilon) \leq c \left(\omega_f(\epsilon) + \Theta_f(\epsilon) + \epsilon \int_\epsilon^d \frac{\omega_f(\xi)}{\xi^2} d\xi \right). \tag{2.49}$$

Proof. For $t \in \gamma$ we have $(S_\gamma f)(t) = 2(C_\gamma^+ f)(t) - f(t)$. Therefore

$$\begin{aligned} & \left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) ((S_\gamma f)(\zeta) - (S_\gamma f)(t)) ds \right| \\ & \leq \Theta_f(\epsilon) + 2 \left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) ((C_\gamma^+ f)(\zeta) - (C_\gamma^+ f)(t)) ds \right|. \end{aligned} \tag{2.50}$$

As a consequence of Lemma 2.4 we have

$$\int_\gamma e_t(\zeta) n_q(\zeta) ((C_\gamma^+ f)(\zeta) - (C_\gamma^+ f)(t)) ds = 0, \tag{2.51}$$

for that

$$\begin{aligned} & \left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) ((C_\gamma^+ f)(\zeta) - (C_\gamma^+ f)(t)) ds \right| \\ & = \left| \int_{\gamma \setminus \gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) ((C_\gamma^+ f)(\zeta) - (C_\gamma^+ f)(t)) ds \right|. \end{aligned} \tag{2.52}$$

Thus, returning to the Lemma 2.4, we obtain

$$\left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) ((C_\gamma^+ f)(\zeta) - (C_\gamma^+ f)(t)) ds \right| \leq c \sup_{z \in \bar{\Omega}_+, |z-t|=\epsilon} |(C_\gamma^+ f)(z) - (C_\gamma^+ f)(t)| \tag{2.53}$$

and the following estimate follows in view of the result of Lemma 2.6:

$$\left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) ((C_\gamma^+ f)(\zeta) - (C_\gamma^+ f)(t)) ds \right| \leq c \left(\omega_f(\epsilon) + \Theta_f(\epsilon) + \epsilon \int_\epsilon^d \frac{\omega_f(\xi)}{\xi^2} d\xi \right). \tag{2.54}$$

Finally

$$\left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) ((S_\gamma f)(\zeta) - (S_\gamma f)(t)) ds \right| \leq c \left(\omega_f(\epsilon) + \Theta_f(\epsilon) + \epsilon \int_\epsilon^d \frac{\omega_f(\xi)}{\xi^2} d\xi \right). \tag{2.55}$$

This completes the proof. □

The following theorem gives sufficient condition for the singular Cauchy integral operator to be bounded on $\mathcal{L}_\varphi(\gamma)$.

THEOREM 2.8. *Suppose φ is a majorant such that for $\epsilon \in (0, d]$,*

$$\epsilon \int_\epsilon^d \frac{\varphi(\xi)}{\xi^2} d\xi \leq c\varphi(\epsilon). \tag{2.56}$$

Then the singular Cauchy integral operator S_γ is a bounded operator on $\mathcal{L}_\varphi(\gamma)$, that is,

$$\|S_\gamma f\|_{\mathcal{L}_\varphi} \leq c\|f\|_{\mathcal{L}_\varphi}, \tag{2.57}$$

for any $f \in \mathcal{L}_\varphi(\gamma)$.

Proof. Suppose $f \in \mathcal{L}_\varphi(\gamma)$, then for $\epsilon \in (0, d]$,

$$\omega_f(\epsilon) \leq \|f\|_{\mathcal{L}_\varphi} \varphi(\epsilon), \quad \Theta_f(\epsilon) \leq \|f\|_{\mathcal{L}_\varphi} \varphi(\epsilon); \tag{2.58}$$

hence

$$\epsilon \int_\epsilon^d \frac{\omega_f(\xi)}{\xi^2} d\xi \leq \epsilon \|f\|_{\mathcal{L}_\varphi} \int_\epsilon^d \frac{\varphi(\xi)}{\xi^2} d\xi \leq c\|f\|_{\mathcal{L}_\varphi} \varphi(\epsilon). \tag{2.59}$$

Combining the previous estimates with Lemma 2.7, we obtain

$$\Theta_{S_\gamma f}(\epsilon) \leq c\|f\|_{\mathcal{L}_\varphi} \varphi(\epsilon), \tag{2.60}$$

and similarly

$$\omega_{S_\gamma f}(\epsilon) \leq c\|f\|_{\mathcal{L}_\varphi} \varphi(\epsilon). \tag{2.61}$$

So we have at once $S_\gamma f \in \mathcal{L}_\varphi(\gamma)$. Besides

$$|S_\gamma f(t)| \leq 2\Theta_f(d) + \|f\|_{\mathcal{L}_\varphi} \leq c\|f\|_{\mathcal{L}_\varphi}, \quad t \in \gamma, \tag{2.62}$$

thus

$$\|S_\gamma f\|_{\mathcal{L}_\varphi} \leq c\|f\|_{\mathcal{L}_\varphi}, \tag{2.63}$$

that is, S_γ is a bounded operator on $\mathcal{L}_\varphi(\gamma)$. □

Notice that the above theorem is a generalization of an analogous result by Bustamante and González in [11] treated in the framework of complex analysis.

3. Riemann boundary value problem

For the Riemann boundary value problem for hyperanalytic functions, due to the technique of canonical factorization, the interested reader can find a good introduction in [8].

At the same time early quite more-general results on complete solution of the Riemann problem for hypercomplex functions, in case where the integration curve is rectifiable and the coefficients are assumed to be in special subspace of continuous function space, have been obtained in [10, 18].

Systematically the study of the solvability of the Riemann boundary problem for analytic functions is technically involved with the Cauchy-type integral which, by taking boundary values, led to the Sokhotski-Plemelj formulas. In such way the singular Cauchy integral operator appearing in these formulas transforms the Riemann problem to a singular integral equation. A good presentation for excellent examples of such study which is close to the present paper could be [4, 5, 14].

The main goal of this paper is to develop a theory of the well-posed continuous Riemann boundary value problem by assuming that the given continuous coefficients defined on γ have to agree with the desired boundary behavior of the solutions, that is, the solutions including their boundary values on γ are continuous functions too.

The results of boundedness of the singular Cauchy integral operator and on the continuous extension of the Cauchy-type integral, as established in the previous section, are now applied to obtain a necessary and sufficient condition for the solvability of the Riemann problem with coefficient G in the case where G admits a canonical factorization

$$G = \frac{H^+}{H^-}, \tag{3.1}$$

where $H^\pm \in \mathcal{A}(\Omega_\pm)$ and the complex parts H_0^\pm never vanish on $\overline{\Omega_\pm}$.

Take an arbitrary fixed point z_0 in Ω_+ . We restrict ourselves to the case of a hypercomplex function G which has complex part G_0 and never vanishes on γ . From this assumption the integer

$$\kappa := \frac{1}{2\pi} [\arg G_0(\zeta)]_\gamma \tag{3.2}$$

has significant importance and is called the index of G with respect to γ , also called index of the Riemann boundary value problem. Note that the index of the function $(W(\zeta) - W(z_0))^\kappa$ with respect to γ is κ , and hence the index of $(W(\zeta) - W(z_0))^{-\kappa}G(\zeta)$ is zero.

We may verify directly that

$$X(z) := \begin{cases} \exp \Gamma(z), & z \in \Omega_+, \\ (W(z) - W(z_0))^{-\kappa} \exp \Gamma(z), & z \in \Omega_-, \end{cases} \tag{3.3}$$

in which

$$\Gamma(z) := \int_\gamma e_z(\zeta) n_q(\zeta) \ln [(W(\zeta) - W(z_0))^{-\kappa} G(\zeta)] ds, \quad z \notin \gamma, \tag{3.4}$$

is a hyperanalytic function in $\mathbb{C} \setminus \gamma$.

Remark about the hypercomplex exponential and logarithmic functions can be found for instance in [8, 9].

The following theorem gives characterization for hypercomplex function to admit a canonical factorization.

THEOREM 3.1. *The function G admits a canonical factorization if and only if $\ln[(W(t) - W(z_0))^{-\kappa}G(t)] \in \mathcal{F}(\gamma)$.*

Proof. Assume G admits a canonical factorization, then we deduce

$$\ln [(W(t) - W(z_0))^{-\kappa}G(t)v] = \ln H^+(t) - \ln [(W(t) - W(z_0))^\kappa H^-(t)], \tag{3.5}$$

for $t \in \gamma$ and H^\pm satisfying (3.1).

Now the necessity follows from Lemma 2.4 and the previous equality.

Conversely, suppose $\ln[(W(t) - W(z_0))^{-\kappa}G(t)] \in \mathcal{F}(\gamma)$. Taking into account (3.5) and Theorem 2.5, it is easy to check that X gives a canonical factorization of G on γ in the form $G = X^+/X^-$. □

Definition 3.2. $p_0 + p_1W(z) + \dots + p_s(W(z))^s$, $s \geq 0$, is called a hypercomplex polynomial.

THEOREM 3.3. *Suppose G admits a canonical factorization. For the homogeneous Riemann boundary value problem ($g(t) \equiv 0$), if $\Phi(\infty) = 0$, then it has κ linearly independent solutions when $\kappa > 0$ and has only the trivial solution when $\kappa \leq 0$. The general solution is given by*

$$\Phi(z) = X(z)P_{\kappa-1}(z), \tag{3.6}$$

where $P_{\kappa-1}$ is an arbitrary hypercomplex polynomial whose degree is not greater than $\kappa - 1$ ($P_{\kappa-1} \equiv 0$ when $\kappa \leq 0$).

Proof. The proof is standard [1, 8, 9, 10] running along similar lines to those in the complex case [4, 5, 7, 12, 13] and will not be given here. □

THEOREM 3.4. *Assume G admits a canonical factorization. The Riemann boundary value problem is solvable if and only if $g/X^+ \in \mathcal{F}(\gamma)$.*

Proof. The boundary condition (1.1) may be rewritten as

$$\frac{\Phi^+(t)}{X^+(t)} - \frac{\Phi^-(t)}{X^-(t)} = \frac{g(t)}{X^+(t)}, \quad t \in \gamma. \tag{3.7}$$

The necessary condition is an immediate consequence of (3.7) and Lemma 2.4. Let now $g/X^+ \in \mathcal{F}(\gamma)$, then obviously the function

$$X(z)(\mathbf{C}_\gamma\{g/X^+\})(z) \tag{3.8}$$

is a solution of the Riemann boundary value problem. □

We consider now $G \in I_0(\gamma)$ and $g \in \mathcal{F}(\gamma)$. Note that

$$\ln [(W(t) - W(z_0))^{-\kappa}G(t)] \in I_0(\gamma) \subset \mathcal{F}(\gamma), \tag{3.9}$$

hence, G admits a canonical factorization.

Taking into account the condition $g \in \mathcal{F}(\gamma)$, it can easily be shown that

$$\begin{aligned} & \int_{\gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta)\left(\frac{g(\zeta)}{X^+(\zeta)} - \frac{g(t)}{X^+(t)}\right)ds \\ &= \int_{\gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta)\left(\frac{g^+(\zeta)}{X^+(\zeta)} - \frac{g^+(t)}{X^+(t)}\right)ds \\ &+ \frac{1}{G(t)} \int_{\gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta)\left(\frac{g^-(\zeta)}{X^-(\zeta)} - \frac{g^-(t)}{X^-(t)}\right)ds \\ &+ \int_{\gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta)\left(\frac{1}{G(\zeta)} - \frac{1}{G(t)}\right)\frac{g^-(\zeta)}{X^-(\zeta)}ds =: J_1 + J_2 + J_3, \quad \epsilon \in (0, d]. \end{aligned} \tag{3.10}$$

In view of the result of Lemma 2.4, J_1, J_2 converge uniformly to zero for $t \in \gamma$, as $\epsilon \rightarrow 0$.

Furthermore, as

$$|J_3| \leq c \int_0^\epsilon \frac{\omega_G(\xi)}{\xi} d\xi, \tag{3.11}$$

we have that $g/X^+ \in \mathcal{F}(\gamma)$.

Taking into account the above and Theorem 3.4, we thus have proved the following theorem.

THEOREM 3.5. *Let $G \in I_0(\gamma)$ and $g \in \mathcal{F}(\gamma)$. Under the requirement $\Phi(\infty) = 0$, the Riemann boundary value problem is solvable when $\kappa \geq 0$ with the general solution*

$$\Phi(z) = X(z)(C_\gamma\{g/X^+\})(z) + X(z)P_{\kappa-1}(z). \tag{3.12}$$

Here $P_{\kappa-1}(z)$ is the same as above; when $\kappa < 0$, it is (uniquely) solvable with the solution (3.12) ($P_{\kappa-1}(z) \equiv 0$ when $\kappa \leq 0$) if and only if the following conditions are satisfied:

$$\int_\gamma \partial_\zeta W(\zeta)n_q(\zeta)\frac{g(\zeta)}{X^+(\zeta)}(W(\zeta))^k ds = 0, \quad k = 0, \dots, -\kappa - 1. \tag{3.13}$$

4. Singular integral equations in the class $\mathcal{L}_\varphi(\gamma)$

This section is devoted to the study of the solvability of the singular integral equation of the type

$$a(t)Y(t) + b(t)(S_\gamma Y)(t) = f(t), \quad t \in \gamma, \tag{4.1}$$

for which we assume that a, b , and f are given hypercomplex functions. For this equation, Y is an unknown hypercomplex function belonging to $\mathcal{L}_\varphi(\gamma)$.

We assume that $a, b \in \mathcal{H}_\psi(\gamma)$, $f \in \mathcal{L}_\varphi(\gamma)$, and $a_0^2(t) - b_0^2(t) \neq 0$ for $t \in \gamma$. Here φ, ψ are given majorants satisfying the following relations:

$$\begin{aligned} & \epsilon \int_\epsilon^d \frac{\varphi(\xi)}{\xi^2} d\xi \leq c\varphi(\epsilon), \\ & \int_0^\epsilon \frac{\psi(\xi)}{\xi} d\xi \leq c\varphi(\epsilon), \quad \epsilon \in (0, d]. \end{aligned} \tag{4.2}$$

We consider the Riemann boundary value problem associated with the singular integral equation (4.1),

$$\Phi^+(t) - \frac{a(t) - b(t)}{a(t) + b(t)} \Phi^-(t) = \frac{f(t)}{a(t) + b(t)}, \quad t \in \gamma, \quad \Phi(\infty) = 0, \tag{4.3}$$

which is similar to (1.1). To simplify notation we write

$$G(t) := \frac{a(t) - b(t)}{a(t) + b(t)}, \quad g(t) := \frac{f(t)}{a(t) + b(t)}, \quad t \in \gamma. \tag{4.4}$$

In view of the indicated relation, the index κ of the function $(a_0(t) - b_0(t))/(a_0(t) + b_0(t))$, and the corresponding functions $\Gamma(z)$ and $X(z)$, we arrive at the following assertions.

THEOREM 4.1. *Let $a, b \in \mathcal{H}_\psi(\gamma)$ and $f \in \mathcal{L}_\varphi(\gamma)$. Then*

- (i) $G \in \mathcal{H}_\psi(\gamma)$,
- (ii) $g, X^\pm \in \mathcal{L}_\varphi(\gamma)$.

Proof. By straightforward computation, the statement (i) can be obtained. On the other hand, since $\psi(\xi)/\xi$ does not increase,

$$\psi(\epsilon) = \int_0^\epsilon \frac{\psi(\epsilon)}{\epsilon} d\xi \leq \int_0^\epsilon \frac{\psi(\xi)}{\xi} d\xi \leq c\varphi(\epsilon), \quad \epsilon \in (0, d], \tag{4.5}$$

we thus have that $g \in \mathcal{H}_\varphi(\gamma)$.

Let now

$$h(t) := (a(t) + b(t))^{-1}, \quad t \in \gamma, \tag{4.6}$$

then

$$\begin{aligned} & \left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) (f(\zeta)h(\zeta) - f(t)h(t)) ds \right| \\ & \leq |h(t)| \left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) (f(\zeta) - f(t)) ds \right| \\ & \quad + \left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) (h(\zeta) - h(t)) f(\zeta) ds \right| \\ & \leq c \left(\Theta_f(\epsilon) + \int_0^\epsilon \frac{\omega_h(\xi)}{\xi} d\xi \right) \\ & \leq c \left(\Theta_f(\epsilon) + \int_0^\epsilon \frac{\psi(\xi)}{\xi} d\xi \right) \leq c\varphi(\epsilon), \quad \epsilon \in (0, d]. \end{aligned} \tag{4.7}$$

Therefore $g \in \mathcal{L}_\varphi(\gamma)$.

We consider now the function X^+ . Suppose $t_1, t_2 \in \gamma$, then

$$\begin{aligned}
 |X^+(t_1) - X^+(t_2)| &\leq c |\Gamma^+(t_1) - \Gamma^+(t_2)| \\
 &\leq c \left(\int_0^\epsilon \frac{\omega_{\ln G}(\xi)}{\xi} d\xi + \epsilon \int_\epsilon^d \frac{\omega_{\ln G}(\xi)}{\xi^2} d\xi + \omega_{\ln G}(\epsilon) \right) \\
 &\leq c \left(\int_0^\epsilon \frac{\omega_G(\xi)}{\xi} d\xi + \epsilon \int_\epsilon^d \frac{\omega_G(\xi)}{\xi^2} d\xi + \omega_G(\epsilon) \right) \\
 &\leq c \left(\int_0^\epsilon \frac{\psi(\xi)}{\xi} d\xi + \epsilon \int_\epsilon^d \frac{\psi(\xi)}{\xi^2} d\xi + \psi(\epsilon) \right) \leq c\varphi(\epsilon),
 \end{aligned}
 \tag{4.8}$$

where $|t_1 - t_2| = \epsilon$.

Thus, we obtain that $X^+ \in \mathcal{H}_\varphi(\gamma)$. Further

$$\begin{aligned}
 &\left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) (X^+(\zeta) - X^+(t)) ds \right| \\
 &= \left| \int_{\gamma \setminus \gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) (X^+(\zeta) - X^+(t)) ds \right| \\
 &\leq c \max_{z \in \Omega_+, |z-t|=\epsilon} |X^+(z) - X^+(t)| \\
 &\leq c \max_{z \in \Omega_+, |z-t|=\epsilon} |\Gamma^+(z) - \Gamma^+(t)|, \quad \epsilon \in (0, d].
 \end{aligned}
 \tag{4.9}$$

Making use of the Lemma 2.6, we get

$$\left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) (X^+(\zeta) - X^+(t)) ds \right| \leq c \left(\omega_{\ln G}(\epsilon) + \Theta_{\ln G}(\epsilon) + \epsilon \int_\epsilon^d \frac{\omega_{\ln G}(\xi)}{\xi^2} d\xi \right).
 \tag{4.10}$$

Taking into account that

$$\left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) (\ln G(\zeta) - \ln G(t)) ds \right| \leq c \int_0^\epsilon \frac{\psi(\xi)}{\xi} d\xi \leq c\varphi(\epsilon),
 \tag{4.11}$$

we have

$$\left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) (X^+(\zeta) - X^+(t)) ds \right| \leq c\varphi(\epsilon).
 \tag{4.12}$$

This implies that $X^+ \in \mathcal{L}_\varphi(\gamma)$, which proves the desired result.

The proof of the assertion concerning X^- is based on the fact that the function G admits a canonical factorization in the form $G = X^+/X^-$ on γ . □

THEOREM 4.2. *Suppose $a, b \in \mathcal{H}_\psi(\gamma)$ and $f \in \mathcal{L}_\varphi(\gamma)$. Then*

- (i) $g/X^+ \in \mathcal{L}_\varphi(\gamma)$,
- (ii) $X^\pm(t) \int_\gamma e_t(\zeta) n_q(\zeta) ((g(\zeta) - g(t))/X^+(\zeta)) ds \in \mathcal{L}_\varphi(\gamma)$, $t \in \gamma$.

Proof. Since $g \in \mathcal{L}_\varphi(\gamma) \subset \mathcal{S}(\gamma)$, then

$$\begin{aligned} & \left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) \left(\frac{g(\zeta)}{X^+(\zeta)} - \frac{g(t)}{X^+(t)} \right) ds \right| \\ & \leq \left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) \left(\frac{g^+(\zeta)}{X^+(\zeta)} - \frac{g^+(t)}{X^+(t)} \right) ds \right| \\ & \quad + \left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) \left(\frac{g^-(\zeta)}{X^+(\zeta)} - \frac{g^-(t)}{X^+(t)} \right) ds \right| =: J_1 + J_2, \quad \epsilon \in (0, d]. \end{aligned} \tag{4.13}$$

According to Lemma 2.4, we get

$$\begin{aligned} J_1 & \leq c \max_{z \in \Omega_+, |z-t|=\epsilon} \left| \frac{g^+(z)}{X^+(z)} - \frac{g^+(t)}{X^+(t)} \right| \\ & \leq c \left(\max_{z \in \Omega_+, |z-t|=\epsilon} |g^+(z) - g^+(t)| + \max_{z \in \Omega_+, |z-t|=\epsilon} |X^+(z) - X^+(t)| \right). \end{aligned} \tag{4.14}$$

By Lemma 2.6

$$\max_{z \in \Omega_+, |z-t|=\epsilon} |g^+(z) - g^+(t)| \leq \left(\omega_g(\epsilon) + \Theta_g(\epsilon) + \epsilon \int_\epsilon^d \frac{\omega_g(\xi)}{\xi^2} d\xi \right). \tag{4.15}$$

Thus, $J_1 \leq c\varphi(\epsilon)$. Now we estimate J_2 :

$$\begin{aligned} J_2 & \leq \left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) \left(\frac{1}{G(\zeta)} - \frac{1}{G(t)} \right) \frac{g^-(\zeta)}{X^-(\zeta)} ds \right| \\ & \quad + \left| \frac{1}{G(t)} \right| \left| \int_{\gamma_\epsilon(t)} e_t(\zeta) n_q(\zeta) \left(\frac{g^-(\zeta)}{X^-(\zeta)} - \frac{g^-(t)}{X^-(t)} \right) ds \right| \\ & \leq c \left(\int_0^\epsilon \frac{\omega_G(\xi)}{\xi} d\xi + \max_{z \in \Omega_-, |z-t|=\epsilon} \left| \frac{g^-(z)}{X^-(z)} - \frac{g^-(t)}{X^-(t)} \right| \right) \leq c\varphi(\epsilon). \end{aligned} \tag{4.16}$$

In view of the previous estimations we get immediately that $g/X^+ \in \mathcal{L}_\varphi(\gamma)$.

We now examine the function

$$\Psi(z) := (\mathbf{C}_\gamma \{g/X^+\})(z). \tag{4.17}$$

Since $g/X^+ \in \mathcal{L}_\varphi(\gamma)$, then the function

$$\Psi^-(t) = (\mathbf{C}_\gamma^- \{g/X^+\})(t) = \int_\gamma e_t(\zeta)n_q(\zeta) \left(\frac{g(\zeta)}{X^+(\zeta)} - \frac{g(t)}{X^+(t)} \right) ds \in \mathcal{L}_\varphi(\gamma). \tag{4.18}$$

Now by applying Lemma 2.4, we also obtain that

$$\begin{aligned} \Psi^-(t) &= (\mathbf{C}_\gamma^- \{g/X^+\})(t) \\ &= \int_\gamma e_t(\zeta)n_q(\zeta) \left(\frac{g(\zeta) - g(t)}{X^+(\zeta)} \right) ds \\ &\quad + g(t) \int_\gamma e_t(\zeta)n_q(\zeta) \left(\frac{1}{X^+(\zeta)} - \frac{1}{X^+(t)} \right) ds \\ &= \int_\gamma e_t(\zeta)n_q(\zeta) \left(\frac{g(\zeta) - g(t)}{X^+(\zeta)} \right) ds \in \mathcal{L}_\varphi(\gamma), \end{aligned} \tag{4.19}$$

consequently $X^+\Psi^- \in \mathcal{H}_\varphi(\gamma)$.

On the other hand,

$$\begin{aligned} &\left| \int_{\gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta) (X^+(\zeta)\Psi^-(\zeta) - X^+(t)\Psi^-(t)) ds \right| \\ &= \left| \int_{\gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta) (G(\zeta) - G(t))X^-(\zeta)\Psi^-(\zeta) ds \right. \\ &\quad \left. + G(t) \int_{\gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta) (X^-(\zeta)\Psi^-(\zeta) - X^-(t)\Psi^-(t)) ds \right| \\ &\leq c \left(\int_0^\epsilon \frac{\omega_G(\xi)}{\xi} d\xi + \left| \int_{\gamma_\epsilon(t)} e_t(\zeta)n_q(\zeta) (X^-(\zeta)\Psi^-(\zeta) - X^-(t)\Psi^-(t)) ds \right| \right) \\ &\leq c\varphi(\epsilon), \quad \epsilon \in (0, d], \end{aligned} \tag{4.20}$$

hence, $X^+\Psi^- \in \mathcal{L}_\varphi(\gamma)$.

Finally, since $X^-\Psi^- = (X^+\Psi^-)/G$, then $X^-\Psi^- \in \mathcal{L}_\varphi(\gamma)$, which completes the proof. □

We can now state our final result.

THEOREM 4.3. *Let $a, b \in \mathcal{H}_\psi(\gamma)$. If $\kappa \geq 0$, the singular integral equation (4.1) is solvable in the class $\mathcal{L}_\varphi(\gamma)$ for all $f \in \mathcal{L}_\varphi(\gamma)$; if $\kappa < 0$, then this equation is solvable in the class $\mathcal{L}_\varphi(\gamma)$ if and only if the following conditions are satisfied:*

$$\int_\gamma \partial_\zeta W(\zeta)n_q(\zeta) \left(\frac{f(\zeta)}{(a(\zeta) + b(\zeta))X^+(\zeta)} \right) (W(\zeta))^k ds = 0, \quad k = 0, \dots, -\kappa - 1. \tag{4.21}$$

The general solution of (4.1) in the class $\mathcal{L}_\varphi(\gamma)$ has the form

$$\begin{aligned} Y(t) = & (X^+(t) - X^-(t)) \int_\gamma e_t(\zeta) n_q(\zeta) \left(\frac{g(\zeta) - g(t)}{X^+(\zeta)} \right) ds \\ & + g(t) + (X^+(t) - X^-(t)) P_{\kappa-1}(t), \quad t \in \gamma, \end{aligned} \quad (4.22)$$

where $P_{\kappa-1}$ is an arbitrary hypercomplex polynomial whose degree is not greater than $\kappa - 1$ for $\kappa > 0$ and $P_{\kappa-1} \equiv 0$ for $\kappa \leq 0$.

Proof. The proof follows directly by using Theorems 3.5, 4.1, and 4.2. \square

Acknowledgments. The research carried out in this paper was initiated while the third author was visiting the Department of Mathematics, University of Oriente, Santiago de Cuba. It was completed during his stay as a Ph.D. student at the Department of Mathematical Analysis, Clifford Research Group, Ghent University, supported by the Special Research Fund of Ghent University under the Doctoral Grant 011/DS 503. He would like to express his sincere gratitude.

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