ON THE PRODUCT AND RATIO OF BESSEL RANDOM VARIABLES

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The distributions of products and ratios of random variables are of interest in many areas of the sciences. In this paper, the exact distributions of the product $|XY|$ and the ratio $|X/Y|$ are derived when X and Y are independent Bessel function random variables. An application of the results is provided by tabulating the associated percentage points.

1. Introduction

For given random variables X and Y, the distributions of the product $|XY|$ and the ratio $|X/Y|$ are of interest in many areas of the sciences.

In traditional portfolio selection models, certain cases involve the product of random variables. The best examples of these are in the case of investment in a number of different overseas markets. In portfolio diversification models (see, e.g., Grubel [6]), not only are prices of shares in local markets uncertain, but also the exchange rates are uncertain so that the value of the portfolio in domestic currency is related to a product of random variables. Similarly in models of diversified production by multinationals (see, e.g., Rugman [21]), there are local production uncertainty and exchange rate uncertainty so that profits in home currency are again related to a product of random variables. An entirely different example is drawn from the econometric literature. In making a forecast from an estimated equation, Feldstein [4] pointed out that both the parameter and the value of the exogenous variable in the forecast period could be considered as random variables. Hence, the forecast was proportional to a product of random variables.

An important example of ratios of random variables is the stress-strength model in the context of reliability. It describes the life of a component which has a random strength Y and is subjected to random stress X. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $Y > X$. Thus, $\Pr(X < Y)$ is a measure of component reliability. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures, and the aging of concrete pressure vessels.

The distributions of $|XY|$ and $|X/Y|$ have been studied by several authors especially when X and Y are independent random variables and come from the same family. With

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respect to products of random variables, see Sakamoto [22] for uniform family, Harter [7] and Wallgren [28] for Student's t family, Springer and Thompson [24] for normal family, Stuart [26] and Podolski [14] for gamma family, Steece [25], Bhargava and Khatri [3], and Tang and Gupta [27] for beta family, Abu-Salih [1] for power function family, and Malik and Trudel [11] for exponential family (see also Rathie and Rohrer [20] for a comprehensive review of known results). With respect to ratios of random variables, see Marsaglia [12] and Korhonen and Narula [9] for normal family, Press [15] for Student's t family, Basu and Lochner [2] for Weibull family, Shcolnick [23] for stable family, Hawkins and Han [8] for noncentral chi-square family, Provost [16] for gamma family, and Pham-Gia [13] for beta family.

In this paper, we study the exact distributions of $|XY|$ and $|X/Y|$ when X and Y are independent Bessel function random variables with pdfs

$$f_X(x) = \frac{|x|^m}{\sqrt{\pi}2^m b^{m+1}\Gamma(m+1/2)} K_m\left(\left|\frac{x}{b}\right|\right), \tag{1.1}$$

$$f_Y(y) = \frac{|y|^n}{\sqrt{\pi}2^n \beta^{n+1}\Gamma(n+1/2)} K_n\left(\left|\frac{y}{\beta}\right|\right), \tag{1.2}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $b > 0$, $\beta > 0$, $m > 1$, and $n > 1$, where

$$K_\nu(x) = \frac{\sqrt{\pi}x^\nu}{2^\nu\Gamma(\nu+1/2)} \int_1^\infty \left(t^2-1\right)^{\nu-1/2} \exp(-xt)\,dt \tag{1.3}$$

is the modified Bessel function of the third kind. Tabulations of the associated percentage points are also provided.

Bessel function distributions have found applications in a variety of areas that range from image and speech recognition and ocean engineering to finance. They are rapidly becoming distributions of first choice whenever "something" heavier than Gaussian tails is observed in the data. Some examples are as follows (see Kotz et al. [10] for further applications).

(1) In communication theory, X and Y could represent the random noises corresponding to two different signals.

(2) In ocean engineering, X and Y could represent distributions of navigation errors.

(3) In finance, X and Y could represent distributions of log-returns of two different commodities.

(4) In image and speech recognition, X and Y could represent "input" distributions.

In each of the examples above, it will be of interest to study the distribution of the ratio $|X/Y|$. For example, in communication theory, $|X/Y|$ could represent the relative strength of the two different signals. In ocean engineering, $|X/Y|$ could represent the relative safety of navigation. In finance, $|X/Y|$ could represent the relative popularity of the two different commodities. The distribution of the product $|XY|$ is considered here for completeness.

The exact expressions for the distributions of the product and ratio are given in Sections 2 and 3 of the paper. The calculations involve the generalized hypergeometric function defined by

$$ {}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;x) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{x^k}{k!}, \tag{1.4} $$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. We also need the following important lemmas.

Lemma 1.1 (Prudnikov et al. [18, equation (2.16.33.5), Volume 2]). *For $b>0$ and $c>0$,*

$$ \begin{aligned} \int_0^\infty & x^{\alpha-1}K_\mu\left(\frac{b}{x}\right)K_\nu(cx)dx \\ &= 2^{\alpha-2\mu-3}b^\mu c^{\mu-\alpha}\Gamma(-\mu)\Gamma\left(\frac{\alpha+\nu-\mu}{2}\right)\Gamma\left(\frac{\alpha-\nu-\mu}{2}\right) \\ &\quad\times {}_0F_3\left(1+\mu,1+\frac{\mu-\nu-\alpha}{2},1+\frac{\nu+\mu-\alpha}{2};\frac{b^2c^2}{16}\right) \\ &\quad+2^{\alpha+2\mu-3}b^{-\mu}c^{-\mu-\alpha}\Gamma(\mu)\Gamma\left(\frac{\alpha+\nu+\mu}{2}\right)\Gamma\left(\frac{\alpha-\nu+\mu}{2}\right) \\ &\quad\times {}_0F_3\left(1-\mu,1-\frac{\alpha+\mu+\nu}{2},1-\frac{\alpha+\mu-\nu}{2};\frac{b^2c^2}{16}\right) \\ &\quad+2^{-\alpha-2\nu-3}b^{\alpha+\nu}c^{\nu}\Gamma(-\nu)\Gamma\left(\frac{\mu-\nu-\alpha}{2}\right)\Gamma\left(-\frac{\mu+\nu+\alpha}{2}\right) \\ &\quad\times {}_0F_3\left(1+\nu,1+\frac{\alpha+\nu-\mu}{2},1+\frac{\alpha+\mu+\nu}{2};\frac{b^2c^2}{16}\right) \\ &\quad+2^{2\nu-\alpha-3}b^{\alpha-\nu}c^{-\nu}\Gamma(\nu)\Gamma\left(\frac{\mu+\nu-\alpha}{2}\right)\Gamma\left(\frac{\nu-\mu-\alpha}{2}\right) \\ &\quad\times {}_0F_3\left(1-\nu,1+\frac{\alpha-\mu-\nu}{2},1+\frac{\alpha+\mu-\nu}{2};\frac{b^2c^2}{16}\right). \end{aligned} \tag{1.5} $$

Lemma 1.2 (Gradshteyn and Ryzhik [5, equation (6.576.4)]). *For $a+b>0$ and $\lambda<1-\mu-\nu$,*

$$ \begin{aligned} \int_0^\infty x^{-\lambda}K_\mu(ax)K_\nu(bx)dx &= \frac{2^{-2-\lambda}a^{-\nu+\lambda-1}b^\nu}{\Gamma(1-\lambda)}\Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \\ &\quad\times\Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right) \\ &\quad\times {}_2F_1\left(\frac{1-\lambda+\mu+\nu}{2},\frac{1-\lambda-\mu+\nu}{2};1-\lambda;1-\frac{b^2}{a^2}\right). \end{aligned} \tag{1.6} $$

Lemma 1.3 (Prudnikov et al. [19, equation (2.21.1.14), Volume 3]). *For $y > 0$, $\beta > 0$, $\alpha+\beta < 1+a$, and $\alpha+\beta < 1+b$,*

$$\begin{aligned}\int_y^\infty & x^{\alpha-1}(x-y)^{\beta-1}{}_2F_1(a,b;c;1-wx)dx \\ &= w^{-a}y^{\alpha+\beta-a-1}\frac{\Gamma(c)\Gamma(b-a)\Gamma(\beta)\Gamma(a-\alpha-\beta+1)}{\Gamma(b)\Gamma(c-a)\Gamma(a-\alpha+1)} \\ &\quad\times {}_3F_2\left(a,c-b,a-\alpha-\beta+1;a-\alpha+1,a-b+1;\frac{1}{wy}\right) \\ &\quad+ w^{-b}y^{\alpha+\beta-b-1}\frac{\Gamma(c)\Gamma(a-b)\Gamma(\beta)\Gamma(b-\alpha-\beta+1)}{\Gamma(a)\Gamma(c-b)\Gamma(b-\alpha+1)} \\ &\quad\times {}_3F_2\left(b,c-a,b-\alpha-\beta+1;b-a+1,b-\alpha+1;\frac{1}{wy}\right).\end{aligned} \tag{1.7}$$

Lemma 1.4 (Prudnikov et al. [19, equation (2.22.2.1), Volume 3]). *For $a > 0$, $\alpha > 0$, $\beta > 0$, and $p \le q+1$,*

$$\begin{aligned}\int_0^a & x^{\alpha-1}(a-x)^{\beta-1}{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;wx)\,dx \\ &= a^{\alpha+\beta-1}B(\alpha,\beta)_{p+1}F_{q+1}(a_1,\dots,a_p,\alpha;b_1,\dots,b_q,\alpha+\beta;aw).\end{aligned} \tag{1.8}$$

Further properties of the generalized hypergeometric function can be found in Prudnikov et al. [17, 18, 19] and Gradshteyn and Ryzhik [5].

2. Product

Theorem 2.1 derives explicit expressions for the distribution of $|XY|$ in terms of the ${}_0F_3$ and ${}_1F_4$ hypergeometric functions.

Theorem 2.1. *Suppose that X and Y are distributed according to (1.1) and (1.2), respectively. The pdf and the cdf of $Z = |XY|$ can be expressed as*

$$\begin{aligned}f_Z(z) = K\{&2^{n-3m-1}b^{-m}\beta^{n-2m}\Gamma^2(-m)\Gamma(n-m)z^{2m}C_1(z) - 2^{m+n}b^m\beta^n C\Gamma(m)\Gamma(n)C_2(z) \\ &+ 2^{m-3n-1}b^{m-2n}\beta^{-n}\Gamma^2(-n)\Gamma(m-n)z^{2n}C_3(z)\},\end{aligned} \tag{2.1}$$

$$\begin{aligned}F_Z(z) = K\Big\{&2^{n-3m-1}b^{-m}\beta^{n-2m}\Gamma^2(-m)\Gamma(n-m)\frac{z^{2m+1}}{2m+1}C_4(z) - 2^{m+n}b^m\beta^n C\Gamma(m)\Gamma(n)zC_5(z) \\ &+ 2^{m-3n-1}b^{m-2n}\beta^{-n}\Gamma^2(-n)\Gamma(m-n)\frac{z^{2n+1}}{2n+1}C_6(z)\Big\},\end{aligned} \tag{2.2}$$

where

$$
\begin{aligned}
C_1(z) &= {}_0F_3\left(1+m, 1+m-n, 1+m; \frac{z^2}{16b^2\beta^2}\right),\\
C_2(z) &= {}_0F_3\left(1-m, 1-n, 1; \frac{z^2}{16b^2\beta^2}\right),\\
C_3(z) &= {}_0F_3\left(1+n, 1+n-m, 1+n; \frac{z^2}{16b^2\beta^2}\right),\\
C_4(z) &= {}_1F_4\left(\frac{1}{2}+m; 1+m, 1+m-n, 1+m, \frac{3}{2}+m; \frac{z^2}{16b^2\beta^2}\right),\\
C_5(z) &= {}_1F_4\left(\frac{1}{2}; 1-m, 1-n, 1, \frac{3}{2}; \frac{z^2}{16b^2\beta^2}\right),\\
C_6(z) &= {}_1F_4\left(\frac{1}{2}+n; 1+n, 1+n-m, 1+n, \frac{3}{2}+n; \frac{z^2}{16b^2\beta^2}\right),\\
\frac{1}{K} &= \pi 2^{m+n} b^{m+1}\beta^{n+1}\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right),
\end{aligned}
\tag{2.3}
$$

and C denotes Euler's constant.

Proof. The pdf of $|XY|$ can be expressed as

$$
\begin{aligned}
f_Z(z) &= 4\int_0^\infty \frac{1}{y} f_X\left(\frac{z}{y}\right) f_Y(y)dy\\
&= 4\int_0^\infty \frac{1}{y}\frac{|z/y|^m}{\sqrt{\pi}2^m b^{m+1}\Gamma(m+1/2)} K_m\left(\left|\frac{z}{by}\right|\right)\frac{|y|^n}{\sqrt{\pi}2^n\beta^{n+1}\Gamma(n+1/2)}K_n\left(\left|\frac{y}{\beta}\right|\right)dy\\
&= \frac{z^m I(m,n)}{\pi 2^{m+n-2} b^{m+1}\beta^{n+1}\Gamma(m+1/2)\Gamma(n+1/2)},
\end{aligned}
\tag{2.4}
$$

where $I(m,n)$ denotes the integral

$$
I(m,n) = \int_0^\infty y^{n-m-1} K_m\left(\frac{z}{by}\right) K_n\left(\frac{y}{\beta}\right) dy. \tag{2.5}
$$

The result in (2.1) follows by direct application of Lemma 1.1 to calculate $I(m,n)$. The cdf of Z can be expressed as

$$
\begin{aligned}
F_Z(z) = K\Big\{&2^{n-3m-1}b^{-m}\beta^{n-2m}\Gamma^2(-m)\Gamma(n-m)\int_0^z w^{2m}C_1(w)dw\\
&- 2^{m+n}b^m\beta^n C\Gamma(m)\Gamma(n)\int_0^z C_2(w)dw\\
&+ 2^{m-3n-1}b^{m-2n}\beta^{-n}\Gamma^2(-n)\Gamma(m-n)\int_0^z w^{2n}C_3(w)dw\Big\}.
\end{aligned}
\tag{2.6}
$$

The result in (2.2) follows by applying Lemma 1.4 to calculate the three integrals in (2.6). □

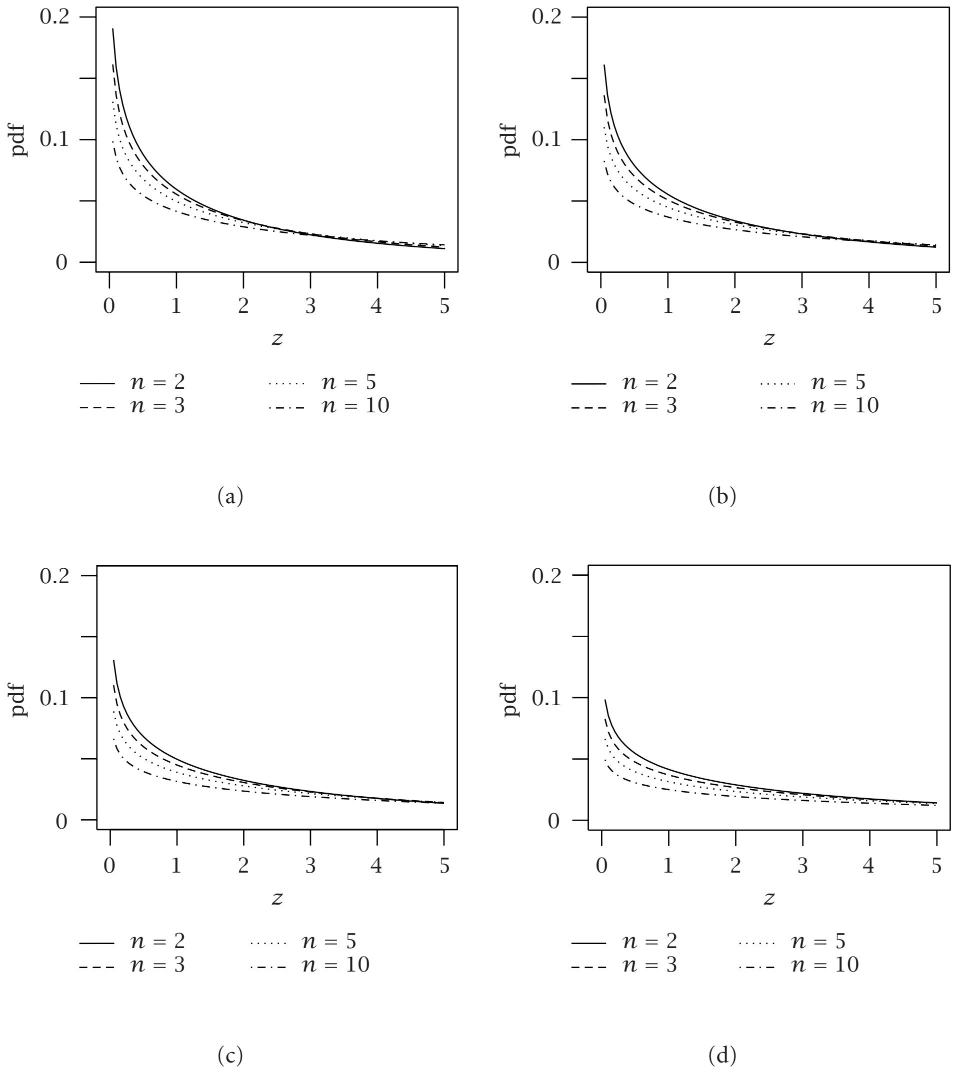

Figure 2.1. Plots of the pdf (2.1) for $b = 1$, $\beta = 1$, and (a) $m = 2$; (b) $m = 3$; (c) $m = 5$; and (d) $m = 10$.

Figure 2.1 illustrates possible shapes of the pdf (2.1) for selected values of m and n. The four curves in each plot correspond to selected values of n. The effect of the parameters is evident.

3. Ratio

Theorem 3.1 derives explicit expressions for the distribution of $|X/Y|$ in terms of the ${}_2F_1$ and ${}_3F_2$ hypergeometric functions.

THEOREM 3.1. *Suppose that X and Y are distributed according to (1.1) and (1.2), respectively. The pdf and the cdf of $Z = |X/Y|$ can be expressed as*

$$f_Z(z) = \frac{2L(\beta/b)^{-2n-1}}{m+n+1} z^{-2n-2} D_1(z), \tag{3.1}$$

$$F_Z(z) = 2L\Gamma(m+n+1)\left\{\frac{\Gamma(-m)D_2(z)}{(2m+1)\Gamma(n+1)} + \frac{\beta z D_3(z)}{mb\Gamma(m+n+1)}\right\}, \tag{3.2}$$

where

$$\begin{aligned} D_1(z) &= {}_2F_1\left(m+n+1, n+1; m+n+2; 1-\frac{b^2}{\beta^2 z^2}\right), \\ D_2(z) &= {}_2F_1\left(m+n+1, m+\frac{1}{2}; m+\frac{3}{2}; \frac{\beta^2 z^2}{b^2}\right), \\ D_3(z) &= {}_3F_2\left(n+1, 1, \frac{1}{2}; 1-m, \frac{3}{2}; \frac{\beta^2 z^2}{b^2}\right), \\ L &= \frac{\Gamma(m+1)\Gamma(n+1)}{\pi\Gamma(m+1/2)\Gamma(n+1/2)}. \end{aligned} \tag{3.3}$$

Proof. The pdf of $Z = |X/Y|$ can be expressed as

$$\begin{aligned} f_Z(z) &= 4\int_0^\infty y f_X(yz) f_Y(y)dy \\ &= 4\int_0^\infty y \frac{(yz)^m}{\sqrt{\pi}2^m\Gamma(m+1/2)} K_m(yz) \frac{y^n}{\sqrt{\pi}2^n\Gamma(n+1/2)} K_n(y)dy \\ &= \frac{z^m I(m,n)}{\pi 2^{m+n-2}\Gamma(m+1/2)\Gamma(n+1/2)}, \end{aligned} \tag{3.4}$$

where $I(m,n)$ denotes the integral

$$I(m,n) = \int_0^\infty y^{m+n+1} K_m(yz) K_n(y) dy. \tag{3.5}$$

The result in (3.1) follows by direct application of Lemma 1.2 to calculate $I(m,n)$. The cdf of Z can be expressed as

$$\begin{aligned} F_Z(z) &= \frac{2L(\beta/b)^{-2n-1}}{m+n+1} \int_0^z w^{-2n-2} D_1(w) dw \\ &= \frac{L}{m+n+1} \int_{b^2/(\beta z)^2}^\infty x^{n-1/2}{}_2F_1(m+n+1, n+1; m+n+2; 1-x)dx, \end{aligned} \tag{3.6}$$

which follows by setting $x = (\beta w/b)^{-2}$. The result in (3.2) follows by applying Lemma 1.3 to calculate the integral in (3.6). □

Using special properties of the $_2F_1$ hypergeometric function, one can derive other equivalent forms and elementary forms for the pdf of $Z = |X/Y|$. This is illustrated in the corollaries below.

Corollary 3.2. *The pdf given by (3.1) can be expressed in the equivalent forms*

$$
\begin{aligned}
f_Z(z) &= \frac{2\beta(\beta z/b)^{2m}\Gamma(m+1)\Gamma(n+1)}{b\pi(m+n+1)\Gamma(m+1/2)\Gamma(n+1/2)}\,{}_2F_1\left(m+n+1, m+1; m+n+2; 1-\frac{\beta^2 z^2}{b^2}\right),\\
f_Z(z) &= \frac{2\beta\Gamma(m+1)\Gamma(n+1)}{b\pi(m+n+1)\Gamma(m+1/2)\Gamma(n+1/2)}\,{}_2F_1\left(n+1, 1; m+n+2; 1-\frac{\beta^2 z^2}{b^2}\right),\\
f_Z(z) &= \frac{2\beta(\beta z/b)^{-2}\Gamma(m+1)\Gamma(n+1)}{b\pi(m+n+1)\Gamma(m+1/2)\Gamma(n+1/2)}\,{}_2F_1\left(1, m+1; m+n+2; 1-\frac{b^2}{\beta^2 z^2}\right).
\end{aligned}
\tag{3.7}
$$

Corollary 3.3. *If $m \geq 2$ and $n \geq 2$ are integers, then (3.1) can be reduced to the elementary form*

$$
\begin{aligned}
f_Z(z) = {} & \frac{2\beta\Gamma(m+1)\Gamma(m+n+1)}{b\pi(\beta^2 z^2/b^2-1)\Gamma(m+1/2)\Gamma(n+1/2)}\\
& \times \Bigg[\sum_{k=1}^{n} \frac{(n-k)!\left(1-\beta^2 z^2/b^2\right)^{1-k}}{(m+n+1-k)!} + \frac{\left(1-\beta^2 z^2/b^2\right)^{1-n}}{m!\beta^2 z^2/b^2}\\
& \times \left\{-2\left(1-\frac{b^2}{\beta^2 z^2}\right)^{-m-1}\log\left(\frac{\beta z}{b}\right) + \sum_{k=1}^{m} \frac{\left(1-b^2/(\beta^2 z^2)\right)^{-k}}{m+1-k}\right\}\Bigg].
\end{aligned}
\tag{3.8}
$$

Corollary 3.4. *If $m-1/2 \geq 1$ and $n-1/2 \geq 1$ are integers, then (3.1) can be reduced to the elementary form*

$$
\begin{aligned}
f_Z(z) = {} & \frac{2b\left(1-\beta^2 z^2/b^2\right)^{-m-n-1}\Gamma(m+1)\Gamma(n+1)}{\beta z^2\pi(m+n+1)(-m-1)_{m+n+2}\Gamma(m+1/2)\Gamma(n+1/2)}\\
& \times \left[\Gamma(-m)\left(\frac{\beta z}{b}\right)^{2m+2} + \sum_{k=1}^{m+n+1} (-m-1)_k(-1)^k\left(1-\frac{b^2}{\beta^2 z^2}\right)^{k-1}(\beta z/b)^{2k}\right].
\end{aligned}
\tag{3.9}
$$

4. Percentiles

Figure 4.1 illustrates possible shapes of the pdf (3.1) for selected values of m and n. The four curves in each plot correspond to selected values of n. The effect of the parameters is evident.

In this section, we provide tabulations of percentage points associated with the derived distributions of $|XY|$ and $|X/Y|$. These values are obtained by numerically solving

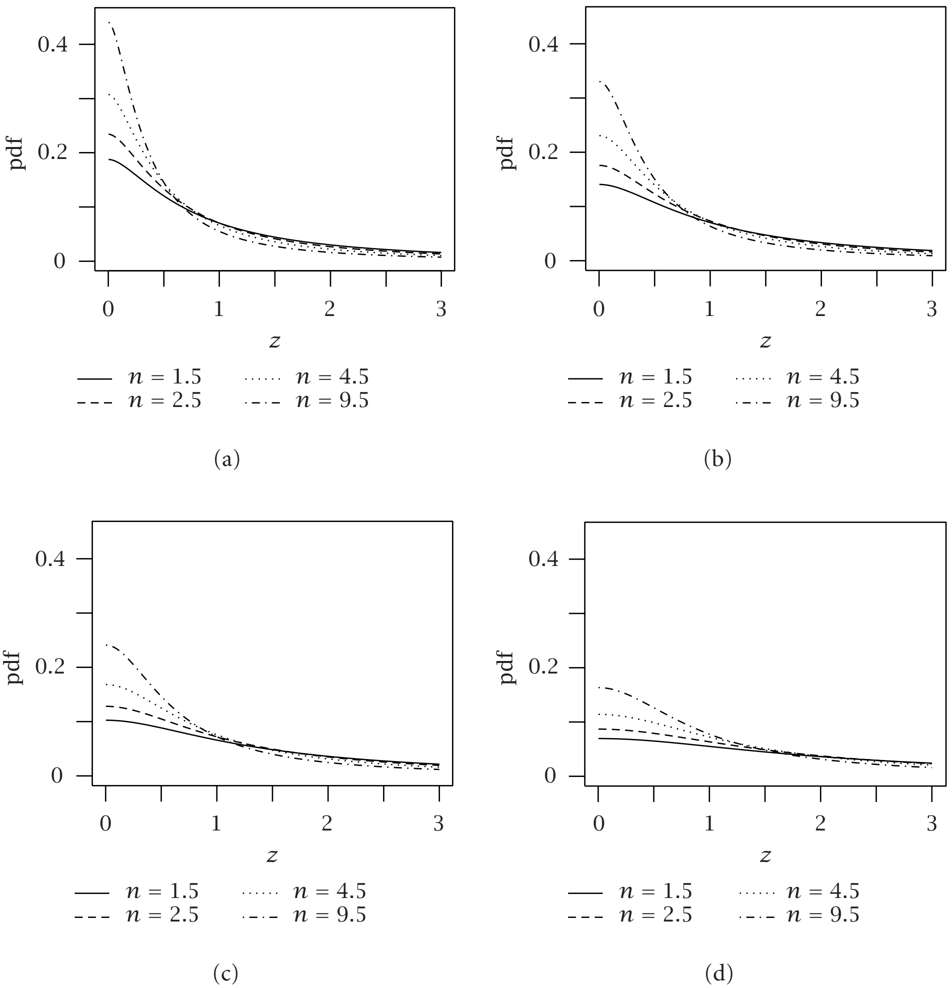

Figure 4.1. Plots of the pdf (3.1) for $b = 1$, $\beta = 1$, and (a) $m = 1.5$; (b) $m = 2.5$; (c) $m = 4.5$; and (d) $m = 9.5$.

the equations

$$K\left\{2^{n-3m-1}b^{-m}\beta^{n-2m}\Gamma^2(-m)\Gamma(n-m)\frac{z_p^{2m+1}}{2m+1}C_4(z_p)-2^{m+n}b^m\beta^n C\Gamma(m)\Gamma(n)z_pC_5(z_p)\right.$$

$$\left.+2^{m-3n-1}b^{m-2n}\beta^{-n}\Gamma^2(-n)\Gamma(m-n)\frac{z_p^{2n+1}}{2n+1}C_6(z_p)\right\}=p,$$

$$2L\Gamma(m+n+1)\left\{\frac{\Gamma(-m)D_2(z_p)}{(2m+1)\Gamma(n+1)}+\frac{\beta z_p D_3(z_p)}{mb\Gamma(m+n+1)}\right\}=p. \tag{4.1}$$

Table 4.1. Percentage points z_p of $Z = |XY|$.

m	n	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 0.95$	$p = 0.99$
2	2	0.02903867	0.0609406	0.1328627	0.3470357	0.6927036	2.098452
2	3	0.07362923	0.1376486	0.2682605	0.6146536	1.130666	3.038813
2	4	0.1100132	0.1998708	0.3777723	0.827011	1.465626	3.755526
2	5	0.1408100	0.2514674	0.4678341	1.000369	1.752475	4.380899
2	6	0.1673438	0.2968515	0.5449642	1.151051	1.991074	4.848271
2	7	0.1906657	0.3359331	0.6139745	1.284624	2.218128	5.323573
2	8	0.2125461	0.372798	0.6770319	1.415690	2.416419	5.745124
2	9	0.2300810	0.4031021	0.7305227	1.517905	2.591786	6.145533
3	3	0.1744381	0.2938927	0.5183827	1.057357	1.798354	4.335123
3	4	0.2572388	0.4184127	0.711219	1.388183	2.282625	5.225012
3	5	0.3263338	0.5227913	0.870987	1.660568	2.691937	6.044048
3	6	0.3860577	0.6124857	1.011584	1.908407	3.049341	6.677126
3	7	0.4394194	0.6929521	1.137077	2.125184	3.382392	7.385786
3	8	0.4865918	0.7648362	1.249936	2.318062	3.661295	7.867155
3	9	0.5306344	0.8335735	1.356932	2.498872	3.923604	8.372842
4	4	0.3773593	0.593275	0.972541	1.816783	2.900670	6.372728
4	5	0.4760256	0.7359029	1.184584	2.162823	3.404096	7.3069
4	6	0.5626443	0.8623236	1.371957	2.474747	3.845828	8.08524
4	7	0.638364	0.9744115	1.541777	2.752022	4.246468	8.858037
4	8	0.7085111	1.076598	1.696747	3.004645	4.632417	9.46566
4	9	0.7702856	1.166607	1.830414	3.240364	4.958135	10.19733
5	5	0.6033154	0.9178398	1.450584	2.584631	3.999764	8.318145
5	6	0.7075169	1.066605	1.668016	2.942446	4.50238	9.176312
5	7	0.8076738	1.211170	1.883969	3.282990	4.993118	10.06218
5	8	0.89244	1.334572	2.062933	3.568706	5.40753	10.82776
5	9	0.974915	1.449774	2.230274	3.847315	5.804434	11.49615
6	6	0.8362517	1.250325	1.934166	3.357292	5.101803	10.24216
6	7	0.9532409	1.417497	2.175916	3.744367	5.625393	11.18685
6	8	1.053948	1.559225	2.37994	4.067588	6.09969	11.93585
6	9	1.151888	1.695543	2.583201	4.397424	6.553556	12.80644
7	7	1.083447	1.595423	2.431431	4.140509	6.192786	12.20569
7	8	1.198694	1.763451	2.676427	4.537504	6.72858	13.04854
7	9	1.308068	1.918944	2.897775	4.865201	7.179527	13.83574
8	8	1.331007	1.948211	2.941492	4.955321	7.298417	14.03725
8	9	1.450212	2.113748	3.176776	5.310468	7.810788	14.88969
9	9	1.588348	2.305376	3.451306	5.742988	8.420277	15.94564

Evidently, this involves computation of the generalized hypergeometric function and routines for this are widely available. We used the function hypergeom $(\cdot)$ in the algebraic manipulation package Maple. Tables 4.1 and 4.2 provide the numerical values of z_p for $b = 1$, $\beta = 1$, $m = 2, 3, \dots, 9$, and $n = m, m+1, \dots, 9$.

Table 4.2. Percentage points z_p of $Z = |X/Y|$.

m	n	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 0.95$	$p = 0.99$
2	2	2.002938	4.334821	11.41224	52.30151	220.7189	5685.296
2	3	0.7445086	1.382358	2.855702	8.214106	21.04093	152.4738
2	4	0.4889538	0.876642	1.710587	4.384554	10.02126	57.84656
2	5	0.3832577	0.6767803	1.286959	3.169977	6.946125	37.36734
2	6	0.3251244	0.5683113	1.068319	2.560906	5.525674	29.00740
2	7	0.2832587	0.4941505	0.9234088	2.196231	4.668944	24.53751
2	8	0.2562869	0.4435011	0.8235854	1.936850	4.135955	21.08758
2	9	0.2353905	0.4071502	0.7504027	1.757447	3.712460	19.04998
3	3	1.647264	2.842614	5.538683	15.08887	37.62435	261.1618
3	4	1.085868	1.791815	3.255642	7.909027	17.61894	98.1088
3	5	0.840817	1.359837	2.407853	5.574137	11.89116	62.47279
3	6	0.7106484	1.133092	1.975254	4.480779	9.4219	49.00014
3	7	0.6216468	0.988768	1.716620	3.826405	7.905925	40.11744
3	8	0.558414	0.8840209	1.517433	3.359672	6.96206	35.21301
3	9	0.5107641	0.8041508	1.376936	3.048617	6.29827	31.90878
4	4	1.547561	2.487375	4.429231	10.55419	23.34162	127.9502
4	5	1.207760	1.900782	3.296398	7.461444	15.79163	83.15728
4	6	1.011248	1.572540	2.678426	5.945279	12.30920	62.89049
4	7	0.8819533	1.364001	2.300262	5.037816	10.45939	53.42483
4	8	0.7944737	1.222424	2.058176	4.470943	9.197412	47.1173
4	9	0.7268028	1.114706	1.866525	4.036168	8.295475	41.95984
5	5	1.499379	2.331841	4.011059	9.034731	19.02830	98.64185
5	6	1.257534	1.935343	3.264778	7.174943	14.79680	76.47803
5	7	1.099045	1.675479	2.801810	6.083041	12.60114	64.14368
5	8	0.990024	1.500218	2.49008	5.374516	11.01383	55.72903
5	9	0.906855	1.371630	2.272028	4.863465	9.93863	50.23113
6	6	1.478265	2.256043	3.782804	8.285063	17.11076	86.97477
6	7	1.28456	1.946632	3.230527	6.954553	14.26699	71.09335
6	8	1.155091	1.741973	2.867585	6.141117	12.57803	62.58595
6	9	1.056264	1.581389	2.598716	5.538108	11.25649	56.78169
7	7	1.459679	2.198993	3.639314	7.815963	15.95768	80.75053
7	8	1.30774	1.956784	3.220993	6.878688	14.07413	71.43641
7	9	1.194723	1.785849	2.932324	6.231386	12.70544	63.87592
8	8	1.441835	2.152747	3.525486	7.523065	15.35245	76.5615
8	9	1.320652	1.960763	3.198957	6.790555	13.74266	69.96122
9	9	1.433036	2.127872	3.470807	7.34273	14.87503	74.51613

We hope these numbers will be of use to the practitioners mentioned in Section 1. Similar tabulations could be easily derived for other values of p, m, n, b, and β by using the hypergeom $(\cdot)$ function in Maple.

Besides being of practical interest, the above tables can be used to check the accuracy of the results derived in Sections 2 and 3. We estimated the relevant percentage points by simulating samples of size 10^8 from the two Bessel function distributions. The estimates were consistent with the tabulated values up to the third decimal place.

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