k-COMPLEMENTING SUBSETS OF NONNEGATIVE INTEGERS

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A collection $\{S_1,S_2,\dots\}$ of nonempty sets is called a complementing system of subsets for a set X of nonnegative integers if every element of X can be uniquely expressed as a sum of elements of the sets $S_1,S_2,\dots$. We present a complete characterization of all complementing systems of subsets for the set of the first n nonnegative integers as well as an explicit enumeration formula.

1. Introduction

Let $S=\{S_1,S_2,\dots\}$ represent a collection of nonempty sets of nonnegative integers in which each member contains the integer 0. Then S is called a *complementing system of subsets* for $X\subseteq\{0,1,\dots\}$ if every $x\in X$ can be uniquely represented as $x=s_1+s_2+\cdots$ with $s_i\in S_i$. We will also write $X=S_1\oplus S_2\oplus\cdots$ and, when necessary, refer to X as the direct sum of the S_i.

We will denote the set of all complementing systems for X by $\mathrm{CS}(X)$. Then $\{X\}\in \mathrm{CS}(X)\neq\emptyset$.

If there is a positive integer k such that $X=S_1\oplus\cdots\oplus S_k$, then $\{S_1,\dots,S_k\}$ will be called a *k-complementing system of subsets*, or a *complementing k-tuple*, for X.

Denote the set of all complementing k-tuples for X by $\mathrm{CS}(k,X)$.

We will address the problem of characterizing all $S\in \mathrm{CS}(k,\mathbb{N}_n)$, where $\mathbb{N}_n=\{0,1,\dots,n-1\}$. The corresponding more general problem for $\mathrm{CS}(\mathbb{N})$ was solved by de Bruijn [2], where $\mathbb{N}=\{0,1,\dots\}$. Long [4] has given a complete solution for $\mathrm{CS}(2,\mathbb{N}_n)$. Since the appearance of Long's paper, no progress seems to have been made to solve the problem for $k>2$. Tijdeman [6] gives a survey of the evolution of this problem and related work.

In Section 2, we give an alternative proof of Long's theorem (Theorem 2.5) followed in Section 3 by its natural extension (Theorem 3.2) and a general structure theorem for $\mathrm{CS}(k,\mathbb{N}_n)$ (Theorem 3.5).

A complementing system $S=\{S_1,S_2,\dots\}\in \mathrm{CS}(\mathbb{N})$ will be called *usual* if for any sequence $g_1,g_2,\dots$ $(g_i>1)$ of integers, each $S_i\in S$ is given by

$$S_i=\{0,m_{i-1},2m_{i-1},\dots,(g_i-1)m_{i-1}\},\tag{1.1}$$

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where $m_0 = 1$, $m_i = g_1 g_2 \cdots g_i$ $(i > 0)$. We will refer to the collection $\{S_1, S_2, \ldots, S_i, \ldots\}$ as the complementing system corresponding to (or generated by) the integers $g_1, g_2, \ldots$.

We will denote the set of all usual complementing systems of subsets for $\mathbb{N}$ by $\mathrm{UCS}(\mathbb{N})$. For positive integers a and c, the set $U = \{0, a, 2a, \ldots, (c-1)a\}$ will be called a *simplex*, written additively (after Tijdeman [6]). We will adopt the notation $U = [a, c]$. Thus, by (1.1) every member of a usual complementing system is a simplex.

We can derive usual complementing systems of subsets for $\mathbb{N}_n$ from the following adaptation of a theorem of Long [4].

Theorem 1.1. *Let $n = g_1 g_2 \cdots g_k$ $(g_i > 1)$ represent any factorization of n as a product of positive integers and let the sets $S_1, \ldots, S_k$ be defined as in (1.1). Then $S = \{S_1, \ldots, S_k\} \in \mathrm{UCS}(\mathbb{N}_n)$.*

Proof. If $k = 1$, then $n = g_1$ and $\{S_1, \ldots, S_k\} = \{S_1\}$ which is clearly a complementing system. For $k = 2$ we have $n = g_1 g_2$ and, by (1.1), $i = 1$ gives $S_1 = [1, g_2]$. If $i = 2$, then for $\{S_1, S_2\}$ to form a complementing pair for $\mathbb{N}_n$, the least nonzero element of S_2 must be g_1 (since S_1 already contains $0, 1, \ldots, g_1 - 1$) and thenceforth elements of S_2 must be consecutively spaced g_1 apart. This shows that S_2 has the form $S_2 = [g_1, g_2]$. Assume that the proposition holds for some fixed integer v and consider the system $\{S_1, \ldots, S_v, S_{v+1}\}$. By the inductive hypothesis, $\{S_1, \ldots, S_v\} \in \mathrm{UCS}([1, m_v])$. So $\{S_1, \ldots, S_v, S_{v+1}\}$ is equivalent to $\{[1, m_v], S_{v+1}\}$; and the case for $k = 2$ shows that S_{v+1} has the required form. Hence the theorem is proved by mathematical induction. □

Remarks 1.2. (i) The proof of Theorem 1.1 also shows that if the sequence $g_1, g_2, \ldots$ generates $\{S_1, S_2, \ldots\} \in \mathrm{UCS}(\mathbb{N})$, then every partial sequence $g_1, g_2, \ldots, g_k$ generates $\{S_1, \ldots, S_k\} \in \mathrm{UCS}(\mathbb{N}_n)$ with $n = g_1 g_2 \cdots g_k$ such that $\{S_1, \ldots, S_k\} \cup \{S_{k+1}, \ldots\} = \{S_1, S_2, \ldots\}$, where S_i is given by (1.1), for $i = 1, \ldots, k$, and for $i > k$ by $S_{k+j} = [m_{k+j-1}, g_{k+j}]$ $(j = 1, 2, \ldots)$. Moreover, $\{S_1, \ldots, S_k\} \subset \{S_1, \ldots, S_{k+1}\}$ for every k.

(ii) It is clear that $S = \{S_1, \ldots, S_k\} \in \mathrm{UCS}(\mathbb{N}_n)$ implies that $aS = \{aS_1, \ldots, aS_k\} \in a\mathrm{UCS}(\mathbb{N}_n)$ $(1 \le a < \infty)$, where $aH = \{ah \mid h \in H\}$. It is natural to define $a\mathrm{UCS}(\mathbb{N}_n) = \mathrm{UCS}(a\mathbb{N}_n)$. Usual complementing systems for finite sets will be taken to include all of the systems $aS = \{aS_1, \ldots, aS_k\} \in \mathrm{UCS}(a\mathbb{N}_n)$ $(a \ge 1)$.

(iii) It follows from Theorem 1.1 that $|\mathrm{UCS}(\mathbb{N}_n)| = f(n)$, where $f(n)$ denotes the number of ordered factorizations of n. A simple bijection is as follows: given any ordered factorization $n = g_1 g_2 \cdots g_k$, then $g_1 g_2 \cdots g_k \leftrightarrow \{[m_0, g_1], [m_1, g_2], \ldots, [m_{k-1}, g_k]\}$.

$f(n)$ can be computed using the recurrence [5, 7]

$$f(n) = \sum_{d \mid n} f(d), \tag{1.2}$$

where $f(1) = 1$ and the sum is over divisors d of n, $d < n$.

If n has the prime factorization $n = p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}$, $1 < p_1 < \cdots < p_r$, $x_i \ge 1$, then $f(n)$ can also be found using MacMahon's formula [7]:

$$f(n) = \sum_{k=1}^{\Omega(n)} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \prod_{j=1}^{r} \binom{x_j + k - i - 1}{x_j}, \tag{1.3}$$

where $\Omega(n) = x_1 + \cdots + x_r$.

We deduce at once that $|\mathrm{UCS}(k,\mathbb{N}_n)| = f(n,k)$, where $\mathrm{UCS}(k,\mathbb{N}_n)$ denotes the set of usual k-complementing systems of subsets for $\mathbb{N}_n$ and $f(n,k)$ is the number of ordered k-factorizations of n.

It follows from (1.3) (see also [1, page 59]) that $f(n,k)$ can be computed from the formula

$$f(n,k) = \sum_{i=0}^{k-1}(-1)^i \binom{k}{i} \prod_{j=1}^{r} \binom{x_j + k - i - 1}{x_j}. \tag{1.4}$$

Definition 1.3. Let $S = \{S_1, S_2, \dots\} \in \mathrm{CS}(\mathbb{N})$. Then every partition p of the set $\{1,2,\dots\}$ of subscripts of members of S induces a $T = \{T_1, T_2, \dots\} \in \mathrm{CS}(\mathbb{N})$ with the property that each $T_j = S_{j1} \oplus S_{j2} \oplus \cdots$ where $j_1, j_2, \dots$ belong to a certain class of p. de Bruijn [2] is followed and T is called a *degeneration* of S. When necessary, T is also said to be induced by the partition or shape p, without reference to S.

The following fundamental classification theorem [2] for complementing systems of subsets for $\mathbb{N}$, which also applies to $\mathbb{N}_n$ via Theorem 1.1, is crucial to all what follows.

Theorem 1.4 (N. G. de Bruijn). *Every complementing system of subsets for $\mathbb{N}$ is the degeneration of a usual complementing system.*

Other relevant properties of usual complementing systems are summarized in the next theorem.

Theorem 1.5. (i) *A collection of sets S is a usual complementing system for a finite set if and only if $S \in \mathrm{CS}(X)$, where X is a simplex.*

(ii) *Let $X = [a_1, c_1] \oplus \cdots \oplus [a_k, c_k]$. Then X is a simplex if and only if the $[a_i, c_i]$ are consecutive simplices of some $S \in \mathrm{UCS}(\mathbb{N})$.*

(iii) *The simplex $[a,n]$ ($a \geq 1$, $n \geq 2$) is the direct sum of more than one simplex if and only if n is composite.*

(iv) *For $S \in \mathrm{UCS}(\mathbb{N})$ to be the degeneration of $T \in \mathrm{UCS}(\mathbb{N})$, where $S \neq T$, it is necessary and sufficient that some member of S has composite cardinality.*

Proof. (i) If S is a usual complementing system for a finite set, then S is generated by a finite sequence of positive integers. Thus by Theorem 1.1 and Remark 1.2(i), $S \in \mathrm{UCS}(X)$, where $X = [1,n]$ for some n. Conversely, if $S \in \mathrm{CS}([a,n])$, where a and n are positive integers, then by Remark 1.2(ii) we can form $\{H_1, \dots, H_v\} \in \mathrm{UCS}([1,n])$. Thus $S = \{aH_1, \dots, aH_v\}$.

(ii) This follows from Remark 1.2(i) and part (i).

(iii) By Theorem 1.1, $\{S_1, S_2\} \in \mathrm{UCS}(\mathbb{N}_n) \Leftrightarrow n = |S_1 \oplus S_2| = |S_1||S_2|$; $\mathbb{N}_n = [1,n]$ and $[a,n] = a[1,n]$.

(iv) This follows from part (iii). $\square$

Remark and Definition 1.6. Theorem 1.5(iv) implies that a fixed $P \in \mathrm{CS}(\mathbb{N})$ is not the nontrivial degeneration of any $T \in \mathrm{UCS}(\mathbb{N})$ if and only if each $P_i \in P$ has prime cardinality, that is, P is a usual complementing system generated by a sequence of prime numbers. P will be called a *prime complementing system* of subsets for $\mathbb{N}$.

Hence we have the following.

Corollary 1.7. *Every usual complementing system of subsets is a degeneration of a prime complementing system.*

We can now state the following complementing subset-systems analogue of the fundamental theorem of arithmetic.

Theorem 1.8. *Every complementing system of subsets is a degeneration of a prime complementing system.*

Proof. The theorem follows by transitivity from Theorem 1.4 and Corollary 1.7. □

Remarks 1.9. (i) Denote the set of prime complementing systems for $\mathbb{N}$ by $\mathrm{PCS}(\mathbb{N})$. It is clear that there are strict inclusions: $\mathrm{PCS}(\mathbb{N}) \subset \mathrm{UCS}(\mathbb{N}) \subset \mathrm{CS}(\mathbb{N})$.

(ii) Theorem 1.8 guarantees that to generate all complementing systems via degenerations it suffices to use the minimal generating set $\mathrm{PCS}(\mathbb{N})$ rather than the whole $\mathrm{UCS}(\mathbb{N})$. The same remark applies to the finite case for the corresponding sets $\mathrm{PCS}(\mathbb{N}_n)$, $\mathrm{CS}(\mathbb{N}_n)$, and $\mathrm{UCS}(\mathbb{N}_n)$. $\mathrm{UCS}(\mathbb{N}_n) = \mathrm{PCS}(\mathbb{N}_n)$ if and only if n is prime.

(iii) If n has the prime factorization $n = p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}$, $1 < p_1 < \cdots < p_r$, $x_i \geq 1$, then it follows from Remark 1.2(iii) that $|\mathrm{PNS}(\mathbb{N}_n)| = (x_1 + \cdots + x_r)!/x_1! \cdots x_r!$.

2. Complementing pairs and the theorem of C. T. Long

Let $\mathrm{SP}(m)$ denote the set of all partitions of $\{1,2,\dots,m\}$ so that $|\mathrm{SP}(m)| = B(m)$, the mth Bell number. Also let $\mathrm{SP}(m,k)$ denote the set of all k-partitions of $\{1,2,\dots,m\}$ so that $|\mathrm{SP}(m,k)| = s2(m,k)$, a Stirling number of the second kind.

An element p of $\mathrm{SP}(m,k)$ will be called nonconsecutive if no member of p contains a pair of consecutive integers. Let $\mathrm{NC}(m,k)$ denote the set of all nonconsecutive k-partitions of $\{1,2,\dots,m\}$, and let $\mathrm{nc}(m,k) = |\mathrm{NC}(m,k)|$.

Theorem 2.1. $\mathrm{nc}(m,k)$ *satisfies the following recurrence:*

$$\begin{gathered} \mathrm{nc}(m,k) = \mathrm{nc}(m-1,k-1) + (k-1)\,\mathrm{nc}(m-1,k), \quad 1 \leq k \leq m, \\ \mathrm{nc}(1,1) = 1, \qquad \mathrm{nc}(2,1) = 0. \end{gathered} \tag{2.1}$$

Proof. To find a $p \in \mathrm{NC}(m,k)$ $(m > k > 2)$, we can either insert the singleton $\{m\}$ into any $p \in \mathrm{NC}(m-1,k-1)$ or put the integer m into any $k-1$ members of a $p \in \mathrm{NC}(m-1,k)$ which do not contain $m-1$. There are clearly $(k-1)\,\mathrm{nc}(m-1,k)$ possibilities in the second case. Hence the main result follows. The boundary conditions are clear from the definition and imply that $\mathrm{nc}(m,1) = 0$ $(m \neq 1)$, $\mathrm{nc}(m,m) = 1$. □

Remark 2.2. A close observation of Table 2.1 shows that the $\mathrm{nc}(m,k)$ are just Stirling numbers of the second kind which have been shifted one step to the right and one step down, that is,

$$s2(m,k) = \mathrm{nc}(m+1,k+1), \quad k \geq 0. \tag{2.2}$$

Table 2.1. Values of $\mathrm{nc}(m,k)$ for $m = 1,\dots,10$, $k = 1,\dots,10$.

$m\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	0	1	1	0	0	0	0	0	0	0
4	0	1	3	1	0	0	0	0	0	0
5	0	1	7	6	1	0	0	0	0	0
6	0	1	15	25	10	1	0	0	0	0
7	0	1	31	90	65	15	1	0	0	0
8	0	1	63	301	350	140	21	1	0	0
9	0	1	127	966	1701	1050	266	28	1	0
10	0	1	255	3025	7770	6951	2646	462	36	1

Thus

$$\mathrm{nc}(m,k) = s2(m-1,k-1), \quad 1 \le k \le m. \tag{2.3}$$

Indeed $\mathrm{nc}(1,1) = 1 = s2(0,0)$, $\mathrm{nc}(2,1) = 0 = s2(1,0)$, and $\mathrm{nc}(2,2) = 1 = s2(1,1)$.

Assume that (2.3) holds for all positive integers up to m. Then Theorem 2.1 gives

$$\begin{aligned} \mathrm{nc}(m+1,k) &= \mathrm{nc}(m,k-1) + (k-1)\cdot \mathrm{nc}(m,k) \\ &= s2(m-1,k-2) + (k-1)s2(m-1,k-1) = s2(m,k-1), \end{aligned} \tag{2.4}$$

where the second equality follows from the inductive hypothesis and the last equality follows from the usual recurrence for $s2(m,k)$. Thus (2.3) is also established by mathematical induction.

Hence the standard formula [3, page 251]

$$s2(m,k) = \sum_{i=1}^{k} \frac{(-1)^{k-i} i^m}{k!} \binom{k}{i} \quad (m,\ k \ge 0) \tag{2.5}$$

yields the following corresponding formula:

$$\mathrm{nc}(m,k) = \sum_{c=1}^{k-1} \frac{(-1)^{k-1-c} c^{m-1}}{(k-1)!} \binom{k-1}{c}, \quad \mathrm{nc}(1,1) = 1,\ 1 \le k \le m. \tag{2.6}$$

If $b^*(m)$ denotes the total number of nonconsecutive partitions of $\{1,2,\dots,m\}$, then it is easily deduced from (2.3) that

$$b^*(m) = B(m-1), \quad m \ge 1, \tag{2.7}$$

where $B(m)$ denotes the mth Bell number.

Lemma 2.3. $\mathrm{nc}(m,2) = 1$, $m > 1$.

Proof. This follows from (2.6) or, more completely, from the proof of Theorem 2.1. $\square$

Notation 2.4. Given any $S \in \mathrm{CS}(\mathbb{N}_n)$, let $\mathrm{Degen}(S)$ denote the set of all degenerations of S. Let $\mathrm{Degen}(S,k)$ denote the set of all k-degenerations of S and let $\mathrm{degen}(S,k)$ be an element of $\mathrm{Degen}(S,k)$. Theorem 1.4 says that $\mathrm{CS}(\mathbb{N}_n) = \cup(\mathrm{Degen}(S),\ S \in \mathrm{UCS}(\mathbb{N}_n))$, and implies that $\mathrm{CS}(k,\mathbb{N}_n) = \cup(\mathrm{Degen}(S,k),\ S \in \mathrm{UCS}(\mathbb{N}_n)$ and $|S| \geq k)$.

It is now straightforward to deduce the following characterization theorem [4, Theorems 1 and 2] for complementing pairs for $\mathbb{N}_n$.

Theorem 2.5 (C. T. Long). (i) $\{A,B\} \in \mathrm{CS}(2,\mathbb{N}_n)$ $(n \geq 2)$ *if and only if there exists a sequence* $g_1, g_2, \ldots, g_v$ *of integers corresponding to the factorization* $n = g_1 \cdots g_v$ *of* n *such that* A *and* B *are sets of all finite sums of the form*

$$a = \sum_{i=0}^{\lfloor (v-1)/2 \rfloor} x_{2i} m_{2i}, \qquad b = \sum_{i=0}^{\lfloor (v-2)/2 \rfloor} x_{2i+1} m_{2i+1}, \tag{2.8}$$

respectively, with $m_0 = 1$, $m_i = g_1 g_2 \cdots g_i$, *and* $0 \leq x_i < m_{i+1}$.

(ii) $|\mathrm{CS}(2,\mathbb{N}_n)| = f(n) - 1$, *where* $f(n)$ *is the number of ordered factorizations of* n.

Proof. (i) This follows from the fact that the unique shape $p = \{\{1,3,\ldots\},\{2,4,\ldots\}\}$ given by Lemma 2.3 induces $\mathrm{degen}(S,2) = \{S_1 \oplus S_3 \oplus \cdots \oplus S_{[(v-1)/2]},\ S_2 \oplus S_4 \oplus \cdots \oplus S_{[(v-2)/2]}\}$ for every $S = \{S_1,\ldots,S_v\} \in \mathrm{UCS}(\mathbb{N}_n)$, $v = 2,3,\ldots,\Omega(n)$. For a fixed $S = \{S_1,\ldots,S_v\}$, any other partition q induces $\mathrm{degen}(S,2)$ if and only if $\{u,u+2,\ldots,x,x+1,\ldots\} \in q (u \in \{1,2\})$ if and only if $S_x \oplus S_{x+1}$ is a simplex by Theorem 1.5(ii) if and only if there exists some $T \in \mathrm{UCS}(\mathbb{N}_n)$ such that $v > |T| \geq 2$ and $\mathrm{degen}(S,2) = \mathrm{degen}(T,2)$ such that $\mathrm{degen}(T,2)$ is induced by p. Thus the action of p on each $S \in \mathrm{UCS}(\mathbb{N}_n)$ with $|S| > 1$ contributes a unique member to $\mathrm{CS}(2,\mathbb{N}_n)$.

(ii) In Remark 1.2(iii) we showed that $|\mathrm{UCS}(\mathbb{N}_n)| = f(n)$ and, by part (i), $|\mathrm{Degen}(S,2)| = 1$ for every $S \in \mathrm{UCS}(\mathbb{N}_n)$ with $|S| > 1$. $\square$

Remark 2.6. In his original theorem, Long [4] states the result of Theorem 2.5(ii) as $|\mathrm{CS}(2,\mathbb{N}_n)| = f(n)$ such that $\{\mathbb{N}_n\} = \{\{0\},\mathbb{N}_n\} \in \mathrm{CS}(2,\mathbb{N}_n)$. However, the strict form given above is more suitable for generalization as shown below.

3. Essential complementing k-tuples and a structure theorem

Definition 3.1. Let $S \in \mathrm{UCS}(v,\mathbb{N}_n)$, $v = 1,2,\ldots,\Omega(n)$. Then any $T \in \mathrm{CS}(k,\mathbb{N}_n)$ $(1 < k \leq v)$ will be called essential if it is induced by the partition p of $\{1,2,\ldots,v\}$ into a complete set of residue classes, modulo k. p is also referred to as essential.

Denote the set of essential k-complementing systems of subsets for $\mathbb{N}_n$ by $\mathrm{ECS}(k,\mathbb{N}_n)$. Then $\mathrm{ECS}(k,\mathbb{N}_n) \neq \varnothing$ since $\{S_1,\ldots,S_k\} \in \mathrm{ECS}(k,\mathbb{N}_n)$. Thus $\mathrm{UCS}(\mathbb{N}_n) \subseteq \mathrm{ECS}(\mathbb{N}_n)$.

We have the following natural extension of Long's theorem.

Theorem 3.2. (i) $\{T_1,\ldots,T_k\} \in \mathrm{ECS}(k,\mathbb{N}_n)$ $(n \geq 2)$ *if and only if there exists a sequence* $g_1, g_2, \ldots, g_v$ *of integers corresponding to the factorization* $n = g_1 \cdots g_v$ $(k \leq v)$ *of* n *such that* *each set T_i consists of all finite sums of the form*

$$t_i = \sum_{j\geq 0} x_{kj+i} m_{kj+i}, \tag{3.1}$$

where $m_0 = 1$, $m_i = g_1 g_2 \cdots g_i$, *and* $0 \leq x_i < m_{i+1}$, $1 \leq i \leq k$.

(ii) $|\operatorname{ECS}(k, \mathbb{N}_n)| = \sum_{i=k}^{\Omega(n)} f(n,i)$, *where $f(n,k)$ is the number of ordered k-factorizations of n.*

Proof. (i) For each k and essential partition p of $\{1,\ldots,v \mid v \geq k\}$ there is an injective degeneration map

$$\operatorname{dgn}(p) : \{S \in \operatorname{UCS}(\mathbb{N}_n) \mid |S| \geq k\} \longrightarrow \operatorname{CS}(k, \mathbb{N}_n). \tag{3.2}$$

Indeed $\operatorname{dgn}(p)(S) \neq \operatorname{dgn}(p)(H) \Rightarrow \{T_i = \oplus S_r \mid r \equiv i(\operatorname{mod} k)\} \neq \{T_i = \oplus H_r \mid r \equiv i(\operatorname{mod} k)\} \Rightarrow S \neq H$. Hence $\operatorname{dgn}(p)$ is a well-defined mapping. The injectivity of $\operatorname{dgn}(p)$ is easily established by following the above implications backward (see also the statement immediately following (3.4) below).

The image of $\operatorname{dgn}(p)$ is clearly $\operatorname{ECS}(k,\mathbb{N}_n)$. We see that the restriction of $\operatorname{dgn}(p)$ to $\operatorname{UCS}(k,\mathbb{N}_n)$ is the identity map.

(ii) By part (i) and Remark 1.2(iii) we have

$$\begin{aligned} |\operatorname{ECS}(k,\mathbb{N}_n)| &= |\operatorname{UCS}(k,\mathbb{N}_n)| + |\operatorname{ECS}(k,\mathbb{N}_n) - \operatorname{UCS}(k,\mathbb{N}_n)| \\ &= f(n,k) + \sum_{i>k} f(n,i). \end{aligned} \tag{3.3}$$

□

We next define the class vector of a set partition [8].

Definitions 3.3. The class vector of a k-partition of $\{1,\ldots,v\}$ is the v-vector $(e_1,\ldots,e_v)$ in which $e_i \in \{1,\ldots,k\}$ and e_i belongs to class i for each i.

For example the partition $\{\{1,7\},\{2,3,5\},\{4,6\}\} \in \operatorname{SP}(7,3)$ is represented by the class vector $(1,2,2,3,2,3,1)$.

Thus if $(e_1,\ldots,e_w)$ represents $p \in \operatorname{NC}(w,k)$, then $e_i \neq e_{i+1}$ for all $i, 1 \leq i < w$; but if $w \leq v$ and $(e_1,\ldots,e_v)$ represents $q \in \operatorname{SP}(v,k) - \operatorname{NC}(w,k)$, then $e_i = e_{i+1} = \cdots = e_{i+c}$ $(0 < c < v)$ for some i.

Thus for $1 \leq k \leq w \leq v$, the contraction map

$$F : \operatorname{SP}(v,k) \longrightarrow \operatorname{NC}(w,k) \tag{3.4}$$

can be defined by setting $F(q) = q$ if $q \in \operatorname{NC}(w,k)$ and $F(q) = p$ if p is represented by the class vector obtained from the class vector h_q of q by replacing every sequence of equal and consecutive components $e_i, e_{i+1}, \ldots, e_{i+c} \in h_q$ with the common value e_i.

It follows that the restriction of F to $\operatorname{NC}(w,k)$ is the identity map.

Since every essential partition p (represented by the class vector $\{1,2,\dots,k,1,2,\dots,k,1,2,\dots\}$) also belongs to $\mathrm{NC}(w,k)$, the last sentence implies that the map (3.2) is indeed injective.

Remark 3.4. Theorem 2.5 also follows from Theorem 3.2 by setting $k=2$ and noting that the degeneration map (3.2) is then surjective. To see this let $T \in \mathrm{CS}(2,\mathbb{N}_n)$. Then by Theorem 1.4 there exists $S=\{S_1,\dots,S_v\} \in \mathrm{UCS}(\mathbb{N}_n)$, $v \geq 2$, and a map $\mathrm{dgn}(q)$ such that $\mathrm{dgn}(q)(S,k)=T$. But Lemma 2.3 shows that $|\mathrm{NC}(w,2)|=1$ which, by (3.4), implies that $q=p$. Hence (3.2) is surjective.

We now turn to the problem of characterizing all complementing k-tuples for $\mathbb{N}_n$. First we observe that it is not possible to state a simple rule for all $T \in \mathrm{CS}(k,\mathbb{N}_n)$, $k>2$, as appeared in (3.1) since the sets in a general $p \in \mathrm{SP}(v,k)$ $(v>k)$ can be constituted quite arbitrarily.

Theorem 1.4 implies that every $T \in \mathrm{CS}(k,\mathbb{N}_n)$ is induced by some $p \in \mathrm{SP}(v,k)$, $k \leq v$. But operationally we need only $\mathrm{NC}(v,k)$, and not $\mathrm{SP}(v,k)$, to determine all of $\mathrm{CS}(k,\mathbb{N}_n)$, using (3.2), in view of the surjective contraction map (3.4) and the fact that Theorem 1.5(ii) enables the automatic coupling of consecutive simplices thus making all partitions in $\mathrm{SP}(v,k)-\mathrm{NC}(v,k)$ redundant. Hence the partitions in $\mathrm{NC}(v,k)$ effectively account for all $S \in \mathrm{CS}(k,\mathbb{N}_n)$, for each $v \geq k$.

Since $\mathrm{nc}(k,k)=1$ and the singleton $\mathrm{NC}(k,k)$ contains the essential partition, it follows from (3.2) that every map $\mathrm{dgn}(q)$ in which $q \in \mathrm{NC}(v,k)$ is not the essential partition is necessarily defined on the reduced domain $\{S \in \mathrm{UCS}(\mathbb{N}_n) \mid |S|>k\}$. Hence for each v $(1 \leq v \leq \Omega(n))$, there exist precisely $\mathrm{nc}(v,k)$ maps, $\mathrm{dgn}(p)$, with $p \in \mathrm{NC}(v,k)$; and so by Remark 1.2(iii), there is a total of $f(n,v)\,\mathrm{nc}(v,k)$ contributions to $\mathrm{CS}(k,\mathbb{N}_n)$. Thus $|\mathrm{CS}(k,\mathbb{N}_n)|$ may be found by summing $f(n,v)\,\mathrm{nc}(v,k)$ over v.

Hence we obtain the following structure theorem for $\mathrm{CS}(k,\mathbb{N}_n)$.

Theorem 3.5. (i) $\{T_1,T_2,\dots,T_k\} \in \mathrm{CS}(k,\mathbb{N}_n)$ $(n \geq 2)$ *if and only if there exists a sequence* $g_1,g_2,\dots,g_v$ *of integers corresponding to the factorization* $n=g_1\cdots g_v$ *of* n *such that each set* T_i *is given by all finite sums of the form*

$$t_i=\sum_{j\geq 0} x_j m_j, \quad j \in p_i \in p \in \mathrm{NC}(v,k), \quad v \geq k, \tag{3.5}$$

where $m_0=1$, $m_i=g_1g_2\cdots g_i$, $0 \leq x_i < m_{i+1}$, $p_i \subset \{1,2,\dots,v\}$, *and* $\mathrm{NC}(v,k)$ *is the set of all* k*-partitions of* $\{1,2,\dots,v\}$ *in which no member of each partition contains a pair of consecutive integers.*

(ii)

$$|\,\mathrm{CS}(k,\mathbb{N}_n)\,|=\sum_{v=k}^{\Omega(n)} f(n,v)\,\mathrm{nc}(v,k)=\sum_{v=k}^{\Omega(n)} f(n,v)s2(v-1,k-1), \tag{3.6}$$

where the second equality follows from (2.3) and $f(n,k)$ *denotes the number of ordered* k*-factorizations of* n.

Remark 3.6. (i) Theorem 3.2 follows from Theorem 3.5 by noting that the essential partition p is the unique member of $\mathrm{NC}(v,k)$ such that $\mathrm{dgn}(p)$ is defined on the maximal domain $\{S \in \mathrm{UCS}(\mathbb{N}_n) \mid |S| \geq k\}$ which forces $\mathrm{nc}(v,k) = 1$ for all $v \geq k$.

(ii) We observe that any fixed $S \in \mathrm{UCS}(k,\mathbb{N}_n)$ gives rise, via degenerations, to a total of $B(k)$ complementing systems $T \in \mathrm{CS}(\mathbb{N}_n)$. In particular, if n is a prime power, then a single Bell number counts the whole of $\mathrm{CS}(\mathbb{N}_n)$ in view of Theorem 1.8, that is,

$$|\,\mathrm{CS}(\mathbb{N}_n)\,| = B(r), \quad n = p^r. \tag{3.7}$$

(iii) Furthermore, $n = p^r$ has a unique ordered prime factorization, which implies that $\mathrm{PCS}(\mathbb{N}_n) = \{S = \{[1,p],[p,p],\ldots,[p^{r-1},p]\}\}$. By Theorem 1.5(ii) each $T \in \mathrm{UCS}(k, \mathbb{N}_n)$ is a degeneration of S induced by a partition of the set $\{1,\ldots,r\}$ into subsets of consecutive integers. For each k $(1 \leq k \leq r)$ this corresponds to the process of putting $k-1$ slashes into any of the $r-1$ possible spaces between r identical symbols. Thus $|\,\mathrm{UCS}(k,\mathbb{N}_n)|$ is counted by the number of k-compositions of r [1, page 55]. Hence

$$\begin{aligned} |\,\mathrm{UCS}(k,\mathbb{N}_n)\,| &= \binom{r-1}{k-1}, \quad n = p^r \ (r > 0), \\ \Longrightarrow |\,\mathrm{UCS}(\mathbb{N}_n)\,| &= 2^{r-1}. \end{aligned} \tag{3.8}$$

(iv) Thus the function $B(m) - 2^{m-1}$, $m = 1,2,\ldots$ [5] also counts the complementing systems of subsets for $\{0,1,\ldots,p^m-1\}$ in which at least one member is not a simplex or, equivalently, the partitions of the set $\{1,2,\ldots,m\}$ in which at least one class of each partition contains a pair of nonconsecutive integers.

(v) Formula (3.7) can be generalized by summing $|\,\mathrm{CS}(k,\mathbb{N}_n)|$ over k, $1 \leq k \leq \Omega(n)$, to give

$$|\,\mathrm{CS}(\mathbb{N}_n)\,| = \sum_{v=1}^{\Omega(n)} f(n,v)b^*(v) = \sum_{v=1}^{\Omega(n)} f(n,v)B(v-1), \tag{3.9}$$

where $b^*(v) = \sum_{k=1}^{\Omega(n)} \mathrm{nc}(v,k)$ and the second equality follows from (2.7).

(vi) Taking (3.9) in conjunction with (3.7) we have $B(r) = \sum_{v=1}^{r} f(n,v)B(v-1)$, where $n = p^r$; and since $f(n,v) = \binom{r-1}{v-1}$ from (3.8), we obtain the familiar recurrence for the Bell numbers:

$$B(r) = \sum_{v=1}^{r} \binom{r-1}{v-1} B(v-1) = \sum_{v=0}^{r-1} \binom{r-1}{v} B(v). \tag{3.10}$$

The distributions of $|\,\mathrm{CS}(\mathbb{N}_{32})|$ and $|\,\mathrm{CS}(\mathbb{N}_{60})|$ are provided as examples in Tables 3.1 and 3.2.

Table 3.1. Distribution of $|\mathrm{CS}(\mathbb{N}_{32})|$.

k	1	2	3	4	5	Sum
$\lvert \mathrm{UCS}(k,\mathbb{N}_{32}) \rvert$	1	4	6	4	1	16
$\lvert \mathrm{CS}(k,\mathbb{N}_{32}) - \mathrm{UCS}(k,\mathbb{N}_{32}) \rvert$	0	11	19	6	0	36
$\lvert \mathrm{CS}(k,\mathbb{N}_{32}) \rvert$	1	15	25	10	1	52

Table 3.2. Distribution of $|\mathrm{CS}(\mathbb{N}_{60})|$.

k	1	2	3	4	Sum
$\lvert \mathrm{UCS}(k,\mathbb{N}_{60}) \rvert$	1	10	21	12	44
$\lvert \mathrm{CS}(k,\mathbb{N}_{60}) - \mathrm{UCS}(k,\mathbb{N}_{60}) \rvert$	0	33	36	0	69
$\lvert \mathrm{CS}(k,\mathbb{N}_{60}) \rvert$	1	43	57	12	113

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