CLASSIFICATION THEOREM ON IRREDUCIBLE REPRESENTATIONS OF THE q-DEFORMED ALGEBRA $U_q'(\mathrm{so}_n)$

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Received 8 August 2004

The aim of this paper is to give a complete classification of irreducible finite-dimensional representations of the nonstandard q-deformation $U_q'(\mathrm{so}_n)$ (which does not coincide with the Drinfel'd-Jimbo quantum algebra $U_q(\mathrm{so}_n)$) of the universal enveloping algebra $U(\mathrm{so}_n(\mathbb{C}))$ of the Lie algebra $\mathrm{so}_n(\mathbb{C})$ when q is not a root of unity. These representations are exhausted by irreducible representations of the classical type and of the nonclassical type. The theorem on complete reducibility of finite-dimensional representations of $U_q'(\mathrm{so}_n)$ is proved.

1. Introduction

Quantum orthogonal groups, quantum Lorentz groups, and the corresponding quantum algebras are of special interest for modern mathematical physics. Jimbo [19] and Drinfel'd [3] defined q-deformations (quantum algebras) $U_q(g)$ for all simple complex Lie algebras g by means of Cartan subalgebras and root subspaces (see also [18, 20]). Reshetikhin et al. [32] defined quantum algebras $U_q(g)$ in terms of the quantum R-matrix satisfying the quantum Yang-Baxter equation. However, these approaches do not give a satisfactory presentation of the quantum algebra $U_q(\mathrm{so}_n)$ from a viewpoint of some problems in quantum physics and representation theory. When considering representations of the quantum algebras $U_q(\mathrm{so}_{n+1})$ and $U_q(\mathrm{so}_{n,1})$, we are interested in reducing them onto the quantum subalgebra $U_q(\mathrm{so}_n)$. This reduction would give an analogue of the Gel'fand-Tsetlin basis for these representations. However, the definitions of quantum algebras mentioned above do not allow the inclusions $U_q(\mathrm{so}_{n+1}) \supset U_q(\mathrm{so}_n)$ and $U_q(\mathrm{so}_{n,1}) \supset U_q(\mathrm{so}_n)$. To be able to exploit such reductions, we have to consider q-deformations of the Lie algebra $\mathrm{so}_{n+1}(\mathbb{C})$ defined in terms of the generators $I_{k,k-1} = E_{k,k-1} - E_{k-1,k}$ (where E_{is} is the matrix with entries $(E_{is})_{rt} = \delta_{ir}\delta_{st}$) rather than by means of Cartan subalgebras and root elements. To construct such deformations, we have to deform trilinear relations for elements $I_{k,k-1}$ instead of Serre's relations (used in the case of the standard quantized universal enveloping algebras). As a result, we obtain the associative algebra, which will be denoted as $U_q'(\mathrm{so}_n)$.

International Journal of Mathematics and Mathematical Sciences 2005:2 (2005) 225–262
DOI: 10.1155/IJMMS.2005.225

This q-deformation was first constructed in [8]. It permits one to construct the reductions of $U_q'(\mathrm{so}_{n,1})$ and $U_q'(\mathrm{so}_{n+1})$ onto $U_q'(\mathrm{so}_n)$. The q-deformed algebra $U_q'(\mathrm{so}_n)$ leads for $n=3$ to the q-deformed algebra $U_q'(\mathrm{so}_3)$ defined by Fairlie [4]. The cyclically symmetric algebra, similar to Fairlie's one, was also considered somewhat earlier by Odesskiĭ [31].

In the classical case, the imbedding $\mathrm{SO}(n)\subset\mathrm{SU}(n)$ (and its infinitesimal analogue) is of great importance for nuclear physics and in the theory of Riemannian symmetric spaces. It is well known that in the framework of quantum groups and Drinfel'd-Jimbo quantum algebras one cannot construct the corresponding embedding. The algebra $U_q'(\mathrm{so}_n)$ allows to define such an embedding [29], that is, it is possible to define the embedding $U_q'(\mathrm{so}_n)\subset U_q(\mathrm{sl}_n)$, where $U_q(\mathrm{sl}_n)$ is the Drinfel'd-Jimbo quantum algebra.

As a disadvantage of the algebra $U_q'(\mathrm{so}_n)$, we have to mention the difficulties with Hopf algebra structure. Nevertheless, $U_q'(\mathrm{so}_n)$ turns out to be a coideal in $U_q(\mathrm{sl}_n)$ (see [29]) and this fact allows us to consider tensor products of finite-dimensional irreducible representations of $U_q'(\mathrm{so}_n)$ for many interesting cases (see [13]).

The algebra $U_q'(\mathrm{so}_n)$ and its representations are interesting in many cases. Main directions of interest are the following:

(1) the theory of orthogonal polynomials and special functions (especially, the theory of q-orthogonal polynomials and basic hypergeometric functions); this direction is not well worked out; some ideas of such applications can be found in [23];
(2) the algebra $U_q'(\mathrm{so}_n)$ (especially its particular case $U_q'(\mathrm{so}_3)$) is related to the algebra of observables in $2+1$ quantum gravity on the Riemmanian surfaces (see papers [2, 5, 28]);
(3) a quantum analogue of the Riemannian symmetric space $\mathrm{SU}(n)/\mathrm{SO}(n)$ is constructed by means of the algebra $U_q'(\mathrm{so}_n)$; this construction is fulfilled in the paper [29] (see also [24]);
(4) a q-analogue of the theory of harmonic polynomials (q-harmonic polynomials on quantum vector space $\mathbb{R}_q^n$) is constructed by using the algebra $U_q'(\mathrm{so}_n)$; in particular, a q-analogue of different separations of variables for the q-Laplace operator on $\mathbb{R}_q^n$ is given by means of this algebra and its subalgebras; this theory is contained in the papers [17, 30];
(5) the algebra $U_q'(\mathrm{so}_n)$ also appears in the theory of links in the algebraic topology (see [1]);
(6) the algebra $U_q'(\mathrm{so}_n)$ is connected with Yangians (see [27] and references therein);
(7) a new quantum analogue of the Brauer algebra is connected with the algebra $U_q'(\mathrm{so}_n)$ (see [26]).

A large class of finite-dimensional irreducible representations of the algebra $U_q'(\mathrm{so}_n)$ was constructed in [8]. The formulas of action of the generators of $U_q'(\mathrm{so}_n)$ upon the basis (which is a q-analogue of the Gel'fand-Tsetlin basis) are given there. A proof of these formulas and some of their corrections were given in [6]. However, the finite-dimensional irreducible representations described in [6, 8] are representations of the classical type. They are q-deformations of the corresponding irreducible representations of the Lie algebra so_n, that is, at $q\to 1$ they turn into representations of so_n.

The algebra $U_q'(\mathrm{so}_n)$ has other classes of finite-dimensional irreducible representations, which have no classical analogue. These representations are singular at the limit $q \to 1$. They are described in [15]. The description of these representations for the algebra $U_q'(\mathrm{so}_3)$ is given in [9]. A classification of irreducible $*$-representations of real forms of the algebra $U_q'(\mathrm{so}_3)$ is given in [33]. The representation theory of $U_q'(\mathrm{so}_n)$ when q is a root of unity is studied in [16].

In this paper, we deal with classification of finite-dimensional irreducible representations of the algebra $U_q'(\mathrm{so}_n)$ when q is not a root of unity. As mentioned above, there were constructed irreducible representations of the algebra $U_q'(\mathrm{so}_n)$ belonging to the classical and to the nonclassical types. However, it was not known that these representations exhaust all irreducible finite-dimensional representations. We started to study this problem in [22]. We show there that these representations are determined by the so-called highest weights (which were defined in [22] and differ from highest weights in the theory of quantized universal enveloping algebras). However, we do not know a correspondence between known representations of the classical and nonclassical types and highest weights. In the present paper, we develop an approach to the problem of classification from other points of view. Namely, we prove that each irreducible finite-dimensional representation of $U_q'(\mathrm{so}_n)$ belongs to the set of representations of the classical type or to the set of representations of the nonclassical type constructed before. For proving this, we use our previous results on structure of the algebra $U_q'(\mathrm{so}_n)$ (tensor operators, Wigner-Eckart theorem, etc.). We also need the theorem on complete reducibility of finite-dimensional representations of $U_q'(\mathrm{so}_n)$. This theorem is proved in this paper. Some ideas from the theory of representations of the Lie algebra $\mathrm{so}_n(\mathbb{C})$ and its real forms are also used.

Note that the problem of classification of irreducible finite-dimensional representations of $U_q'(\mathrm{so}_n)$ is much more complicated than in the case of Drinfel'd-Jimbo quantum algebras since in $U_q'(\mathrm{so}_n)$ we do not have an analogue of a Cartan subalgebra and root elements. The set of all irreducible finite-dimensional representations of $U_q'(\mathrm{so}_n)$ is wider than in the case of $U_q(\mathrm{so}_n)$.

Everywhere below we assume that q is not a root of unity.

2. The q-deformed algebra $U_q'(\mathrm{so}_n)$

The universal enveloping algebra $U(\mathrm{so}_n(\mathbb{C}))$ is generated by the elements $I_{ij} = E_{ij} - E_{ji}$, $i > j$. But in order to generate the algebra $U(\mathrm{so}_n(\mathbb{C}))$, it is enough to take only the elements $I_{21}, I_{32}, \dots, I_{n,n-1}$. It is a minimal set of elements necessary for generating $U(\mathrm{so}_n(\mathbb{C}))$. These elements satisfy the relations

$$\begin{gathered} I_{i,i-1}^2 I_{i+1,i} - 2I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I_{i,i-1}^2 = -I_{i+1,i}, \\ I_{i,i-1}I_{i+1,i}^2 - 2I_{i+1,i}I_{i,i-1}I_{i+1,i} + I_{i+1,i}^2 I_{i,i-1} = -I_{i,i-1}, \\ I_{i,i-1}I_{j,j-1} - I_{j,j-1}I_{i,i-1} = 0 \quad \text{for } |i-j| > 1. \end{gathered} \tag{2.1}$$

The following theorem is true for $U(\mathrm{so}_n(\mathbb{C}))$ (see [21]): *the enveloping algebra $U(\mathrm{so}_n(\mathbb{C}))$ is isomorphic to the complex associative algebra (with a unit element) generated by the elements $I_{21}, I_{32}, \dots, I_{n,n-1}$ satisfying the above relations.*

We make a q-deformation of these relations by fulfilling the deformation of the integer 2 as $2 \to [2]_q := (q^2 - q^{-2})/(q - q^{-1}) = q + q^{-1}$. As a result, we obtain the relations

$$I_{i,i-1}^2 I_{i+1,i} - (q+q^{-1}) I_{i,i-1} I_{i+1,i} I_{i,i-1} + I_{i+1,i} I_{i,i-1}^2 = -I_{i+1,i}, \tag{2.2}$$

$$I_{i,i-1} I_{i+1,i}^2 - (q+q^{-1}) I_{i+1,i} I_{i,i-1} I_{i+1,i} + I_{i+1,i}^2 I_{i,i-1} = -I_{i,i-1}, \tag{2.3}$$

$$I_{i,i-1} I_{j,j-1} - I_{j,j-1} I_{i,i-1} = 0 \quad \text{for } |i-j| > 1. \tag{2.4}$$

The q-deformed algebra $U'_q(\mathrm{so}_n)$ is defined as the complex unital (i.e., with a unit element) associative algebra generated by elements $I_{21}, I_{32}, \ldots, I_{n,n-1}$ satisfying relations (2.2), (2.3), and (2.4). It is a q-deformation of the universal enveloping algebra $U(\mathrm{so}_n(\mathbb{C}))$, different from the Drinfel'd-Jimbo quantized universal enveloping algebra $U_q(\mathrm{so}_n)$. For this algebra, the inclusions $U'_q(\mathrm{so}_n) \supset U'_q(\mathrm{so}_{n-1})$ and $U_q(\mathrm{sl}_n) \supset U'_q(\mathrm{so}_n)$ are constructed, where $U_q(\mathrm{sl}_n)$ is the well-known Drinfel'd-Jimbo quantum algebra (see the introduction).

An analogue of the skew-symmetric matrices $I_{ij} = E_{ij} - E_{ji}$, $i > j$, constituting a basis of the Lie algebra $\mathrm{so}_n(\mathbb{C})$, can be introduced into $U'_q(\mathrm{so}_n)$ (see [7, 30]). For $k > l+1$, they are defined recursively by the formulas

$$I_{kl} := [I_{l+1,l}, I_{k,l+1}]_q \equiv q^{1/2} I_{l+1,l} I_{k,l+1} - q^{-1/2} I_{k,l+1} I_{l+1,l}. \tag{2.5}$$

The elements I_{kl}, $k > l$, satisfy the commutation relations

$$[I_{lr}, I_{kl}]_q = I_{kr}, \quad [I_{kl}, I_{kr}]_q = I_{lr}, \quad [I_{kr}, I_{lr}]_q = I_{kl} \quad \text{for } k > l > r, \tag{2.6}$$

$$[I_{kl}, I_{sr}] = 0 \quad \text{for } k > l > s > r,\ k > s > r > l, \tag{2.7}$$

$$[I_{kl}, I_{sr}]_q = (q - q^{-1})(I_{lr} I_{ks} - I_{kr} I_{sl}) \quad \text{for } k > s > l > r. \tag{2.8}$$

For $q = 1$, they coincide with the corresponding commutation relations for the Lie algebra $\mathrm{so}_n(\mathbb{C})$.

The algebra $U'_q(\mathrm{so}_n)$ can be also defined as a unital associative algebra generated by I_{kl}, $1 \le l < k \le n$, satisfying the relations (2.6), (2.7), and (2.8). In fact, the relations (2.6), (2.7), and (2.8) can be reduced to the relations (2.2), (2.3), and (2.4) for $I_{21}, I_{32}, \ldots, I_{n,n-1}$.

The Poincaré-Birkhoff-Witt theorem for the algebra $U'_q(\mathrm{so}_n)$ can be formulated as follows (a proof of this theorem is given in [16]): *the elements*

$$I_{21}{}^{m_{21}} I_{31}{}^{m_{31}} \cdots I_{n1}{}^{m_{n1}} I_{32}{}^{m_{32}} I_{42}{}^{m_{42}} \cdots I_{n2}{}^{m_{n2}} \cdots I_{n,n-1}{}^{m_{n,n-1}}, \quad m_{ij} = 0, 1, 2, \ldots, \tag{2.9}$$

form a basis of the algebra $U'_q(\mathrm{so}_n)$.

In $U'_q(\mathrm{so}_n)$, the commutative subalgebra $\mathcal{A}$ generated by the elements $I_{21}, I_{43}, I_{65}, \ldots, I_{n-1,n-2}$ (or $I_{n,n-1}$) can be separated. So, this subalgebra is generated by $\lfloor n/2 \rfloor$ elements, where $\lfloor n/2 \rfloor$ is an integral part of the number $n/2$. However, there exist no root elements in the algebra $U'_q(\mathrm{so}_n)$ with respect to this commutative subalgebra. This leads to the fact that properties of $U'_q(\mathrm{so}_n)$ are not similar to those of the Drinfel'd-Jimbo algebra $U_q(\mathrm{so}_n)$.

3. Irreducible representations of the classical and nonclassical types

In this section, we give known facts on irreducible representations of $U_q'(\mathrm{so}_n)$, which will be used below. The corresponding references are given in the introduction.

Two types of irreducible finite-dimensional representations are known for $U_q'(\mathrm{so}_n)$:

(a) representations of the classical type;
(b) representations of the nonclassical type.

Known irreducible representations of the classical type are q-deformations of the irreducible finite-dimensional representations of the Lie algebra so_n. There is a one-to-one correspondence between these irreducible representations of the algebra $U_q'(\mathrm{so}_n)$ and irreducible finite-dimensional representations of the Lie algebra so_n. Moreover, formulas for representations of the classical type of $U_q'(\mathrm{so}_n)$ turn into the corresponding formulas for the representations of Lie algebra so_n at $q \to 1$.

There exists no classical analogue for representations of the nonclassical type: representation operators $T(a)$, $a \in U_q'(\mathrm{so}_n)$, have singularities at $q = 1$.

We describe known irreducible finite-dimensional representations of the algebras $U_q'(\mathrm{so}_n)$, $n \geq 3$, which belong to the classical type. As in the classical case, they are given by sets $\mathbf{m}_n$ of $\lfloor n/2 \rfloor$ numbers $m_{1,n}, m_{2,n}, \dots, m_{\lfloor n/2 \rfloor,n}$ (here $\lfloor n/2 \rfloor$ denotes the integral part of $n/2$), which are all integral or all half-integral and satisfy the dominance conditions

$$\begin{gathered} m_{1,2k+1} \geq m_{2,2k+1} \geq \cdots \geq m_{k,2k+1} \geq 0, \\ m_{1,2k} \geq m_{2,2k} \geq \cdots \geq m_{k-1,2k} \geq |m_{k,2k}| \end{gathered} \tag{3.1}$$

for $n = 2k+1$ and $n = 2k$, respectively. These representations are denoted by $T_{\mathbf{m}_n}$. We take a q-analogue of the Gel'fand-Tsetlin basis in the representation space, which is obtained by successive reduction of the representation $T_{\mathbf{m}_n}$ to the subalgebras $U_q'(\mathrm{so}_{n-1}), U_q'(\mathrm{so}_{n-2}), \dots, U_q'(\mathrm{so}_3)$, $U_q'(\mathrm{so}_2) := U(\mathrm{so}_2)$. As in the classical case, its elements are labelled by the Gel'fand-Tsetlin tableaux

$$\{\alpha_n\} \equiv \begin{Bmatrix} \mathbf{m}_n \\ \mathbf{m}_{n-1} \\ \vdots \\ \mathbf{m}_2 \end{Bmatrix} \equiv \{\mathbf{m}_n, \alpha_{n-1}\} \equiv \{\mathbf{m}_n, \mathbf{m}_{n-1}, \alpha_{n-2}\}, \tag{3.2}$$

where, as in the nondeformed case, the components of $\mathbf{m}_s$ and $\mathbf{m}_{s-1}$ satisfy the "betweenness" conditions

$$\begin{gathered} m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq \cdots \geq m_{p,2p+1} \geq m_{p,2p} \geq -m_{p,2p+1}, \\ m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq \cdots \geq m_{p-1,2p-1} \geq |m_{p,2p}|. \end{gathered} \tag{3.3}$$

Sometimes, the basis elements, defined by a tableau $\{\alpha_n\}$, are denoted as $|\alpha_{n-1}\rangle$ or as $|\mathbf{m}_{n-1}, \alpha_{n-2}\rangle$, that is, we will omit the first row $\mathbf{m}_n$ in a tableau.

It is convenient to introduce the so-called l-coordinates

$$l_{j,2p+1} = m_{j,2p+1} + p - j + 1, \qquad l_{j,2p} = m_{j,2p} + p - j, \tag{3.4}$$

for the numbers $m_{i,k}$. The operator $T_{\mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ acts upon Gel'fand-Tsetlin basis elements, labelled by (3.2), as

$$T_{\mathbf{m}_n}(I_{2p+1,2p})\,|\,\alpha_n\rangle = \sum_{j=1}^{p} \frac{A^j_{2p}(\alpha_n)}{a(l_{j,2p})}\,\big|\,(\alpha_n)^{+j}_{2p}\big\rangle - \sum_{j=1}^{p} \frac{A^j_{2p}\big((\alpha_n)^{-j}_{2p}\big)}{a(l_{j,2p}-1)}\,\big|\,(\alpha_n)^{-j}_{2p}\big\rangle \tag{3.5}$$

and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ acts as

$$\begin{aligned} T_{\mathbf{m}_n}(I_{2p,2p-1})\,|\,\alpha_n\rangle = {} & \sum_{j=1}^{p-1} \frac{B^j_{2p-1}(\alpha_n)}{b(l_{j,2p-1})[l_{j,2p-1}]}\,\big|\,(\alpha_n)^{+j}_{2p-1}\big\rangle \\ & - \sum_{j=1}^{p-1} \frac{B^j_{2p-1}\big((\alpha_n)^{-j}_{2p-1}\big)}{b(l_{j,2p-1}-1)[l_{j,2p-1}-1]}\,\big|\,(\alpha_n)^{-j}_{2p-1}\big\rangle + \mathrm{i}C_{2p-1}(\alpha_n)\,|\,\alpha_n\rangle. \end{aligned} \tag{3.6}$$

In these formulas, $(\alpha_n)^{\pm j}_s$ means the tableau (3.2) in which jth component $m_{j,s}$ in $\mathbf{m}_s$ is replaced by $m_{j,s}\pm 1$, respectively. The coefficients A^j_{2p}, B^j_{2p-1}, C_{2p-1}, a and b in (3.5) and (3.6) are given by the expressions

$$\begin{aligned} & A^j_{2p}(\alpha_n) \\ & \quad = \left(\frac{\prod_{i=1}^{p}[l_{i,2p+1}+l_{j,2p}][l_{i,2p+1}-l_{j,2p}-1]\prod_{i=1}^{p-1}[l_{i,2p-1}+l_{j,2p}][l_{i,2p-1}-l_{j,2p}-1]}{\prod_{i\neq j}^{p}[l_{i,2p}+l_{j,2p}][l_{i,2p}-l_{j,2p}][l_{i,2p}+l_{j,2p}+1][l_{i,2p}-l_{j,2p}-1]}\right)^{1/2}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} & B^j_{2p-1}(\alpha_n) \\ & \quad = \left(\frac{\prod_{i=1}^{p}[l_{i,2p}+l_{j,2p-1}][l_{i,2p}-l_{j,2p-1}]\prod_{i=1}^{p-1}[l_{i,2p-2}+l_{j,2p-1}][l_{i,2p-2}-l_{j,2p-1}]}{\prod_{i\neq j}^{p-1}[l_{i,2p-1}+l_{j,2p-1}][l_{i,2p-1}-l_{j,2p-1}][l_{i,2p-1}+l_{j,2p-1}-1][l_{i,2p-1}-l_{j,2p-1}-1]}\right)^{1/2}, \end{aligned} \tag{3.8}$$

$$C_{2p-1}(\alpha_n) = \frac{\prod_{s=1}^{p}[l_{s,2p}]\prod_{s=1}^{p-1}[l_{s,2p-2}]}{\prod_{s=1}^{p-1}[l_{s,2p-1}][l_{s,2p-1}-1]}, \tag{3.9}$$

$$\begin{aligned} a(l_{j,2p}) &= \big\{\big(q^{l_{j,2p}+1}+q^{-l_{j,2p}-1}\big)\big(q^{l_{j,2p}}+q^{-l_{j,2p}}\big)\big\}^{1/2}, \\ b(l_{j,2p-1}) &= \big([2l_{j,2p-1}+1][2l_{j,2p-1}-1]\big)^{1/2}. \end{aligned} \tag{3.10}$$

The numbers in square brackets in formulas (3.6), (3.7), (3.8), and (3.9) mean q-numbers defined by

$$[a] \equiv [a]_q := \frac{q^a - q^{-a}}{q - q^{-1}}. \tag{3.11}$$

It is seen from formula (3.9) that the coefficient C_{2p-1} vanishes if $m_{p,2p} \equiv l_{p,2p} = 0$.

The following assertion is well known [8]: *the representations $T_{\mathbf{m}_n}$ are irreducible. The representations $T_{\mathbf{m}_n}$ and $T_{\mathbf{m}'_n}$ are pairwise nonequivalent for $\mathbf{m}_n \neq \mathbf{m}'_n$.*

Irreducible finite-dimensional representations of the nonclassical type are given by sets $\epsilon := (\epsilon_2, \epsilon_3, \dots, \epsilon_n)$, $\epsilon_i = \pm 1$, and by sets $\mathbf{m}_n$ consisting of $\lfloor n/2 \rfloor$ *half-integral* (but not integral) numbers $m_{1,n}, m_{2,n}, \dots, m_{\lfloor n/2 \rfloor,n}$ that satisfy the dominance conditions

$$m_{1,n} \geq m_{2,n} \geq \cdots \geq m_{\lfloor n/2 \rfloor,n} \geq \frac{1}{2}. \tag{3.12}$$

These representations are denoted by $T_{\epsilon,\mathbf{m}_n}$.

For a basis in the representation space, we use an analogue of the basis of the previous case. Its elements are labelled by tableaux (3.2), where the components of $\mathbf{m}_s$ and $\mathbf{m}_{s-1}$ satisfy the "betweenness" conditions

$$\begin{gathered} m_{1,2p+1} \geq m_{1,2p} \geq m_{2,2p+1} \geq m_{2,2p} \geq \cdots \geq m_{p,2p+1} \geq m_{p,2p} \geq \frac{1}{2}, \\ m_{1,2p} \geq m_{1,2p-1} \geq m_{2,2p} \geq m_{2,2p-1} \geq \cdots \geq m_{p-1,2p-1} \geq m_{p,2p}. \end{gathered} \tag{3.13}$$

The corresponding basis elements are denoted by the same symbols as in the previous case. The l-coordinates for $m_{j,s}$ are introduced by the same formulas as before.

The operator $T_{\epsilon,\mathbf{m}_n}(I_{2p+1,2p})$ of the representation $T_{\epsilon,\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ acts upon the basis elements $|\alpha_n\rangle$ by the formulas

$$\begin{aligned} T_{\epsilon,\mathbf{m}_n}(I_{2p+1,2p})\,|\alpha_n\rangle = {} & \delta_{m_{p,2p},1/2}\frac{\epsilon_{2p+1}}{q^{1/2}-q^{-1/2}}D_{2p}(\alpha_n)\,|\alpha_n\rangle \\ & + \sum_{j=1}^{p}\frac{A^j_{2p}(\alpha_n)}{a'(l_{j,2p})}\,\big|(\alpha_n)^{+j}_{2p}\big\rangle - \sum_{j=1}^{p}\frac{A^j_{2p}\big((\alpha_n)^{-j}_{2p}\big)}{a'(l_{j,2p}-1)}\,\big|(\alpha_n)^{-j}_{2p}\big\rangle, \end{aligned} \tag{3.14}$$

where the summation in the last sum must be from 1 to $p-1$ if $m_{p,2p} = 1/2$, and the operator $T_{\mathbf{m}_n}(I_{2p,2p-1})$ acts as

$$\begin{aligned} T_{\epsilon,\mathbf{m}_n}(I_{2p,2p-1})\,|\alpha_n\rangle = {} & \sum_{j=1}^{p-1}\frac{B^j_{2p-1}(\alpha_n)}{b(l_{j,2p-1})[l_{j,2p-1}]_+}\,\big|(\alpha_n)^{+j}_{2p-1}\big\rangle \\ & - \sum_{j=1}^{p-1}\frac{B^j_{2p-1}\big((\alpha_n)^{-j}_{2p-1}\big)}{b(l_{j,2p-1}-1)[l_{j,2p-1}-1]_+}\,\big|(\alpha_n)^{-j}_{2p-1}\big\rangle + \epsilon_{2p}\hat{C}_{2p-1}(\alpha_n)\,|\alpha_n\rangle, \end{aligned} \tag{3.15}$$

where

$$[a]_+ = \frac{q^a+q^{-a}}{q-q^{-1}}. \tag{3.16}$$

As before, $(\alpha_n)_s^{\pm j}$ means the tableau (3.2) in which jth component $m_{j,s}$ in $\mathbf{m}_s$ is replaced by $m_{j,s} \pm 1$, respectively. The expressions for A_{2p}^j, B_{2p-1}^j, and b are given by the same formulas as in (3.5) and (3.6),

$$\begin{aligned} a'(l_{j,2p}) &= \{(q^{l_{j,2p}+1} - q^{-l_{j,2p}-1})(q^{l_{j,2p}} - q^{-l_{j,2p}})\}^{1/2}, \\ \widehat{C}_{2p-1}(\alpha_n) &= \frac{\prod_{s=1}^{p}[l_{s,2p}]_+ \prod_{s=1}^{p-1}[l_{s,2p-2}]_+}{\prod_{s=1}^{p-1}[l_{s,2p-1}]_+[l_{s,2p-1}-1]_+}, \\ D_{2p}(\alpha_n) &= \frac{\prod_{i=1}^{p}[l_{i,2p+1}-1/2]\prod_{i=1}^{p-1}[l_{i,2p-1}-1/2]}{\prod_{i=1}^{p-1}[l_{i,2p}+1/2][l_{i,2p}-1/2]}. \end{aligned} \tag{3.17}$$

The following assertion is true (see [15]): *the representations $T_{\epsilon,\mathbf{m}_n}$ are irreducible. The representations $T_{\epsilon,\mathbf{m}_n}$ and $T_{\epsilon',\mathbf{m}'_n}$ are pairwise nonequivalent for $(\epsilon,\mathbf{m}_n) \neq (\epsilon',\mathbf{m}'_n)$. For any admissible $(\epsilon,\mathbf{m}_n)$ and $\mathbf{m}'_n$, the representations $T_{\epsilon,\mathbf{m}_n}$ and $T_{\mathbf{m}'_n}$ are pairwise nonequivalent.*

Remark 3.1. As in the case of irreducible representations of the Lie algebra so_n, it follows from the explicit description of irreducible representations $T_{\mathbf{m}_n}$ and $T_{\epsilon,\mathbf{m}_n}$ of $U'_q(\mathrm{so}_n)$ that the restriction of $T_{\mathbf{m}_n}$ onto the subalgebra $U'_q(\mathrm{so}_{n-1})$ decomposes into a direct sum of irreducible representations of this subalgebra belonging to the classical type, and the restriction of $T_{\epsilon,\mathbf{m}_n}$ onto $U'_q(\mathrm{so}_{n-1})$ decomposes into a direct sum of irreducible representations belonging to the nonclassical type. Formulas for the representations determine explicitly these decompositions.

4. Vector operators and Wigner-Eckart theorem

In this section, we define vector operators for irreducible representations of $U'_q(\mathrm{so}_n)$ and give the Wigner-Eckart theorem for them. This information will be used under proving our main results.

The algebra $U'_q(\mathrm{so}_n)$ is not a Hopf algebra. For this reason, we cannot define a tensor product of its representations. However, $U'_q(\mathrm{so}_n)$ can be embedded into the Hopf algebra $U_q(\mathrm{sl}_n)$ (see [29, 30]). Using this embedding, a tensor product of the irreducible representations T_1 and T of $U'_q(\mathrm{so}_n)$ is determined, where T_1 is a vector representation (i.e., a representation of the classical type characterized by the numbers $(1,0,\ldots,0)$) and T is an arbitrary irreducible finite-dimensional representation [13]. The decomposition of this tensor product into irreducible constituents is given by the formulas as in the classical case if the representation T belongs to the classical type (i.e., the decomposition of $T_1 \otimes T_{\mathbf{m}_n}$ contains the irreducible representations of the classical type characterized by $\mathbf{m}_n^{+j}$, $\mathbf{m}_n^{-j}$, $j = 1,2,\ldots,\lfloor n/2 \rfloor$, and also the representation $T_{\mathbf{m}_n}$ if $n = 2k+1$ and $m_{k,2k+1} \neq 0$). For the representations $T = T_{\epsilon,\mathbf{m}_n}$ of the nonclassical type, we have

$$T_1 \otimes T_{\epsilon,\mathbf{m}_n} = \bigoplus_{\mathbf{m}'_n \in S_\epsilon(\mathbf{m}_n)} T_{\epsilon,\mathbf{m}'_n}, \tag{4.1}$$

where

$$S_\epsilon(\mathbf{m}_{2p+1}) = \bigcup_{j=1}^{p} \left\{T_{\epsilon,\mathbf{m}_{2p+1}^{+j}}\right\} \cup \bigcup_{j=1}^{p} \left\{T_{\epsilon,\mathbf{m}_{2p+1}^{-j}}\right\} \cup \left\{T_{\epsilon,\mathbf{m}_{2p+1}}\right\},$$
$$S_\epsilon(\mathbf{m}_{2p}) = \bigcup_{j=1}^{p} \left\{T_{\epsilon,\mathbf{m}_{2p}^{+j}}\right\} \cup \bigcup_{j=1}^{p} \left\{T_{\epsilon,\mathbf{m}_{2p}^{-j}}\right\}. \tag{4.2}$$

As before, $\mathbf{m}_n^{\pm j}$ is the set of numbers $\mathbf{m}_n$ with m_{jn} replaced by $m_{jn} \pm 1$, respectively. Note that each representation $T_{\mathbf{m}_n'}$ and each representation $T_{\epsilon,\mathbf{m}_n'}$ for which $\mathbf{m}_n'$ does not satisfy the dominance conditions must be omitted. Proofs of these decompositions can be found in [14]. As in the case of quantized universal enveloping algebras (see [20, Chapter 7]), decompositions of the above tensor products are fulfilled by means of matrices whose entries are called *Clebsch-Gordan coefficients*.

We define a vector operator (it is a set of n operators), which transforms under the vector representation of the algebra $U_q'(\mathrm{so}_n)$. This operator acts on a linear space $\mathcal{H}$ on which some representation T of $U_q'(\mathrm{so}_n)$ acts. We will consider only the case when $\mathcal{H}$ is a finite-dimensional space. We also suppose that $\mathcal{H}$ decomposes into a direct sum of irreducible invariant (with respect to $U_q'(\mathrm{so}_n)$) subspaces, where only irreducible representations of the classical type or only irreducible representations of the nonclassical type are realized. This assumption is explained by the fact that a vector operator cannot map a subspace on which an irreducible representation of the classical type is realized into a subspace on which a representation of the nonclassical type is realized, or vise versa.

The set A_r, $r = 1,2,\dots,n$, of operators on $\mathcal{H}$ is called a *vector operator* for the algebra $U_q'(\mathrm{so}_n)$ if

$$\left[A_{j-1}, T(I_{j,j-1})\right]_q = A_j, \qquad \left[T(I_{j,j-1}), A_j\right]_q = A_{j-1},$$
$$\left[T(I_{j,j-1}), A_k\right]_q = 0, \quad k \neq j, j-1, \tag{4.3}$$

where $[X,Y]_q \equiv q^{1/2}XY - q^{-1/2}YX$ and T is a fixed representation of $U_q'(\mathrm{so}_n)$ acting on $\mathcal{H}$.

We represent the space $\mathcal{H}$ as a direct sum of irreducible invariant (with respect to $U_q'(\mathrm{so}_n)$) subspaces

$$\mathcal{H} = \bigoplus_{\epsilon,\mathbf{m}_n,i} \mathcal{V}_{\epsilon,\mathbf{m}_n,i}, \tag{4.4}$$

where $\mathcal{V}_{\epsilon,\mathbf{m}_n,i}$ is a subspace, on which an irreducible representation of $U_q'(\mathrm{so}_n)$ characterized by ϵ and $\mathbf{m}_n$ is realized, and i separates multiple irreducible representations of $U_q'(\mathrm{so}_n)$ in the decomposition. If irreducible representations belong to the classical type, then ϵ must be omitted.

We take a Gel'fand-Tsetlin basis in each subspace $\mathcal{V}_{\epsilon,\mathbf{m}_n,i}$ and denote these basis vectors by $|\epsilon,\mathbf{m}_n,i,\alpha\rangle$, where $\alpha \equiv \alpha_{n-1}$ are the corresponding Gel'fand-Tsetlin tableaux. Then the subspaces

$$\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_n} = \bigoplus_i \mathbb{C}\,|\,\epsilon,\mathbf{m}_n,i,\alpha\rangle \tag{4.5}$$

can be defined.

The Wigner-Eckart theorem for vector operators $\{A_j\}$ (proved in [14]) states that the matrix elements of A_j are of the form

$$\langle\epsilon',\mathbf{m}'_n,i',\alpha'\,|\,A_j\,|\,\epsilon,\mathbf{m}_n,i,\alpha\rangle = C^{\epsilon',\mathbf{m}'_n,\alpha'}_{j;\epsilon,\mathbf{m}_n,\alpha}\langle\epsilon',\mathbf{m}'_n,i'\,\|A\|\,\epsilon,\mathbf{m}_n,i\rangle, \tag{4.6}$$

where $C^{\epsilon',\mathbf{m}'_{n-1},\alpha'}_{j;\epsilon,\mathbf{m}_{n-1},\alpha}$ are Clebsch-Gordan coefficients of the tensor product $T_1 \otimes T_{\epsilon,\mathbf{m}_n}$ (these coefficients are given in an explicit form in [14]), and $\langle\epsilon',\mathbf{m}'_{n-1},i'\,|A|\,\epsilon,\mathbf{m}_{n-1},i\rangle$ are called *reduced matrix elements* of the vector operator $\{A_j\}$. These reduced matrix elements depend only on numbers characterizing the representations and on the indices separating multiple representations, and are independent of basis elements of irreducible invariant subspaces. They are also independent of the number j of the operator A_j. In the above formulas, ϵ must be omitted if we deal only with representations of the classical type.

Due to the formulas for decompositions of the tensor products $T_1 \otimes T_{\mathbf{m}_n}$ and $T_1 \otimes T_{\epsilon,\mathbf{m}_n}$, we find that matrix elements $\langle\epsilon',\mathbf{m}'_n,i',\alpha'\,|A_j|\,\epsilon,\mathbf{m}_n,i,\alpha\rangle$ can be nonvanishing only if $\epsilon' = \epsilon$ and also $\mathbf{m}'_n = \mathbf{m}_n^{\pm s}$ or $\mathbf{m}'_n = \mathbf{m}_n$ (since only for these cases, the corresponding Clebsch-Gordan coefficients can be nonvanishing). Due to the above formulas for decompositions of tensor products of representations, a vector operator cannot map a subspace of an irreducible representation of the classical type (of the nonclassical type) into subspaces on which irreducible representations of the nonclassical type (of the classical type) are realized. Therefore, in matrix elements (4.6), both indices ϵ and ϵ' exist or both are absent.

We can define the operators

$$A^{\mathbf{m}_n}_{\mathbf{m}_n} : \mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_n} \longrightarrow \mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_n}, \quad A^{\mathbf{m}_n^{+j}}_{\mathbf{m}_n} : \mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_n} \longrightarrow \mathcal{V}^{\alpha'}_{\epsilon,\mathbf{m}_n^{+j}}, \quad A^{\mathbf{m}_n^{-j}}_{\mathbf{m}_n} : \mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_n} \longrightarrow \mathcal{V}^{\alpha'}_{\epsilon,\mathbf{m}_n^{-j}}, \tag{4.7}$$

which have matrix elements coinciding with reduced matrix elements of the tensor operator $\{A_j\}$:

$$\begin{aligned}
\langle\epsilon,\mathbf{m}_n,i',\alpha\,|\,A^{\mathbf{m}_n}_{\mathbf{m}_n}\,|\,\epsilon,\mathbf{m}_n,i,\alpha\rangle &= \langle\epsilon,\mathbf{m}_n,i'\,\|A\|\,\epsilon,\mathbf{m}_n,i\rangle,\\
\langle\epsilon,\mathbf{m}_n^{+j},i',\alpha'\,|\,A^{\mathbf{m}_n^{+j}}_{\mathbf{m}_n}\,|\,\epsilon,\mathbf{m}_n,i,\alpha\rangle &= \langle\epsilon,\mathbf{m}_n^{+j},i'\,\|A\|\,\epsilon,\mathbf{m}_n,i\rangle,\\
\langle\epsilon,\mathbf{m}_n^{-j},i',\alpha'\,|\,A^{\mathbf{m}_n^{-j}}_{\mathbf{m}_n}\,|\,\epsilon,\mathbf{m}_n,i,\alpha\rangle &= \langle\epsilon,\mathbf{m}_n^{-j},i'\,\|A\|\,\epsilon,\mathbf{m}_n,i\rangle.
\end{aligned} \tag{4.8}$$

(The symbol ϵ must be omitted in these formulas if necessary.) It follows from the Wigner-Eckart theorem that for any irreducible representation $T_{\epsilon,\mathbf{m}_n}$ contained in the representation T, these operators satisfy the following relations:

$$
\begin{aligned}
T_{\epsilon,\mathbf{m}_n}(a)A_{\mathbf{m}_n}^{\mathbf{m}_n} &= A_{\mathbf{m}_n}^{\mathbf{m}_n}T_{\epsilon,\mathbf{m}_n}(a), \quad a \in U_q'(\mathrm{so}_n),\\
T_{\epsilon,\mathbf{m}_n}(a)A_{\mathbf{m}_n^{\mp j}}^{\mathbf{m}_n}A_{\mathbf{m}_n}^{\mathbf{m}_n^{\pm j}} &= A_{\mathbf{m}_n^{\mp j}}^{\mathbf{m}_n}A_{\mathbf{m}_n}^{\mathbf{m}_n^{\pm j}}T_{\epsilon,\mathbf{m}_n}(a), \quad a \in U_q'(\mathrm{so}_n),
\end{aligned}
\tag{4.9}
$$

where $A_{\mathbf{m}_n^{\mp j}}^{\mathbf{m}_n}A_{\mathbf{m}_n}^{\mathbf{m}_n^{\pm j}}$ is considered as operators from $\mathcal{V}_{\epsilon,\mathbf{m}_n}^{\alpha}$ into $\mathcal{V}_{\epsilon,\mathbf{m}_n}^{\alpha}$.

Proposition 4.1. *Let $\xi \in \mathcal{H}$ belong to a subspace $\mathcal{H}_{\mathbf{m}_n}$ of the irreducible representation $T_{\mathbf{m}_n}$ of $U_q'(\mathrm{so}_n)$. Then $A_{\mathbf{m}_n}^{\mathbf{m}_n^{+j}}\xi$ and $A_{\mathbf{m}_n}^{\mathbf{m}_n^{-j}}\xi$ belong to some subspaces $\mathcal{H}_{\mathbf{m}_n^{+j}}$ and $\mathcal{H}_{\mathbf{m}_n^{-j}}$ of $\mathcal{H}$, on which the irreducible representations $T_{\mathbf{m}_n^{+j}}$ and $T_{\mathbf{m}_n^{-j}}$ of $U_q'(\mathrm{so}_n)$ are realized, respectively. All the vectors $A_{\mathbf{m}_n}^{\mathbf{m}_n^{\pm j}}(T_{\mathbf{m}_n}(a)\xi)$, $a \in U_q'(\mathrm{so}_n)$, also belong to these subspaces $\mathcal{H}_{\mathbf{m}_n^{\pm j}}$, respectively.*

Proof. The assertion follows from the definition of vector operators and from formula (4.6). □

5. Auxiliary propositions

As stated above, the algebra $U_q'(\mathrm{so}_n)$ has a commutative subalgebra $\mathcal{A}$ generated by the elements $I_{2s,2s-1}$, $s = 1,2,\dots,r$, where $r = \lfloor n/2 \rfloor$ is the integral part of $n/2$.

Proposition 5.1. (a) *If T is a finite-dimensional representation of the algebra $U_q'(\mathrm{so}_n)$, then the operators*

$$
T(I_{21}), T(I_{43}), \dots, T(I_{2k,2k-1}),
\tag{5.1}
$$

where $n = 2k$ or $n = 2k+1$, are simultaneously diagonalizable.

(b) *Possible eigenvalues of any of these operators can be only as* $\mathrm{i}[m]$, *$m \in (1/2)\mathbb{Z}$*, $\mathrm{i} = \sqrt{-1}$, *or $[m]_+$, $m \in \mathbb{Z} + 1/2$, where*

$$
[m] \equiv [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \qquad [m]_+ = \frac{q^m + q^{-m}}{q - q^{-1}}.
\tag{5.2}
$$

Proof. This proposition is true for the algebra $U_q'(\mathrm{so}_3)$. It follows from complete reducibility of finite-dimensional representations of $U_q'(\mathrm{so}_3)$ (see [12]) and from the fact that representations of the classical and of the nonclassical types exhaust all irreducible representations of $U_q'(\mathrm{so}_3)$ (see [11]). Each of the elements $I_{21}, I_{43}, \dots, I_{2k,2k-1}$ can be included into some subalgebra $U_q'(\mathrm{so}_3)$ as one of its generating elements. Therefore, each of the operators $T(I_{2j,2j-1})$, $j = 1,2,\dots,k$, can be diagonalized and has eigenvalues indicated in assertion (b). This means that these operators are semisimple. Semisimple operators on a finite-dimensional space can be simultaneously diagonalized if they commute with each other. The proposition is proved. □

Eigenvalues of the form $\mathrm{i}[m]$ are called *eigenvalues of the classical type*. Eigenvalues of the form $[m]_+$ are called *eigenvalues of the nonclassical type*.

Remark 5.2. In the formulation of Proposition 5.1, we could take for the algebra $U'_q(\mathrm{so}_{2k+1})$ the operators $T(I_{32}), T(I_{54}), \ldots, T(I_{2k+1,2k})$ instead of $T(I_{21}), T(I_{43}), \ldots, T(I_{2k,2k-1})$.

In Propositions 5.3, 5.4, and 5.5 below, we suppose that the following assumption is fulfilled: *each finite-dimensional representation of $U'_q(\mathrm{so}_{n-1})$ is completely reducible and irreducible finite-dimensional representations of $U'_q(\mathrm{so}_{n-1})$ are exhausted by the irreducible representations of the classical and nonclassical types described in Section 3.* Note that for $U'_q(\mathrm{so}_3)$ and $U'_q(\mathrm{so}_4)$, this assumption is true (see [10, 11, 12]).

Proposition 5.3. *The restriction of any irreducible finite-dimensional representation T of the algebra $U'_q(\mathrm{so}_n)$ onto the subalgebra $U'_q(\mathrm{so}_{n-1})$ is completely reducible representation of $U'_q(\mathrm{so}_{n-1})$ and decomposes into irreducible representations of this subalgebra which belong only to the classical type or only to the nonclassical type.*

Proof. The restriction of T to the subalgebra $U'_q(\mathrm{so}_{n-1})$ is completely reducible due to the assumption. Let $T\downarrow_{U'_q(\mathrm{so}_{n-1})} = \bigoplus_i R_i$, where R_i are irreducible representations of $U'_q(\mathrm{so}_{n-1})$, and let $\mathcal{H} = \bigoplus_i \mathcal{V}_i$ be the corresponding decomposition of the space $\mathcal{H}$ of the representation T. The subspaces $\mathcal{V}_i$ are invariant with respect to the operators $T(I_{j,j-1})$, $j = 2,3,\ldots,n-1$, corresponding to the elements of $U'_q(\mathrm{so}_{n-1})$. Only the operator $T(I_{n,n-1})$ maps vectors of any of the subspaces $\mathcal{V}_i$ to linear combinations of vectors from other subspaces $\mathcal{V}_i$. Since the representation T is irreducible, then acting repeatedly by $T(I_{n,n-1})$ upon any vector of any subspace $\mathcal{V}_i$ we obtain linear combinations of vectors from all other subspaces $\mathcal{V}_i$. Let some irreducible representation R_{i_0} of $U'_q(\mathrm{so}_{n-1})$ in the decomposition of T belong to the classical type. We state then that all other representations R_i in the decomposition belong to the classical type. This follows from the following reasoning. We take the operators $T(I_{n,s})$, $s = 1,2,\ldots,n-1$. It follows from the commutation relations (2.6), (2.7), and (2.8) for the elements $I_{r,s}$, $r > s$, given in Section 2, that these operators constitute a vector operator for the subalgebra $U'_q(\mathrm{so}_{n-1})$ (generated by $I_{21}, I_{32}, \ldots, I_{n-1,n-2}$) acting on the space $\mathcal{H}$. Then due to the Wigner-Eckart theorem, the action of operators $T(I_{n,s})$, $s = 1,2,\ldots,n-1$, on vectors of $\mathcal{V}_{i_0}$ gives linear combinations of vectors of subspaces $\mathcal{V}_i$ on which only irreducible representations of the classical type are realized. Repeated application of $T(I_{n,s})$ again gives representations of the same type. Therefore, in this case, all representations R_i belong to the classical type. If R_{i_0} belongs to the nonclassical type, then (by the same reasoning) all representations R_i belong to the nonclassical type. The proposition is proved. □

We write down the decomposition $T\downarrow_{U'_q(\mathrm{so}_{n-1})} = \bigoplus_i R_i$ from the above proof in the form $T\downarrow_{U'_q(\mathrm{so}_{n-1})} = \bigoplus_{\mathbf{m}_{n-1}} d_{\mathbf{m}_{n-1}} T_{\mathbf{m}_{n-1}}$ if the decomposition contains representations of the classical type, where $T_{\mathbf{m}_{n-1}}$ are irreducible representations of $U'_q(\mathrm{so}_{n-1})$ from Section 3 and $d_{\mathbf{m}_{n-1}}$ are multiplicities of these representations. If the decomposition contains irreducible representations of the nonclassical type, then $T\downarrow_{U'_q(\mathrm{so}_{n-1})} = \bigoplus_{\epsilon,\mathbf{m}_{n-1}} d_{\epsilon,\mathbf{m}_{n-1}} T_{\epsilon,\mathbf{m}_{n-1}}$, where $T_{\epsilon,\mathbf{m}_{n-1}}$ are irreducible representations of the nonclassical type.

Proposition 5.4. *The action of the operator $T(I_{n,n-1})$ upon a vector of a subspace, on which the representation $T_{\mathbf{m}_{n-1}}$ (the representation $T_{\epsilon,\mathbf{m}_{n-1}}$) of $U'_q(\mathrm{so}_{n-1})$ is realized, gives* *a linear combination of vectors belonging only to subspaces of the irreducible representations of* $U_q'(\mathrm{so}_{n-1})$ *contained in the decomposition into irreducible components of the tensor product* $T_1 \otimes T_{\mathbf{m}_{n-1}}$ *(of the tensor product* $T_1 \otimes T_{\epsilon,\mathbf{m}_{n-1}}$*), where* T_1 *is the vector representation of* $U_q'(\mathrm{so}_{n-1})$.

Proof. The operators $T(I_{n,s})$, $s = 1,2,\dots,n-1$, constitute a vector operator for the subalgebra $U_q'(\mathrm{so}_{n-1})$. Now the proposition follows from the Wigner-Eckart theorem. □

Proposition 5.5. *Let* T *be a finite-dimensional irreducible representation of* $U_q'(\mathrm{so}_n)$*. Then all operators* $T(I_{2i,2i-1})$ *from Proposition 5.1 have eigenvalues only of the classical type or only of the nonclassical type.*

Proof. The proposition is true for the algebra $U_q'(\mathrm{so}_4)$. Namely, eigenvalues of $T(I_{21})$ and $T(I_{43})$ of an irreducible representation T of $U_q'(\mathrm{so}_4)$ are of the classical type if T is a representation of the classical type and of the nonclassical type if T is a representation of the nonclassical type (see [10]). We restrict the representation T of $U_q'(\mathrm{so}_n)$ successively to $U_q'(\mathrm{so}_{n-1}), U_q'(\mathrm{so}_{n-2}),\dots, U_q'(\mathrm{so}_4)$ and decompose it into irreducible constituents. (Moreover, the chain of these subalgebras can be taken in such a way that the last subalgebra $U_q'(\mathrm{so}_4)$ contains any two fixed neighbouring operators from Proposition 5.1(a).) Applying Proposition 5.3 at the first step, we obtain in the decomposition of T irreducible representations of $U_q'(\mathrm{so}_{n-1})$ all belonging to the classical type or all belonging to the nonclassical type. Due to the assumption before Proposition 5.3 and Remark 3.1 at the end of Section 3, on each next step, we obtain only irreducible representations of the classical type or only irreducible representations of the nonclassical type, described in Section 3. Thus, restriction of T onto any subalgebra $U_q'(\mathrm{so}_4)$ decomposes into irreducible representations of $U_q'(\mathrm{so}_4)$ all belonging to the classical type or all belonging to the nonclassical type. Our proposition follows from this assertion. The proposition is proved. □

An irreducible representation T of $U_q'(\mathrm{so}_n)$ for which all the operators $T(I_{2i,2i-1})$, $i = 1,2,\dots,\lfloor n/2\rfloor$, have eigenvalues of the classical type (of the nonclassical type) is called a *representation of the classical type* (*of the nonclassical type*). The algebra $U_q'(\mathrm{so}_n)$ does not have irreducible finite-dimensional representations of other types. In Section 3, irreducible representations of the classical and of the nonclassical type are given. But we do not know yet that they exhaust all irreducible representations of these types. Our aim is to prove that the irreducible representations of Section 3 exhaust all irreducible finite-dimensional representations of $U_q'(\mathrm{so}_n)$.

6. Reduced matrix elements for the classical type representations

The theorem on classification of irreducible finite-dimensional representations of the algebra $U_q'(\mathrm{so}_n)$ will be proved by means of mathematical induction. Namely, we make an assumption on irreducible finite-dimensional representations of the subalgebra $U_q'(\mathrm{so}_{n-1})$ (which is true for the subalgebra $U_q'(\mathrm{so}_4)$) and then prove that this assumption is true for the algebra $U_q'(\mathrm{so}_n)$.

Assumption 6.1. Each finite-dimensional representation of $U_q'(\mathrm{so}_{n-1})$ is completely reducible and irreducible finite-dimensional representations of $U_q'(\mathrm{so}_{n-1})$ are exhausted by irreducible representations of the classical and nonclassical types described in Section 3.

This assumption is true for the algebras $U_q'(\mathrm{so}_3)$ and $U_q'(\mathrm{so}_4)$ (see [10, 11]).

As we know from the previous section, irreducible finite-dimensional representations T of $U_q'(\mathrm{so}_n)$ are divided into two classes—irreducible representations of the classical type and irreducible representations of the nonclassical type. For deriving the theorem on classification of irreducible representations belonging to the classical type, we need the results on reduced matrix elements of the tensor operator $T(I_{n,r})$, $k=1,2,\dots,n-1$, for the subalgebra $U_q'(\mathrm{so}_{n-1})$.

Let T be an irreducible finite-dimensional representation of $U_q'(\mathrm{so}_n)$ belonging to the classical type. According to our assumption and Proposition 5.3, this representation decomposes under the restriction onto the subalgebra $U_q'(\mathrm{so}_{n-1})$ as a direct sum of irreducible representations of the classical type from Section 3. For the space $\mathscr{H}$ of the representation T, we have

$$\mathscr{H}=\bigoplus_{\mathbf{m}_{n-1},i}\mathscr{V}_{\mathbf{m}_{n-1},i}, \tag{6.1}$$

where $\mathscr{V}_{\mathbf{m}_{n-1},i}$ is a linear subspace, on which the irreducible representation $T_{\mathbf{m}_{n-1}}$ of $U_q'(\mathrm{so}_{n-1})$ from Section 3 is realized, and i separates multiple irreducible representations of $U_q'(\mathrm{so}_{n-1})$ in the decomposition. Let

$$\mathscr{V}_{\mathbf{m}_{n-1}}=\bigoplus_{i}\mathscr{V}_{\mathbf{m}_{n-1},i}. \tag{6.2}$$

We take a Gel'fand-Tsetlin basis in each subspace $\mathscr{V}_{\mathbf{m}_{n-1},i}$ and denote these basis vectors by $|\mathbf{m}_{n-1},i,\alpha\rangle$, where $\alpha\equiv\alpha_{n-2}$ are the corresponding Gel'fand-Tsetlin tableaux. Then the subspaces

$$\mathscr{V}^{\alpha}_{\mathbf{m}_{n-1}}=\bigoplus_{i}\mathbb{C}\,|\,\mathbf{m}_{n-1},i,\alpha\rangle \tag{6.3}$$

can be defined. We know from Proposition 5.4 that the operator $T(I_{n,n-1})$ maps the vector $|\mathbf{m}_{n-1},i,\alpha\rangle$ into a linear combination of vectors of the subspaces $\mathscr{V}_{\mathbf{m}_{n-1}}$ and $\mathscr{V}_{\mathbf{m}^{\pm s}_{n-1}}$, $s=1,2,\dots,k$, where $n-1=2k$ or $n-1=2k+1$. Since the operator $T(I_{n,n-1})$ commutes with all the operators $T(I_{s,s-1})$, $s=2,3,\dots,n-2$ (i.e., with operators corresponding to elements of the subalgebra $U_q'(\mathrm{so}_{n-2})$), it maps the subspace $\mathscr{V}^{\alpha}_{\mathbf{m}_{n-1}}$ into a sum of subspaces $\mathscr{V}^{\alpha}_{\mathbf{m}'_{n-1}}$ with the same α.

Due to Proposition 5.4 and Wigner-Eckart theorem (see formula (4.6)), the action of the operator $T(I_{n,n-1})$ on the subspace $\mathscr{V}^{\alpha}_{\mathbf{m}_{n-1}}$ can be represented in the form

$$\begin{aligned}
T(I_{2p+2,2p+1})\downarrow_{\mathscr{V}^{\alpha}_{\mathbf{m}_{2p+1}}}={}&\sum_{j=1}^{p}\Bigg(\prod_{r=1}^{p}[l_{j,2p+1}+l_{r,2p}][l_{j,2p+1}-l_{r,2p}]\Bigg)^{1/2}\rho_j(\mathbf{m}_{2p+1})\\
&+\sum_{j=1}^{p}\Bigg(\prod_{r=1}^{p}[l_{j,2p+1}+l_{r,2p}-1][l_{j,2p+1}-l_{r,2p}-1]\Bigg)^{1/2}\tau_j(\mathbf{m}_{2p+1})\\
&+\Bigg(\prod_{r=1}^{p}[l_{r,2p}]\Bigg)\sigma(\mathbf{m}_{2p+1})
\end{aligned} \tag{6.4}$$

if $n = 2p+2$ and in the form

$$
\begin{aligned}
T(I_{2p+1,2p}) \downarrow_{\mathcal{V}^{\alpha}_{\mathbf{m}_{2p}}} = & \sum_{j=1}^{p} \left(\prod_{r=1}^{p-1} [l_{j,2p} + l_{r,2p-1}][l_{j,2p} - l_{r,2p-1} + 1] \right)^{1/2} \rho'_j(\mathbf{m}_{2p}) \\
& + \sum_{j=1}^{p} \left(\prod_{r=1}^{p-1} [l_{j,2p} + l_{r,2p-1} - 1][l_{j,2p} - l_{r,2p-1}] \right)^{1/2} \tau'_j(\mathbf{m}_{2p})
\end{aligned} \tag{6.5}
$$

if $n = 2p+1$, where $\rho_j(\mathbf{m}_{2p+1})$, $\rho'_j(\mathbf{m}_{2p})$, $\tau_j(\mathbf{m}_{2p+1})$, $\tau'_j(\mathbf{m}_{2p})$, and $\sigma(\mathbf{m}_{2p+1})$ are the operators such that

$$
\begin{gathered}
\rho_j(\mathbf{m}_{2p+1}) : \mathcal{V}^{\alpha}_{\mathbf{m}_{2p+1}} \longrightarrow \mathcal{V}^{\alpha}_{\mathbf{m}^{+j}_{2p+1}}, \qquad \rho'_j(\mathbf{m}_{2p}) : \mathcal{V}^{\alpha}_{\mathbf{m}_{2p}} \longrightarrow \mathcal{V}^{\alpha}_{\mathbf{m}^{+j}_{2p}}, \\
\tau_j(\mathbf{m}_{2p+1}) : \mathcal{V}^{\alpha}_{\mathbf{m}_{2p+1}} \longrightarrow \mathcal{V}^{\alpha}_{\mathbf{m}^{-j}_{2p+1}}, \qquad \tau'_j(\mathbf{m}_{2p}) : \mathcal{V}^{\alpha}_{\mathbf{m}_{2p}} \longrightarrow \mathcal{V}^{\alpha}_{\mathbf{m}^{-j}_{2p}}, \\
\sigma(\mathbf{m}_{2p+1}) : \mathcal{V}^{\alpha}_{\mathbf{m}_{2p+1}} \longrightarrow \mathcal{V}^{\alpha}_{\mathbf{m}_{2p+1}}
\end{gathered} \tag{6.6}
$$

(they are the operators $A^{\mathbf{m}^{\pm j}_{n-1}}_{\mathbf{m}_{n-1}}$ and $A^{\mathbf{m}_{n-1}}_{\mathbf{m}_{n-1}}$ from Section 4). The last summand in (6.4) must be omitted if $l_{p,2p+1} = 1$ (in this case the representation $T_{\mathbf{m}_{2p+1}}$ does not occur in the tensor product $T_1 \otimes T_{\mathbf{m}_{2p+1}}$). The coefficients in (6.4) and (6.5) are the corresponding Clebsch-Gordan coefficients of the algebra $U'(\mathrm{so}_{n-1})$ taken from [14]. As we know from the Wigner-Eckart theorem, $\rho_j(\mathbf{m}_{2p+1})$, $\rho'_j(\mathbf{m}_{2p})$, $\tau_j(\mathbf{m}_{2p+1})$, $\tau'_j(\mathbf{m}_{2p})$, and $\sigma(\mathbf{m}_{2p+1})$ are independent of α. A dependence on α is contained in the Clebsch-Gordan coefficients.

We first consider the case of the algebra $U'_q(\mathrm{so}_{2p+2})$. We act by both parts of the relation

$$
I_{2p+1,2p}I^2_{2p+2,2p+1} - (q+q^{-1})I_{2p+2,2p+1}I_{2p+1,2p}I_{2p+2,2p+1} + I^2_{2p+2,2p+1}I_{2p+1,2p} = -I_{2p+1,2p}, \tag{6.7}
$$

taken for the representation T, upon vectors of the subspace $\mathcal{V}^{\alpha}_{\mathbf{m}_{2p+1}}$ with fixed $\mathbf{m}_{2p+1}$ and α, and take into account formula (6.4). Comparing terms with the same resulting subspaces $\mathcal{V}^{\alpha}_{\mathbf{m}'_{2p+1}}$, we obtain for $\rho_j(\mathbf{m}_{2p+1})$, $\tau_j(\mathbf{m}_{2p+1})$, and $\sigma(\mathbf{m}_{2p+1})$, the relations

$$
[l_{i,2p+1} - l_{j,2p+1} + 1]\rho_j(\mathbf{m}^{+i}_{2p+1})\rho_i(\mathbf{m}_{2p+1}) - [l_{i,2p+1} - l_{j,2p+1} - 1]\rho_i(\mathbf{m}^{+j}_{2p+1})\rho_j(\mathbf{m}_{2p+1}) = 0, \tag{6.8}
$$

$$
[l_{i,2p+1} + l_{j,2p+1}]\tau_i(\mathbf{m}^{+j}_{2p+1})\rho_j(\mathbf{m}_{2p+1}) - [l_{i,2p+1} + l_{j,2p+1} - 2]\rho_j(\mathbf{m}^{-i}_{2p+1})\tau_i(\mathbf{m}_{2p+1}) = 0, \tag{6.9}
$$

$$
[l_{i,2p+1} - l_{j,2p+1} + 1]\tau_i(\mathbf{m}^{-j}_{2p+1})\tau_j(\mathbf{m}_{2p+1}) - [l_{i,2p+1} - l_{j,2p+1} - 1]\tau_j(\mathbf{m}^{-i}_{2p+1})\tau_i(\mathbf{m}_{2p+1}) = 0, \tag{6.10}
$$

$$
[l_{j,2p+1} + 1]\sigma(\mathbf{m}^{+j}_{2p+1})\rho_j(\mathbf{m}_{2p+1}) - [l_{j,2p+1} - 1]\rho_j(\mathbf{m}_{2p+1})\sigma(\mathbf{m}_{2p+1}) = 0, \tag{6.11}
$$

$$[l_{j,2p+1}]\tau_j(\mathbf{m}_{2p+1})\sigma(\mathbf{m}_{2p+1}) - [l_{j,2p+1}-2]\sigma(\mathbf{m}_{2p+1}^{-j})\tau_j(\mathbf{m}_{2p+1}) = 0, \tag{6.12}$$

$$\begin{aligned}\sum_{i=1}^{p}\Bigg(&-[2l_{i,2p+1}+1]\prod_{\substack{r=1\\ r\neq k}}^{p}\left([l_{i,2p+1}]^2-[l_{r,2p}]^2\right)\tau_i(\mathbf{m}_{2p+1}^{+i})\rho_i(\mathbf{m}_{2p+1})\\ &+[2l_{i,2p+1}-3]\prod_{\substack{r=1\\ r\neq k}}^{p}\left([l_{i,2p+1}-1]^2-[l_{r,2p}]^2\right)\rho_i(\mathbf{m}_{2p+1}^{-i})\tau_i(\mathbf{m}_{2p+1})\Bigg)\\ &+\prod_{\substack{r=1\\ r\neq k}}^{p}[l_{r,2p}]^2\cdot\sigma^2(\mathbf{m}_{2p+1}) = -E,\end{aligned} \tag{6.13}$$

where $i \neq j$, E is the unit operator on $\mathcal{V}^{\alpha}_{\mathbf{m}_{2p+1}}$ and k is a fixed number from the set $\{1,2,\dots,p\}$. Note that the last term on the left-hand side of (6.13) must be omitted if $l_{p,2p+1}=1$.

The irreducible representations $T_{\mathbf{m}_{2p+1}}$ of $U_q'(\mathrm{so}_{2p+1})$ under restriction to $U_q'(\mathrm{so}_{2p})$ decompose into irreducible representations $T_{\mathbf{m}_{2p}}$ of this subalgebra such that the numbers $\mathbf{m}_{2p}$ satisfy the inequalities determined by the Gel'fand-Tsetlin tableaux (see Section 3). Under this, each of the numbers $l_{r,2p}$ runs over a certain set of values. Assuming that none of $l_{r,2p}$, $r \neq p$, is a constant for the representation $T_{\mathbf{m}_{2p+1}}$, we equate in (6.13) terms with the same dependence on $[l_{r,2p}]^2$, $r = 1,2,\dots,p$, and obtain the relations

$$\begin{aligned}\sum_{i=1}^{p}(-1)^p\left([2l_{i,2p+1}+1]\tau_i(\mathbf{m}_{2p+1}^{+i})\rho_i(\mathbf{m}_{2p+1}) - [2l_{i,2p+1}-3]\rho_i(\mathbf{m}_{2p+1}^{-i})\tau_i(\mathbf{m}_{2p+1})\right)&\\ = -\sigma^2(\mathbf{m}_{2p+1}),&\end{aligned} \tag{6.14}$$

$$\begin{aligned}\sum_{i=1}^{p}\Big([2l_{i,2p+1}+1][l_{i,2p+1}]^{2(p-\nu-1)}\tau_i(\mathbf{m}_{2p+1}^{+i})\rho_i(\mathbf{m}_{2p+1})&\\ -[2l_{i,2p+1}-3][l_{i,2p+1}-1]^{2(p-\nu-1)}\rho_i(\mathbf{m}_{2p+1}^{-i})\tau_i(\mathbf{m}_{2p+1})\Big) = 0, \quad \nu = 1,2,\dots,p-2,&\end{aligned} \tag{6.15}$$

$$\begin{aligned}\sum_{i=1}^{p}\Big([2l_{i,2p+1}+1][l_{i,2p+1}]^{2p-2}\tau_i(\mathbf{m}_{2p+1}^{+i})\rho_i(\mathbf{m}_{2p+1})&\\ -[2l_{i,2p+1}-3][l_{i,2p+1}-1]^{2p-2}\rho_i(\mathbf{m}_{2p+1}^{-i})\tau_i(\mathbf{m}_{2p+1})\Big) = E.&\end{aligned} \tag{6.16}$$

If s parameters $l_{r,2p}$, $r \neq p$, are constant for the representation $T_{\mathbf{m}_{2p+1}}$, then the corresponding $\rho_r(\mathbf{m}_{2p+1})$ and $\tau_r(\mathbf{m}_{2p+1})$ vanish and the number of the relations (6.15) and (6.16) is decreased by s.

In a similar way, it is proved that $\rho_i'(\mathbf{m}_{2p})$ and $\tau_i'(\mathbf{m}_{2p})$ from formula (6.5) satisfy the relations

$$[l_{i,2p}-l_{j,2p}+1]\rho'_j(\mathbf{m}_{2p}^{+i})\rho'_i(\mathbf{m}_{2p})-[l_{i,2p}-l_{j,2p}-1]\rho'_i(\mathbf{m}_{2p}^{+j})\rho'_j(\mathbf{m}_{2p})=0,\quad i\neq j, \tag{6.17}$$

$$[l_{i,2p}+l_{j,2p}+1]\tau'_i(\mathbf{m}_{2p}^{+j})\rho'_j(\mathbf{m}_{2p})-[l_{i,2p}+l_{j,2p}-1]\rho'_j(\mathbf{m}_{2p}^{-i})\tau'_i(\mathbf{m}_{2p})=0,\quad i\neq j, \tag{6.18}$$

$$[l_{i,2p}-l_{j,2p}+1]\tau'_i(\mathbf{m}_{2p}^{-j})\tau'_j(\mathbf{m}_{2p})-[l_{i,2p}-l_{j,2p}-1]\tau'_j(\mathbf{m}_{2p}^{-i})\tau'_i(\mathbf{m}_{2p})=0,\quad i\neq j, \tag{6.19}$$

$$\begin{aligned}\sum_i\Bigg(&-\frac{[2l_{i,2p}+2]}{[l_{i,2p}][l_{i,2p}+1]}\prod_{r=1}^{p-1}\big([l_{i,2p}][l_{i,2p}+1]-[l_{r,2p-1}][l_{r,2p-1}-1]\big)\tau'_i(\mathbf{m}_{2p}^{+i})\rho'_i(\mathbf{m}_{2p})\\&+\frac{[2l_{i,2p}-2]}{[l_{i,2p}][l_{i,2p}-1]}\prod_{r=1}^{p-1}\big([l_{i,2p}][l_{i,2p}-1]-[l_{r,2p-1}][l_{r,2p-1}-1]\big)\rho'_i(\mathbf{m}_{2p}^{-i})\tau'_i(\mathbf{m}_{2p})\Bigg)=-E,\end{aligned} \tag{6.20}$$

and the last equality leads to the system of equations

$$\begin{aligned}\sum_{i=1}^{p}\Big(&[2l_{i,2p}+2]\big([l_{i,2p}][l_{i,2p}+1]\big)^{p-\nu-2}\tau'_i(\mathbf{m}_{2p}^{+i})\rho'_i(\mathbf{m}_{2p})\\&-[2l_{i,2p}-2]\big([l_{i,2p}][l_{i,2p}-1]\big)^{p-\nu-2}\rho'_i(\mathbf{m}_{2p}^{-i})\tau'_i(\mathbf{m}_{2p})\Big)=0,\quad \nu=1,2,\dots,p-1,\end{aligned} \tag{6.21}$$

$$\begin{aligned}\sum_{i=1}^{p}\Big(&[2l_{i,2p}+2]\big([l_{i,2p}][l_{i,2p}+1]\big)^{p-2}\tau'_i(\mathbf{m}_{2p}^{+i})\rho'_i(\mathbf{m}_{2p})\\&-[2l_{i,2p}-2]\big([l_{i,2p}][l_{i,2p}-1]\big)^{p-2}\rho'_i(\mathbf{m}_{2p}^{-i})\tau'_i(\mathbf{m}_{2p})\Big)=E.\end{aligned} \tag{6.22}$$

It follows from the last relations of Section 4 that for any $a\in U'_q(\mathrm{so}_{2p+1})$ the operators $\rho_i(\mathbf{m}_{2p+1})$, $\tau_i(\mathbf{m}_{2p+1})$, and $\sigma(\mathbf{m}_{2p+1})$ satisfy the relations

$$T_{\mathbf{m}_{2p+1}}(a)\sigma(\mathbf{m}_{2p+1})=\sigma(\mathbf{m}_{2p+1})T_{\mathbf{m}_{2p+1}}(a), \tag{6.23}$$

$$\rho_i(\mathbf{m}_{2p+1}^{-i})\tau_i(\mathbf{m}_{2p+1})T_{\mathbf{m}_{2p+1}}(a)=T_{\mathbf{m}_{2p+1}}(a)\rho_i(\mathbf{m}_{2p+1}^{-i})\tau_i(\mathbf{m}_{2p+1}). \tag{6.24}$$

Similar relations are satisfied by $\rho'_i(\mathbf{m}_{2p})$ and $\tau'_i(\mathbf{m}_{2p})$.

Remark 6.2. Relations (6.8), (6.9), (6.10), (6.11), (6.12), and (6.13) and relations (6.17), (6.18), (6.19), and (6.20) are consequences of the relation (2.3) with $i=n-1$. Other relations from (2.2), (2.3), and (2.4) containing $I_{n,n-1}$ are satisfied by the operators (6.4) and (6.5). It is a consequence of the fact that $I_{n,n-1}$ is a component of the vector operator.

Proposition 6.3. *Let $\xi\in\mathcal{H}$ belong to a subspace $\mathcal{H}_{\mathbf{m}_{2p+1}}$, on which the irreducible representation $T_{\mathbf{m}_{2p+1}}$ of $U'_q(\mathrm{so}_{2p+1})$ is realized. Then $\rho_j(\mathbf{m}_{2p+1})\xi\in\mathcal{H}_{\mathbf{m}_{2p+1}^{+j}}$ and $\tau_j(\mathbf{m}_{2p+1})\xi\in\mathcal{H}_{\mathbf{m}_{2p+1}^{-j}}$, where $\mathcal{H}_{\mathbf{m}_{2p+1}^{\pm j}}$ are subspaces of $\mathcal{H}$, on which the irreducible representations $T_{\mathbf{m}_{2p+1}^{\pm j}}$ of $U'_q(\mathrm{so}_{2p+1})$ are realized, respectively. All the vectors $\rho_j(\mathbf{m}_{2p+1})(T_{\mathbf{m}_{2p+1}}(a)\xi)$, $a\in U'_q(\mathrm{so}_{2p+1})$, and all the vectors $\tau_j(\mathbf{m}_{2p+1})(T_{\mathbf{m}_{2p+1}}(a)\xi)$, $a\in U'_q(\mathrm{so}_{2p+1})$, belong to these subspaces $\mathcal{H}_{\mathbf{m}_{2p+1}^{+j}}$ and $\mathcal{H}_{\mathbf{m}_{2p+1}^{-j}}$, respectively.*

This proposition is a corollary of Proposition 4.1.

THEOREM 6.4. *If the above assumption is true, then the restriction of an irreducible representation T of $U_q'(\mathrm{so}_n)$ to the subalgebra $U_q'(\mathrm{so}_{n-1})$ contains each irreducible representation of this subalgebra not more than once.*

Proof. We prove the theorem for the algebra $U_q'(\mathrm{so}_{2p+2})$. For the algebra $U_q'(\mathrm{so}_{2p+1})$, the proof is the same. We consider the decomposition

$$T\downarrow_{U_q'(\mathrm{so}_{2p+1})} = \bigoplus_{\mathbf{m}_{2p+1}} d_{\mathbf{m}_{2p+1}} T_{\mathbf{m}_{2p+1}}, \tag{6.25}$$

where $d_{\mathbf{m}_{2p+1}}$ denotes a multiplicity of the representation $T_{\mathbf{m}_{2p+1}}$ in the decomposition. The decomposition $\mathcal{H} = \bigoplus_{\mathbf{m}_{2p+1},\alpha} \mathcal{V}^{\alpha}_{\mathbf{m}_{2p+1}}$ corresponds to the decomposition (6.25), where, as in Section 4, α numerates elements of the Gel'fand-Tsetlin basis for the representation $T_{\mathbf{m}_{2p+1}}$. Let $T_{\mathbf{m}'_{2p+1}} \equiv T_{\mathbf{m}^{\max}_{2p+1}}$ be a maximal irreducible representation of $U_q'(\mathrm{so}_{2p+1})$ in the decomposition (6.25), that is, such that $\rho_j(\mathbf{m}'_{2p+1}) = 0$, $j = 1,2,\ldots,p$. Due to the relations (6.8), (6.9), and (6.10), the operators ρ_i and ρ_j, as well as the operators ρ_i and τ_j, $i \neq j$, and the operators τ_i and τ_j, commute (up to a constant) with each other. For this reason, each of the parameters $l_{i,2p+1}$, $i = 1,2,\ldots,p$, in the set of the representations $T_{\mathbf{m}_{2p+1}}$ from the decomposition (6.25) runs over some set of numbers independent of values of other parameters $l_{j,2p+1}$, $j \neq i$.

We take one of the subspaces $\mathcal{V}^{\alpha}_{\mathbf{m}'_{2p+1}}$, where $\mathbf{m}'_{2p+1} \equiv \mathbf{m}^{\max}_{2p+1}$. Its dimension is equal to the multiplicity $d_{\mathbf{m}'_{2p+1}}$ of the representation $T_{\mathbf{m}'_{2p+1}}$ in the decomposition (6.25). Then $\sigma(\mathbf{m}'_{2p+1})$ is an operator on $\mathcal{V}^{\alpha}_{\mathbf{m}'_{2p+1}}$. Clearly, $\sigma(\mathbf{m}'_{2p+1})$ has at least one eigenvector ξ_0 in $\mathcal{V}^{\alpha}_{\mathbf{m}'_{2p+1}}$. According to (6.23), all the vectors $T_{\mathbf{m}'_{2p+1}}(a)\xi_0$, $a \in U_q'(\mathrm{so}_{2p+1})$, are eigenvectors of $\sigma(\mathbf{m}'_{2p+1})$. The vectors $T_{\mathbf{m}'_{2p+1}}(a)\xi_0$, $a \in U_q'(\mathrm{so}_{2p+1})$, constitute a subspace $\mathcal{V}^{\mathrm{ir}}_{\mathbf{m}'_{2p+1}}$, where the irreducible representation $T_{\mathbf{m}'_{2p+1}}$ of $U_q'(\mathrm{so}_{2p+1})$ is realized. Let $\xi_j = \tau_j(\mathbf{m}'_{2p+1})\xi_0$, $j = 1,2,\ldots,p$. Then $\xi_j \in \mathcal{V}^{\alpha}_{\mathbf{m}'^{-j}_{2p+1}}$ and, due to (6.9), $\rho_i(\mathbf{m}'^{-j}_{2p+1}) = 0$ for $i \neq j$. It follows from (6.12) that ξ_j is an eigenvector of the operator $\sigma(\mathbf{m}'^{-j}_{2p+1})$. Due to Proposition 6.3, the vector $T_{\mathbf{m}'_{2p+1}}(a)\xi_0$ is mapped by the operator $\tau_j(\mathbf{m}'_{2p+1})$ into the subspace $\mathcal{V}^{\mathrm{ir}}_{\mathbf{m}'^{-j}_{2p+1}}$. Hence, the operator $\tau_j(\mathbf{m}'_{2p+1})$ maps $\mathcal{V}^{\mathrm{ir}}_{\mathbf{m}'_{2p+1}}$ into $\{0\}$ or into the subspace $\mathcal{V}^{\mathrm{ir}}_{\mathbf{m}'^{-j}_{2p+1}}$, on which the irreducible representation $T_{\mathbf{m}'^{-j}_{2p+1}}$ is realized.

Under a restriction to $U_q'(\mathrm{so}_{2p})$, the representation $T_{\mathbf{m}'_{2p+1}}$ decomposes into a sum of irreducible representations $T_{\mathbf{m}_{2p}}$, $\mathbf{m}_{2p} = (m_{1,2p},\ldots,m_{p,2p})$. With the numbers $m_{i,2p}$ we associate numbers $l_{i,2p}$ (see Section 3). Suppose that none of $l_{r,2p}$ is a constant for the representation $T_{\mathbf{m}'_{2p+1}}$. We apply both sides of the relations (6.14), (6.15), and (6.16) to the vector ξ_0 and obtain p equations with p unknown $\rho_i(\mathbf{m}'^{-i}_{2p+1})\tau_i(\mathbf{m}'_{2p+1})\xi_0$, $i = 1,2,\ldots,p$. (Note that $\rho_j(\mathbf{m}'_{2p+1}) = 0$, $j = 1,2,\ldots,p$.) Since $l_{1,2p+1} > l_{2,2p+1} > \cdots > l_{p,2p+1}$ and q is not a root of unity, the form of coefficients in (6.14), (6.15), and (6.16) shows that the determinant of this system is not equal to 0. (In fact, this determinant is proportional to the Vandermond determinant for $[l_{i,2p+1}]^2$, $i = 1,2,\ldots,p$.) Solving this system, we obtain its (unique) solution. Since the right-hand side of (6.13) is $-E$, this means that the vectors $\rho_i(\mathbf{m}'^{-i}_{2p+1})\tau_i(\mathbf{m}'_{2p+1})\xi_0$, $i = 1,2,\ldots,p$, are multiple to the vector ξ_0. Since $\tau_i(\mathbf{m}'_{2p+1})\xi_0 = \xi_i$, the vector $\rho_i(\mathbf{m}'^{-i}_{2p+1})\xi_i$ is a multiple to the vector ξ_0. Therefore, due to (6.24), the operator

$\rho_i(\mathbf{m}'^{-i}_{2p+1})$ maps the subspace $\mathcal{V}^{\mathrm{ir}}_{\mathbf{m}'^{-i}_{2p+1}}$ into $\{0\}$ or into $\mathcal{V}^{\mathrm{ir}}_{\mathbf{m}'_{2p+1}}$. If some of the parameters $l_{r,2p}$ are constant, then the number of (6.14), (6.15), and (6.16) is smaller than p. As it is easy to see, in this case, the system of equations also has a unique solution and the conclusion remains true.

Let $\xi_{j,i} = \tau_j(\mathbf{m}'^{-i}_{2p+1})\xi_i$, $i = 1,2,\ldots,p$. As above, it is shown that the subspace $\mathcal{V}^{\mathrm{ir}}_{\mathbf{m}'^{-j,-i}_{2p+1}}$ spanned by the vectors $T_{\mathbf{m}'^{-j,-i}_{2p+1}}\xi_{j,i}$ is irreducible for $U'(\mathrm{so}_{2p+1})$ and consists of eigenvectors of the operator $\sigma(\mathbf{m}'^{-j,-i}_{2p+1})$. It is mapped by the operator $\rho_j(\mathbf{m}'^{-j,-i}_{2p+1})$ into $\{0\}$ or into $\mathcal{V}^{\mathrm{ir}}_{\mathbf{m}'^{-i}_{2p+1}}$. Moreover, due to (6.9), up to a constant we have

$$\tau_j(\mathbf{m}'^{-i}_{2p+1})\xi_i = \xi_{j,i} = \tau_i(\mathbf{m}'^{-j}_{2p+1})\xi_j = \xi_{i,j}. \tag{6.26}$$

Hence, the subspaces constructed by means of the vectors $\xi_{j,i}$ and $\xi_{i,j}$ coincide. Note that if $\mathbf{m}'^{-i}_{2p+1}$, $\mathbf{m}'^{-j}_{2p+1}$, and $\mathbf{m}'^{-j,-i}_{2p+1}$ satisfy the dominance conditions, then the constant in (6.26) is not vanishing.

We continue this reasoning further applying successively the operators τ_j and ρ_j with appropriate values of the numbers $\mathbf{m}_{2p+1}$. Due to the relations (6.8), (6.9), and (6.10), the operators ρ_i and ρ_j, as well as the operators ρ_i and τ_j, $i \neq j$, and the operators τ_i and τ_j, commute (up to a constant) with each other. Therefore, as a result of such continuation, we obtain the set of subspaces $\mathcal{V}^{\mathrm{ir}}_{\mathbf{m}_{2p+1}}$ of the representation space $\mathcal{H}$, on which nonequivalent irreducible representations of the subalgebra $U'_q(\mathrm{so}_{2p+1})$ are realized and which consist of eigenvectors of the operators $\sigma(\mathbf{m}_{2p+1})$. These subspaces are mapped by the operators ρ_i and τ_i into subspaces of this set. We consider the subspace $\mathcal{H}'$ of the space $\mathcal{H}$, which is a direct sum of these subspaces $\mathcal{V}^{\mathrm{ir}}_{\mathbf{m}_{2p+1}}$. It follows from the expression (6.4) for $T(I_{2p+2,2p+1})$ that this operator leaves $\mathcal{H}'$ invariant. Due to irreducibility of the representation T, we have $\mathcal{H}' = \mathcal{H}$. This completes the proof for the algebra $U'_q(\mathrm{so}_{2p+2})$. As is noted above, for $U'_q(\mathrm{so}_{2p+1})$, the proof is the same. The only difference is that instead of relations (6.8), (6.9), (6.10), (6.11), (6.12), (6.13), (6.14), (6.15), and (6.16), we have to use relations (6.17), (6.18), (6.19), (6.20), (6.21), and (6.22). The theorem is proved. □

The fact that any irreducible representation T of $U'_q(\mathrm{so}_n)$ contains each irreducible representation of the subalgebra $U'_q(\mathrm{so}_{n-1})$ not more than once means that the operators $\rho_j(\mathbf{m}_{2p+1})$, $\tau_j(\mathbf{m}_{2p+1})$, $\sigma_j(\mathbf{m}_{2p+1})$, $\rho'_j(\mathbf{m}_{2p})$, and $\tau'_j(\mathbf{m}_{2p})$ in (6.4) and (6.5) are numerical functions. Thus, the formula (6.4) can be represented in the form

$$\begin{aligned}
T(I_{2p+2,2p+1})\,|\mathbf{m}_{2p+1},\alpha\rangle &= \sum_j\left(\prod_{r=1}^{p}\left([l_{j,2p+1}]^2-[l_{r,2p}]^2\right)\right)^{1/2}\rho_j(\mathbf{m}_{2p+1})\,|\mathbf{m}^{+j}_{2p+1},\alpha\rangle\\
&\quad+\sum_j\left(\prod_{r=1}^{p}\left([l_{j,2p+1}-1]^2-[l_{r,2p}]^2\right)\right)^{1/2}\tau_j(\mathbf{m}_{2p+1})\,|\mathbf{m}^{-j}_{2p+1},\alpha\rangle\\
&\quad+\left(\prod_{r=1}^{p}[l_{r,2p}]\right)\sigma(\mathbf{m}_{2p+1})\,|\mathbf{m}^{+j}_{2p+1},\alpha\rangle
\end{aligned}\tag{6.27}$$

and the formula (6.5) in the form

$$T(I_{2p+1,2p})\,|\,\mathbf{m}_{2p},\alpha\rangle = \sum_j \left(\prod_{r=1}^{p-1}\left(\left[l_{j,2p}+\frac{1}{2}\right]^2-\left[l_{r,2p-1}-\frac{1}{2}\right]^2\right)\right)^{1/2}\rho'_j(\mathbf{m}_{2p})\,|\,\mathbf{m}_{2p}^{+j},\alpha\rangle$$
$$+\sum_j \left(\prod_{r=1}^{p-1}\left(\left[l_{j,2p}-\frac{1}{2}\right]^2-\left[l_{r,2p-1}-\frac{1}{2}\right]^2\right)\right)^{1/2}\tau'_j(\mathbf{m}_{2p})\,|\,\mathbf{m}_{2p}^{-j},\alpha\rangle, \tag{6.28}$$

where $\rho_j(\mathbf{m}_{2p+1})$, $\tau_j(\mathbf{m}_{2p+1})$, $\sigma_j(\mathbf{m}_{2p+1})$, $\rho'_j(\mathbf{m}_{2p})$, and $\tau'_j(\mathbf{m}_{2p+1})$ are appropriate numerical functions.

Remark 6.5. We have seen under proving Theorem 6.4 that in the set of the representations $T_{\mathbf{m}_{2p+1}}$ from the decomposition (6.25) each of the parameters $m_{i,2p+1}$, $i=1,2,\dots,p$, runs over some set of numbers independent of values of other parameters $m_{j,2p+1}$, $j\neq i$. It is easy to show by means of formula (6.27) that in an irreducible representation T of $U'_q(\mathrm{so}_{2p+2})$ each $m_{i,2p+1}$, $i=1,2,\dots,p$, takes all values from the set $m^{\min}_{i,2p+1}, m^{\min}_{i,2p+1}+1,\dots,m^{\max}_{i,2p+1}$ without any omitting. A similar assertion is true for irreducible finite-dimensional representations of $U'_q(\mathrm{so}_{2p+1})$.

We find an explicit form of the functions ρ_j, τ_j, σ, ρ'_j and τ'_j from (6.27) and (6.28). We first consider the case of $U'_q(\mathrm{so}_{2p+2})$. From (6.11), we obtain the relation $[l_{j,2p+1}+1]\sigma(\mathbf{m}^{+j}_{2p+1})=[l_{j,2p+1}-1]\sigma(\mathbf{m}_{2p+1})$. This means that $\prod_{j=1}^{p}[l_{j,2p+1}][l_{j,2p+1}-1]\cdot\sigma(\mathbf{m}_{2p+1})$ is independent of $l_{j,2p+1}$, $j=1,2,\dots,p$, that is,

$$\sigma(\mathbf{m}_{2p+1})=\prod_{j=1}^{p}\left([l_{j,2p+1}][l_{j,2p+1}-1]\right)^{-1}\cdot\sigma, \tag{6.29}$$

where σ is a constant. (Note that if $l_{p,2p+1}=1$, then $\sigma(\mathbf{m}_{2p+1})\equiv 0$.)

We derive from (6.8), (6.9), and (6.10) the relation

$$\begin{aligned}&[l_{i,2p+1}-l_{j,2p+1}+1][l_{i,2p+1}+l_{j,2p+1}+1]\rho_j(\mathbf{m}^{+i}_{2p+1})\tau_j(\mathbf{m}^{+i+j}_{2p+1})\\&\quad=[l_{i,2p+1}-l_{j,2p+1}-1][l_{i,2p+1}+l_{j,2p+1}-1]\rho_j(\mathbf{m}_{2p+1})\tau_j(\mathbf{m}^{+j}_{2p+1}),\end{aligned} \tag{6.30}$$

which shows (after multiplication of both sides by $[l_{i,2p+1}]^2-[l_{j,2p+1}]^2$) that the expression

$$\left([l_{i,2p+1}]^2-[l_{j,2p+1}]^2\right)\left([l_{i,2p+1}-1]^2-[l_{j,2p+1}]^2\right)\rho_j(\mathbf{m}_{2p+1})\tau_j(\mathbf{m}^{+j}_{2p+1}) \tag{6.31}$$

is independent of $l_{i,2p+1}$. Therefore, the expression

$$\begin{aligned}\beta_j(l_{j,2p+1})&=\rho_j(\mathbf{m}_{2p+1})\tau_j(\mathbf{m}^{+j}_{2p+1})[l_{j,2p+1}]^2[2l_{j,2p+1}-1][2l_{j,2p+1}+1]\\&\quad\times\prod_{r\neq j}\left([l_{r,2p+1}]^2-[l_{j,2p+1}]^2\right)\left([l_{r,2p+1}-1]^2-[l_{j,2p+1}]^2\right)\end{aligned} \tag{6.32}$$

depends only on $l_{j,2p+1}$.

In order to find $\beta_j(l_{j,2p+1})$ we rewrite the relations (6.14), (6.15), and (6.16) for $\beta_i(l_{i,2p+1})$:

$$\sum_{i=1}^{p}\frac{1}{[2l_{i,2p+1}-1]}\left(\frac{\beta_i(l_{i,2p+1})}{[l_{i,2p+1}]^2c_i(l_{i,2p+1})}-\frac{\beta_i(l_{i,2p+1}-1)}{[l_{i,2p+1}-1]^2c_i(l_{i,2p+1}-1)}\right)$$
$$=(-1)^{p+1}\frac{\sigma^2}{\prod_{r=1}^{p}[l_{r,2p+1}]^2[l_{r,2p+1}-1]^2},\tag{6.33}$$

$$\sum_{i=1}^{p}\frac{1}{[2l_{i,2p+1}-1]}\left(\frac{[l_{i,2p+1}]^{2\nu}\beta_i(l_{i,2p+1})}{c_i(l_{i,2p+1})}-\frac{[l_{i,2p+1}-1]^{2\nu}\beta_i(l_{i,2p+1}-1)}{c_i(l_{i,2p+1}-1)}\right)=0, \quad \nu=0,1,2,\dots,p-3,\tag{6.34}$$

$$\sum_{i=1}^{p}\frac{1}{[2l_{i,2p+1}-1]}\left(\frac{[l_{i,2p+1}]^{2p-4}\beta_i(l_{i,2p+1})}{c_i(l_{i,2p+1})}-\frac{[l_{i,2p+1}-1]^{2p-4}\beta_i(l_{i,2p+1}-1)}{c_i(l_{i,2p+1}-1)}\right)=1,\tag{6.35}$$

where

$$c_i(l_{i,2p+1})=\prod_{r\neq i}\left([l_{r,2p+1}]^2-[l_{i,2p+1}]^2\right)\left([l_{r,2p+1}-1]^2-[l_{i,2p+1}]^2\right).\tag{6.36}$$

For each fixed σ, this system of equations has a unique solution $\beta_i(l_{i,2p+1})$, $i=1,2,\dots,p$, since the determinant of this system is nonvanishing. In order to give this solution, we take into account the constants

$$l_{r+1,2p+2}=l_{r,2p+1}^{\min}-1,\quad r=1,2,\dots,p,\tag{6.37}$$

where $l_{r,2p+1}^{\min}$, $r=1,2,\dots,p$, are minimal values of $l_{r,2p+1}$ in the decomposition (6.25), and represent σ (without loss of a generality) in the form

$$\sigma=\mathrm{i}\prod_{r=1}^{p+1}[l_{r,2p+2}],\tag{6.38}$$

where $l_{1,2p+2}$ is a number, which is determined by σ.

From the definition of numbers $l_{r,2p+2}$, $r=2,3,\dots,p+1$, and from Remark 6.5 after Theorem 6.4, it follows that

$$l_{2,2p+2}>l_{3,2p+2}>\cdots>l_{p+1,2p+2}.\tag{6.39}$$

Proposition 6.6. *Solutions of the system (6.33), (6.34), and (6.35) are given by the expressions*

$$\beta_i(l_{i,2p+1})=\prod_{r=1}^{p+1}\left([l_{i,2p+1}]^2-[l_{r,2p+2}]^2\right)$$
$$=\sum_{j=0}^{p+1}(-1)^je_{p-j+1}\left([l_{1,2p+2}]^2,\dots,[l_{p+1,2p+2}]^2\right)[l_{i,2p+1}]^{2j},\tag{6.40}$$

where $e_r(x_1,\dots,x_{p+1})$ are elementary symmetric polynomials in $x_1,\dots,x_{p+1}$.

Proof. In order to prove this proposition, we use the relations

$$\sum_{i=1}^{s} \frac{z_i^m}{\prod_{r=1,r\neq i}^{s}(z_i - z_r)} = \begin{cases} 1 & \text{if } m = s-1, \\ 0 & \text{if } 0 \le m \le s-2, \end{cases} \tag{6.41}$$

$$\sum_{i=1}^{s} \frac{1}{z_i \prod_{r=1,r\neq i}^{s}(z_i - z_r)} = \frac{(-1)^{s-1}}{z_1 \cdots z_s} \tag{6.42}$$

(see, e.g., [25]). We put in these relations $s = 2p$ and use the notations $z_i = x_i, z_{i+p} = y_i$, $i = 1,2,\dots,p$. Then they can be written as

$$\begin{aligned} &\sum_{i=1}^{p} \frac{1}{x_i - y_i} \left(\frac{x_i^m}{\prod_{r\neq i}(x_r - x_i)(y_r - x_i)} - \frac{y_i^m}{\prod_{r\neq i}(x_r - y_i)(y_r - y_i)} \right) \\ &\qquad = \begin{cases} 1 & \text{if } m = 2p-1, \\ 0 & \text{if } 0 \le m \le 2p-2, \end{cases} \end{aligned} \tag{6.43}$$

$$\begin{aligned} &\sum_{i=1}^{p} \frac{1}{x_i - y_i} \left(\frac{1}{x_i \prod_{r\neq i}(x_r - x_i)(y_r - x_i)} - \frac{1}{y_i \prod_{r\neq i}(x_r - y_i)(y_r - y_i)} \right) \\ &\quad = \frac{-1}{x_1 \cdots x_p y_1 \cdots y_p}. \end{aligned} \tag{6.44}$$

We put into the relations (6.33), (6.34), and (6.35) $l_{j,2p+1} = l_{j,2p+1}^{\min}$, $j = 1,2,\dots,p$, where $l_{j,2p+1}^{\min}$ is a minimal value of $l_{j,2p+1}$ in the decomposition (6.25). Taking into account that $\beta_j(l_{j,2p+1}^{\min} - 1) = 0$, $j = 1,2,\dots,p$, we see that (6.33), (6.34), and (6.35) turn into a system of p equations for $\beta_j(l_{j,2p+1}^{\min})$, $j = 1,2,\dots,p$. We substitute into this system the expressions (6.40) for $\beta_i(l_{i,2p+1}^{\min})$ and then cancel $p-1$ multipliers from the expression for $\beta_i(l_{i,2p+1}^{\min})$ with the corresponding parts of the expressions for $c_i(l_{i,2p+1}^{\min})$, which are in the denominators. As a result, we obtain a system of relations, which contains only the multiplier $([l_{1,2p+2}]^2 - [l_{i,2p+1}^{\min}]^2)$ from $\beta_i(l_{i,2p+1}^{\min})$. Our expressions for $\beta_i(l_{i,2p+1}^{\min})$ are correct if these relations are true. It is easy to see that they are reduced to the relations (6.41) and (6.42) at $s = p$ if we set $z_i = [l_{i,2p+1}^{\min}]^2$, $i = 1,2,\dots,p$.

Further we prove the correctness of the expressions (6.40) for $\beta_i(l_{i,2p+1})$ by induction. Namely, we first put $l_{j,2p+1} = l_{j,2p+1}^{\min}$, $j \neq 1$, and successively conduct the proof for $\beta_1(l_{1,2p+1}^{\min} + 1), \beta_1(l_{1,2p+1}^{\min} + 2), \dots, \beta_1(l_{1,2p+1}^{\max} - 1)$. Then we put $l_{j,2p+1} = l_{j,2p+1}^{\min}$, $j \neq 1,2$, and conduct the proof for $\beta_2(l_{2,2p+1}^{\min} + 1), \beta_2(l_{2,2p+1}^{\min} + 2), \dots, \beta_2(l_{2,2p+1}^{\max} - 1)$ under any value of $l_{1,2p+1}$. We continue this procedure up to $\beta_p(l_{p,2p+1})$. On each step, this proof is conducted by using the relations (6.43) and (6.44). Namely, we put in these relations $x_i = [l_{i,2p+1}]^2$ and $y_i = [l_{i,2p+1} - 1]^2$, then multiply each of them by the corresponding symmetric polynomial from (6.40), and sum up them termwise in order to obtain the relation (6.33), then the relations (6.34) for $\nu = 0,1,2,\dots,p-3$, and at last the relation (6.35). This proves that $\beta_j(l_{j,2p+1})$, $j = 1,2,\dots,p$, for given values of $l_{j,2p+1}$ satisfy the relations (6.33), (6.34), and (6.35). Note that $\beta_i(l_{i,2p+1}^{\max}) = 0$ since in this case $\rho_i(\mathbf{m}_{2p+1}^{\max}) = 0$. The proposition is proved. $\square$

Thus, we have found the expressions for $\beta_j(l_{j,2p+1})$, $j = 1,2,\ldots,p$, depending on $l_{1,2p+2}$, and the corresponding values of σ. In order to separate $\rho_j(\mathbf{m}_{2p+1})$ and $\tau_j(\mathbf{m}^{+j}_{2p+1})$ in expression (6.32) for $\beta_j(l_{j,2p+1})$, we note that these functions are not determined uniquely by the representation. Ambiguity in a choice of $\rho_j(\mathbf{m}_{2p+1})$ and $\tau_j(\mathbf{m}^{+j}_{2p+1})$ is related to a choice of basis elements. Namely, in the basis

$$|\mathbf{m}_{2p+1},\alpha\rangle' = \prod_{r=1}^{p}\omega_r(l_{r,2p+1})\cdot|\mathbf{m}_{2p+1},\alpha\rangle, \tag{6.45}$$

where $\omega_r(l_{r,2p+1})$ is a numerical multiplier depending only on $l_{r,2p+1}$, we obtain somewhat different formulas for the operator $T(I_{2p+2,2p+1})$. Actually, if to pass to the basis $\{|\mathbf{m}_{2p+1},\alpha\rangle'\}$ in formula (6.27), then the coefficient $\sigma(\mathbf{m}_{2p+1})$ remains without any change, and $\rho_j(\mathbf{m}_{2p+1})$ and $\tau_j(\mathbf{m}_{2p+1})$ are transformed into

$$\begin{aligned}\hat{\rho}_j(\mathbf{m}_{2p+1}) &= \frac{\omega_j(l_{j,2p+1})}{\omega_j(l_{j,2p+1}+1)}\rho_j(\mathbf{m}_{2p+1}),\\ \hat{\tau}_j(\mathbf{m}_{2p+1}) &= \frac{\omega_j(l_{j,2p+1})}{\omega_j(l_{j,2p+1}-1)}\tau_j(\mathbf{m}_{2p+1}).\end{aligned} \tag{6.46}$$

Moreover, we have

$$\hat{\rho}_j(\mathbf{m}_{2p+1})\hat{\tau}_j(\mathbf{m}^{+j}_{2p+1}) = \rho_j(\mathbf{m}_{2p+1})\tau_j(\mathbf{m}^{+j}_{2p+1}). \tag{6.47}$$

It is clear that the multiplier $\omega(l_{j,2p+1})$ can be chosen in such a way that $\hat{\rho}_j(\mathbf{m}_{2p+1}) = -\hat{\tau}_j(\mathbf{m}^{+j}_{2p+1})$, that is,

$$\frac{\omega_j(l_{j,2p+1})}{\omega_j(l_{j,2p+1}+1)}\rho_j(\mathbf{m}_{2p+1}) = -\frac{\omega_j(l_{j,2p+1}+1)}{\omega_j(l_{j,2p+1})}\tau_j(\mathbf{m}^{+j}_{2p+1}). \tag{6.48}$$

We obtain from here that

$$\left(\frac{\omega_j(l_{j,2p+1})}{\omega_j(l_{j,2p+1}+1)}\right)^2 = -\frac{\tau_j(\mathbf{m}^{+j}_{2p+1})}{\rho_j(\mathbf{m}_{2p+1})}. \tag{6.49}$$

Taking this relation for $l_{j,2p+1} = l^{\min}_{j,2p+1}, l^{\min}_{j,2p+1}+1, l^{\min}_{j,2p+1}+2,\ldots$, we find that

$$\omega_j(l_{j,2p+1}) = c\left(\prod_{l=l^{\min}_{j,2p+1}}^{l_{j,2p+1}-1}\frac{\rho_j(\mathbf{m}_{2p+1})}{\tau_j(\mathbf{m}^{+j}_{2p+1})}\right)^{1/2}, \tag{6.50}$$

where c is a constant. Thus, we may consider that from the very beginning we have a basis for which

$$\rho_j(\mathbf{m}_{2p+1}) = -\tau_j(\mathbf{m}_{2p+1}^{+j}). \tag{6.51}$$

Then it follows from (6.32), (6.40), and (6.51) that

$$\rho_j(\mathbf{m}_{2p+1}) = \left(\frac{[l_{j,2p+1}]^{-2}[2l_{j,2p+1}-1]^{-1}\prod_{r=1}^{p+1}\left([l_{r,2p+2}]^2-[l_{j,2p+1}]^2\right)}{[l_{j,2p+1}+1]\prod_{r\neq j}\left([l_{r,2p+1}]^2-[l_{j,2p+1}]^2\right)\left([l_{r,2p+1}-1]^2-[l_{j,2p+1}]^2\right)} \right)^{1/2}, \tag{6.52}$$

where $l_{r+1,2p+2} = l_{r,2p+1}^{\min} - 1$, $r = 1,2,\ldots,p$, and $l_{1,2p+2}$ is a parameter which together with $l_{r,2p+2}$, $r = 2,3,\ldots,p+1$, must determine irreducible representations. In the next section, we will find a domain of the parameters $l_{r,2p+2}$, $r = 1,2,\ldots,p+1$.

Substituting the expressions (6.51) and (6.52) for $\rho_j(\mathbf{m}_{2p+1})$ and $\tau_j(\mathbf{m}_{2p+1})$ into (6.27), we obtain

$$\begin{aligned} T(I_{2p+2,2p+1})\,|\mathbf{m}_{2p+1},\alpha\rangle &= \sum_{j=1}^{p}\frac{B_{2p+1}^{j}(\mathbf{m}_{2p+1})}{b(l_{j,2p+1})[l_{j,2p+1}]}\,|\mathbf{m}_{2p+1}^{+j},\alpha\rangle \\ &\quad - \sum_{j=1}^{p}\frac{B_{2p+1}^{j}(\mathbf{m}_{2p+1}^{-j})}{b(l_{j,2p+1}-1)[l_{j,2p+1}-1]}\,|\mathbf{m}_{2p+1}^{-j},\alpha\rangle \\ &\quad + \mathrm{i}\,C_{2p+1}(\mathbf{m}_{2p+1})\,|\mathbf{m}_{2p+1},\alpha\rangle, \end{aligned} \tag{6.53}$$

where $b(l_{j,2p+1}) = ([2l_{j,2p+1}+1][2l_{j,2p+1}-1])^{1/2}$ and

$$\begin{aligned} &B_{2p+1}^{j}(\mathbf{m}_{2p+1}) \\ &\quad = \left(\frac{\prod_{i=1}^{p+1}[l_{i,2p+2}+l_{j,2p+1}][l_{i,2p+2}-l_{j,2p+1}]\prod_{i=1}^{p}[l_{i,2p}+l_{j,2p+1}][l_{i,2p}-l_{j,2p+1}]}{\prod_{i\neq j}^{p}[l_{i,2p+1}+l_{j,2p+1}][l_{i,2p+1}-l_{j,2p+1}][l_{i,2p+1}+l_{j,2p+1}-1][l_{i,2p+1}-l_{j,2p+1}-1]} \right)^{1/2}, \\ &C_{2p+1}(\mathbf{m}_{2p+1}) = \frac{\prod_{s=1}^{p+1}[l_{s,2p+2}]\prod_{s=1}^{p}[l_{s,2p}]}{\prod_{s=1}^{p}[l_{s,2p+1}][l_{s,2p+1}-1]}. \end{aligned} \tag{6.54}$$

This formula coincides with (3.6) if we replace $p+1$ by p. We have to determine admissible values of the parameters $l_{i,2p+2}$, $i = 1,2,\ldots,p+1$.

Now we consider the case of $U_q'(\mathrm{so}_{2p+1})$. We have to find possible expressions for $\rho_j'(\mathbf{m}_{2p})$ and $\tau_j'(\mathbf{m}_{2p})$ in (6.28).

We derive from (6.17), (6.18), and (6.19) the relation

$$\begin{aligned} &[l_{i,2p}+l_{j,2p}][l_{i,2p}-l_{j,2p}-1][l_{i,2p}+l_{j,2p}+1][l_{i,2p}-l_{j,2p}]\rho_j'(\mathbf{m}_{2p})\tau_j'(\mathbf{m}_{2p}^{+j}) \\ &\quad = [l_{i,2p}+l_{j,2p}][l_{i,2p}-l_{j,2p}-1][l_{i,2p}+l_{j,2p}-1][l_{i,2p}-l_{j,2p}-2]\rho_j'(\mathbf{m}_{2p}^{-i})\tau_j'(\mathbf{m}_{2p}^{-i+j}), \end{aligned} \tag{6.55}$$

which shows that the expression

$$([l_{i,2p}][l_{i,2p}-1]-[l_{j,2p}][l_{j,2p}+1])([l_{i,2p}+1][l_{i,2p}]-[l_{j,2p}][l_{j,2p}+1])\rho'_j(\mathbf{m}_{2p})\tau'_j(\mathbf{m}_{2p}^{+j}) \tag{6.56}$$

is independent of $l_{i,2p}$. Therefore, the expression

$$\begin{aligned}\beta'_j(l_{j,2p}) &= \rho'_j(\mathbf{m}_{2p})\tau'_j(\mathbf{m}_{2p}^{+j})(q^{l_{j,2p}}+q^{-l_{j,2p}})(q^{l_{j,2p}+1}+q^{-l_{j,2p}-1})\\ &\quad\times\prod_{r\neq j}([l_{r,2p}][l_{r,2p}-1]-[l_{j,2p}][l_{j,2p}+1])([l_{r,2p}+1][l_{r,2p}]-[l_{j,2p}][l_{j,2p}+1])\end{aligned} \tag{6.57}$$

depends only on $l_{j,2p}$. Then we rewrite the relations (6.21) and (6.22) for $\beta'_j(l_{j,2p})$ and in the same way as in Proposition 6.6, using the equalities (6.41) and (6.43), derive the following proposition.

Proposition 6.7. *Solutions of the system of equations for $\beta'_j(l_{j,2p})$ are given by the expressions*

$$\begin{aligned}\beta'_j(l_{j,2p}) &= \prod_{r=1}^{p}([l_{j,2p}][l_{j,2p}+1]-[l_{r,2p+1}][l_{r,2p+1}-1])\\ &= \prod_{r=1}^{p}[l_{r,2p+1}+l_{j,2p}][l_{r,2p+1}-l_{j,2p}-1]\\ &= \sum_{j=0}^{p}(-1)^{p-j}e_{p-j}([l_{1,2p+1}][l_{1,2p+1}-1],\dots,[l_{p,2p+1}][l_{p,2p+1}-1])([l_{j,2p}][l_{j,2p}+1])^j,\end{aligned} \tag{6.58}$$

where $l_{i,2p+1}=l_{i,2p}^{\max}+1$, $i=1,2,\dots,p$, and $e_r(x_1,\dots,x_p)$ are elementary symmetric polynomials in $x_1,\dots,x_p$.

Separating $\rho'_j(\mathbf{m}_{2p})$ and $\tau'_j(\mathbf{m}_{2p}^{+j})$ from $\beta'_j(l_{j,2p})$ as in the previous case, for the operator $T(I_{2p+1,2p})$ of an irreducible representation T of $U'_q(\mathrm{so}_{2p+1})$, we obtain

$$T(I_{2p+1,2p})|\mathbf{m}_{2p},\alpha\rangle = \sum_{j=1}^{p}\frac{A_{2p}^{j}(\mathbf{m}_{2p})}{a(l_{j,2p})}|\mathbf{m}_{2p}^{+j},\alpha\rangle - \sum_{j=1}^{p}\frac{A_{2p}^{j}(\mathbf{m}_{2p}^{-j})}{a(l_{j,2p}-1)}|\mathbf{m}_{2p}^{-j},\alpha\rangle, \tag{6.59}$$

where $a(l_{j,2p})=\{(q^{l_{j,2p}+1}+q^{-l_{j,2p}-1})(q^{l_{j,2p}}+q^{-l_{j,2p}})\}^{1/2}$ and

$$\begin{aligned}&A_{2p}^{j}(\mathbf{m}_{2p})\\ &\quad=\left(\frac{\prod_{i=1}^{p}[l_{i,2p+1}+l_{j,2p}][l_{i,2p+1}-l_{j,2p}-1]\prod_{i=1}^{p-1}[l_{i,2p-1}+l_{j,2p}][l_{i,2p-1}-l_{j,2p}-1]}{\prod_{i\neq j}^{p}[l_{i,2p}+l_{j,2p}][l_{i,2p}-l_{j,2p}][l_{i,2p}+l_{j,2p}+1][l_{i,2p}-l_{j,2p}-1]}\right)^{1/2}.\end{aligned} \tag{6.60}$$

Thus, we derived an explicit form of the operator $T(I_{n,n-1})$ of an irreducible representation of $U'_q(\mathrm{so}_n)$. In order to obtain a classification of irreducible representations of the classical type, we have (by using (6.53) and (6.59)) to derive a domain of the parameters $l_{1n}, l_{2n}, \ldots, l_{pn}$, $p = \lfloor n/2 \rfloor$.

7. Reduced matrix elements for the nonclassical type representations

We assume that Assumption 6.1 of Section 6 is acting.

Proposition 7.1. *Let T be an irreducible finite-dimensional representation of $U'_q(\mathrm{so}_n)$ belonging to the nonclassical type. Then the decomposition of $T\downarrow_{U'_q(\mathrm{so}_{n-1})}$ into irreducible constituents contains irreducible representations $T_{\epsilon,\mathbf{m}_{n-1}}$ with the same ϵ.*

Proof. The proposition follows from Proposition 5.4 and from the fact that the decomposition of the tensor products $T_1 \otimes T_{\epsilon,\mathbf{m}_{n-1}}$ (where T_1 is a vector representation) into irreducible constituents contains irreducible representations of the nonclassical type with ϵ coinciding with ϵ in $T_{\epsilon,\mathbf{m}_{n-1}}$. The proposition is proved. □

Let T be such as in Proposition 7.1 and let $\mathcal{H}$ be a space on which T acts. Let

$$\mathcal{H} = \bigoplus_{\mathbf{m}_{n-1},i} \mathcal{V}_{\epsilon,\mathbf{m}_{n-1},i}, \tag{7.1}$$

where $\mathcal{V}_{\epsilon,\mathbf{m}_{n-1},i}$ is a linear subspace, on which an irreducible representation $T_{\epsilon,\mathbf{m}_{n-1}}$ of $U'_q(\mathrm{so}_{n-1})$ is realized, and i separates multiple irreducible representations in the decomposition. We also introduce the subspaces

$$\mathcal{V}_{\epsilon,\mathbf{m}_{n-1}} = \bigoplus_{i} \mathcal{V}_{\epsilon,\mathbf{m}_{n-1},i}, \tag{7.2}$$

We take a Gel'fand-Tsetlin basis in each subspace $\mathcal{V}_{\epsilon,\mathbf{m}_{n-1},i}$ and denote the basis vectors by $|\epsilon, \mathbf{m}_{n-1}, i, \alpha\rangle$, where $\alpha \equiv \alpha_{n-2}$ are the corresponding Gel'fand-Tsetlin tableaux. Let

$$\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{n-1}} = \bigoplus_{i} \mathbb{C}\,|\,\epsilon, \mathbf{m}_{n-1}, i, \alpha\rangle. \tag{7.3}$$

We know from Proposition 5.4 that the operator $T(I_{n,n-1})$ transforms the vector $|\epsilon, \mathbf{m}_{n-1}, i, \alpha\rangle$ into a linear combination of vectors of the subspaces $\mathcal{V}_{\epsilon,\mathbf{m}_{n-1}}$ and $\mathcal{V}_{\epsilon,\mathbf{m}^{\pm s}_{n-1}}$, $s = 1, 2, \ldots, k$, where $k = \lfloor (1/2)(n-1) \rfloor$. Since the operator $T(I_{n,n-1})$ commutes with all the operators $T(I_{s,s-1})$, $s = 2, 3, \ldots, n-2$ (i.e., with operators corresponding to elements of the subalgebra $U'_q(\mathrm{so}_{n-2})$), it maps subspaces $\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{n-1}}$ into a sum of subspaces $\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}'_{n-1}}$ with the same α.

Due to Wigner-Eckart theorem (see formula (4.6)), the action of the operator $T(I_{n,n-1})$ on the subspace $\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{n-1}}$ can be represented in the form

$$\begin{aligned} T(I_{2p+2,2p+1})\downarrow_{\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{2p+1}}} = &\sum_{j=1}^{p}\left(\prod_{r=1}^{p}[l_{j,2p+1}+l_{r,2p}][l_{j,2p+1}-l_{r,2p}]\right)^{1/2}\rho_j(\epsilon,\mathbf{m}_{2p+1}) \\ &+\sum_{j=1}^{p}\left(\prod_{r=1}^{p}[l_{j,2p+1}+l_{r,2p}-1][l_{j,2p+1}-l_{r,2p}-1]\right)^{1/2}\tau_j(\epsilon,\mathbf{m}_{2p+1}) \\ &+\left(\prod_{r=1}^{p}[l_{r,2p}]_+\right)\sigma(\epsilon,\mathbf{m}_{2p+1}), \end{aligned} \tag{7.4}$$

if $n=2p+2$ and in the form

$$\begin{aligned} T(I_{2p+1,2p})\downarrow_{\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{2p}}} = &\sum_{j=1}^{p}\left(\prod_{r=1}^{p-1}[l_{j,2p}+l_{r,2p-1}][l_{j,2p}-l_{r,2p-1}+1]\right)^{1/2}\rho'_j(\epsilon,\mathbf{m}_{2p}) \\ &+\sum_{j=1}^{p}\left(\prod_{r=1}^{p-1}[l_{j,2p}+l_{r,2p-1}-1][l_{j,2p}-l_{r,2p-1}]\right)^{1/2}\tau'_j(\epsilon,\mathbf{m}_{2p}) \end{aligned} \tag{7.5}$$

if $n=2p+1$, where $\rho_j(\epsilon,\mathbf{m}_{2p+1})$, $\rho'_j(\epsilon,\mathbf{m}_{2p})$, $\tau_j(\epsilon,\mathbf{m}_{2p+1})$, $\tau'_j(\epsilon,\mathbf{m}_{2p})$, and $\sigma(\epsilon,\mathbf{m}_{2p+1})$ are the operators such that

$$\begin{gathered} \rho_j(\epsilon,\mathbf{m}_{2p+1}):\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{2p+1}}\longrightarrow\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}^{+j}_{2p+1}},\qquad \rho'_j(\epsilon,\mathbf{m}_{2p}):\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{2p}}\longrightarrow\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}^{+j}_{2p}}, \\ \tau_j(\epsilon,\mathbf{m}_{2p+1}):\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{2p+1}}\longrightarrow\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}^{-j}_{2p+1}}, \\ \tau'_j(\epsilon,\mathbf{m}_{2p}):\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{2p}}\longrightarrow\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}^{-j}_{2p}},\quad \text{if } j\neq p \text{ or } m_{p,2p}\geq\frac{3}{2}, \\ \tau'_p(\epsilon,\mathbf{m}_{2p}):\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{2p}}\longrightarrow\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{2p}},\quad \text{if } m_{p,2p}=\frac{1}{2}, \\ \sigma(\epsilon,\mathbf{m}_{2p+1}):\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{2p+1}}\longrightarrow\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{2p+1}}. \end{gathered} \tag{7.6}$$

The coefficients in (7.4) and (7.5) are the corresponding Clebsch-Gordan coefficients of the algebra $U'(\mathrm{so}_{n-1})$ taken from [14]. As we know from the Wigner-Eckart theorem, $\rho_j(\epsilon,\mathbf{m}_{2p+1})$, $\rho'_j(\epsilon,\mathbf{m}_{2p})$, $\tau_j(\epsilon,\mathbf{m}_{2p+1})$, $\tau'_j(\epsilon,\mathbf{m}_{2p})$, and $\sigma(\epsilon,\mathbf{m}_{2p+1})$ are independent of α. A dependence on α is contained in the Clebsch-Gordan coefficients.

We first consider the case of the algebra $U'_q(\mathrm{so}_{2p+2})$. We act by both parts of the relation

$$I_{2p+1,2p}I^2_{2p+2,2p+1}-(q+q^{-1})I_{2p+2,2p+1}I_{2p+1,2p}I_{2p+2,2p+1}+I^2_{2p+2,2p+1}I_{2p+1,2p}=-I_{2p+1,2p} \tag{7.7}$$

upon vectors of the subspace $\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{2p+1}}$ with fixed ϵ, $\mathbf{m}_{2p+1}$, α and take into account formula (7.4). As a result, we obtain for $\rho_j(\epsilon,\mathbf{m}_{2p+1})$, $\tau_j(\epsilon,\mathbf{m}_{2p+1})$, and $\sigma(\epsilon,\mathbf{m}_{2p+1})$ the relations

$$\begin{aligned}&[l_{i,2p+1}-l_{j,2p+1}+1]\rho_j(\epsilon,\mathbf{m}_{2p+1}^{+i})\rho_i(\epsilon,\mathbf{m}_{2p+1})\\&\qquad-[l_{i,2p+1}-l_{j,2p+1}-1]\rho_i(\epsilon,\mathbf{m}_{2p+1}^{+j})\rho_j(\epsilon,\mathbf{m}_{2p+1})=0,\end{aligned}\tag{7.8}$$

$$\begin{aligned}&[l_{i,2p+1}+l_{j,2p+1}]\tau_i(\epsilon,\mathbf{m}_{2p+1}^{+j})\rho_j(\epsilon,\mathbf{m}_{2p+1})\\&\qquad-[l_{i,2p+1}+l_{j,2p+1}-2]\rho_j(\epsilon,\mathbf{m}_{2p+1}^{-i})\tau_i(\epsilon,\mathbf{m}_{2p+1})=0,\end{aligned}\tag{7.9}$$

$$\begin{aligned}&[l_{i,2p+1}-l_{j,2p+1}+1]\tau_i(\epsilon,\mathbf{m}_{2p+1}^{-j})\tau_j(\epsilon,\mathbf{m}_{2p+1})\\&\qquad-[l_{i,2p+1}-l_{j,2p+1}-1]\tau_j(\epsilon,\mathbf{m}_{2p+1}^{-i})\tau_i(\epsilon,\mathbf{m}_{2p+1})=0,\end{aligned}\tag{7.10}$$

$$\begin{aligned}&[l_{j,2p+1}+1]_+\sigma(\epsilon,\mathbf{m}_{2p+1}^{+j})\rho_j(\epsilon,\mathbf{m}_{2p+1})\\&\qquad-[l_{j,2p+1}-1]_+\rho_j(\epsilon,\mathbf{m}_{2p+1})\sigma(\epsilon,\mathbf{m}_{2p+1})=0,\end{aligned}\tag{7.11}$$

$$\begin{aligned}&[l_{j,2p+1}]_+\tau_j(\epsilon,\mathbf{m}_{2p+1})\sigma(\epsilon,\mathbf{m}_{2p+1})\\&\qquad-[l_{j,2p+1}-2]_+\sigma(\epsilon,\mathbf{m}_{2p+1}^{-j})\tau_j(\epsilon,\mathbf{m}_{2p+1})=0,\end{aligned}\tag{7.12}$$

$$\begin{aligned}&\sum_{i=1}^{p}\Bigg(-[2l_{i,2p+1}+1]\prod_{\substack{r=1\\r\neq k}}^{p}\left([l_{i,2p+1}]_+^2-[l_{r,2p}]_+^2\right)\tau_i(\epsilon,\mathbf{m}_{2p+1}^{+i})\rho_i(\epsilon,\mathbf{m}_{2p+1})\\&\qquad+[2l_{i,2p+1}-3]\prod_{\substack{r=1\\r\neq k}}^{p}\left([l_{i,2p+1}-1]_+^2-[l_{r,2p}]_+^2\right)\rho_i(\epsilon,\mathbf{m}_{2p+1}^{-i})\tau_i(\epsilon,\mathbf{m}_{2p+1})\Bigg)\\&\qquad-\prod_{\substack{r=1\\r\neq k}}^{p}[l_{r,2p}]_+^2\cdot\sigma^2(\epsilon,\mathbf{m}_{2p+1})=-E,\end{aligned}\tag{7.13}$$

where $i\neq j$, E is the unit operator on $\mathcal{V}^{\alpha}_{\epsilon,\mathbf{m}_{2p+1}}$ and k is a fixed number from the set $\{1,2,\dots,p\}$.

The irreducible representations $T_{\epsilon,\mathbf{m}_{2p+1}}$ of $U_q'(\mathrm{so}_{2p+1})$ under restriction to $U_q'(\mathrm{so}_{2p})$ decompose into irreducible representations $T_{\epsilon,\mathbf{m}_{2p}}$ of this subalgebra such that the numbers $\mathbf{m}_{2p}$ satisfy the inequalities determined by the Gel'fand-Tsetlin tableaux. Under this, each of the numbers $l_{r,2p}$ runs over a certain set of values. Assuming that none of $l_{r,2p}$, $r\neq p$, is a constant for the representation $T_{\epsilon,\mathbf{m}_{2p+1}}$, we equate in (7.13) terms with the same dependence on $[l_{r,2p}]_+^2$ and obtain the relations

$$\begin{aligned}&\sum_{i=1}^{p}\left([2l_{i,2p+1}+1]\tau_i(\epsilon,\mathbf{m}_{2p+1}^{+i})\rho_i(\epsilon,\mathbf{m}_{2p+1})-[2l_{i,2p+1}-3]\rho_i(\epsilon,\mathbf{m}_{2p+1}^{-i})\tau_i(\epsilon,\mathbf{m}_{2p+1})\right)\\&\qquad=(-1)^p\sigma^2(\epsilon,\mathbf{m}_{2p+1}),\end{aligned}\tag{7.14}$$

$$\begin{aligned}&\sum_{i=1}^{p}\Big([2l_{i,2p+1}+1][l_{i,2p+1}]_+^{2(p-\nu-1)}\tau_i(\epsilon,\mathbf{m}_{2p+1}^{+i})\rho_i(\epsilon,\mathbf{m}_{2p+1})\\&\qquad-[2l_{i,2p+1}-3][l_{i,2p+1}-1]_+^{2(p-\nu-1)}\rho_i(\epsilon,\mathbf{m}_{2p+1}^{-i})\tau_i(\epsilon,\mathbf{m}_{2p+1})\Big)=0,\\&\qquad\qquad\nu=1,2,\dots,p-2,\end{aligned}\tag{7.15}$$

$$\sum_{i=1}^{p}\Big([2l_{i,2p+1}+1][l_{i,2p+1}]_+^{2p-2}\tau_i(\epsilon,\mathbf{m}_{2p+1}^{+i})\rho_i(\epsilon,\mathbf{m}_{2p+1}) \\ -[2l_{i,2p+1}-3][l_{i,2p+1}-1]_+^{2p-2}\rho_i(\epsilon,\mathbf{m}_{2p+1}^{-i})\tau_i(\epsilon,\mathbf{m}_{2p+1})\Big)=E. \tag{7.16}$$

If k parameters $l_{r,2p}$, $r\neq p$, are constant for the representation $T_{\epsilon,\mathbf{m}_{2p+1}}$, then the number of the relations (7.14), (7.15), and (7.16) is decreased by k.

In a similar way it is proved that $\rho_i'(\epsilon,\mathbf{m}_{2p})$ and $\tau_i'(\epsilon,\mathbf{m}_{2p})$ from formula (7.5) satisfy the relations

$$[l_{i,2p}-l_{j,2p}+1]\rho_j'(\epsilon,\mathbf{m}_{2p}^{+i})\rho_i'(\epsilon,\mathbf{m}_{2p}) \\ -[l_{i,2p}-l_{j,2p}-1]\rho_i'(\epsilon,\mathbf{m}_{2p}^{+j})\rho_j'(\epsilon,\mathbf{m}_{2p})=0,\quad i\neq j, \tag{7.17}$$

$$[l_{i,2p}+l_{j,2p}+1]\tau_i'(\epsilon,\mathbf{m}_{2p}^{+j})\rho_j'(\epsilon,\mathbf{m}_{2p}) \\ -[l_{i,2p}+l_{j,2p}-1]\rho_j'(\epsilon,\mathbf{m}_{2p}^{-i})\tau_i'(\epsilon,\mathbf{m}_{2p})=0,\quad i\neq j, \tag{7.18}$$

$$[l_{i,2p}-l_{j,2p}+1]\tau_i'(\epsilon,\mathbf{m}_{2p}^{-j})\tau_j'(\epsilon,\mathbf{m}_{2p}) \\ -[l_{i,2p}-l_{j,2p}-1]\tau_j'(\epsilon,\mathbf{m}_{2p}^{-i})\tau_i'(\epsilon,\mathbf{m}_{2p})=0,\quad i\neq j, \tag{7.19}$$

$$\sum_{i=1}^{p}\Bigg(-\frac{[2l_{i,2p}+2]}{[l_{i,2p}]_+[l_{i,2p}+1]_+}\prod_{r=1}^{p-1}\Big([l_{i,2p}]_+[l_{i,2p}+1]_+ \\ -[l_{r,2p-1}]_+[l_{r,2p-1}-1]_+\Big)\tau_i'(\epsilon,\mathbf{m}_{2p}^{+i})\rho_i'(\epsilon,\mathbf{m}_{2p}) \\ +\frac{[2l_{i,2p}-2]}{[l_{i,2p}]_+[l_{i,2p}-1]_+}\prod_{r=1}^{p-1}\Big([l_{i,2p}]_+[l_{i,2p}-1]_+ \\ -[l_{r,2p-1}]_+[l_{r,2p-1}-1]_+\Big)\rho_i'(\epsilon,\mathbf{m}_{2p}^{-i})\tau_i'(\epsilon,\mathbf{m}_{2p})\Bigg)=-E. \tag{7.20}$$

If $l_{p,2p}\equiv m_{p,2p}=1/2$, then $\rho_p'(\epsilon,\mathbf{m}_{2p}^{-p})\tau_p'(\epsilon,\mathbf{m}_{2p})$ must be replaced by $(\tau_p'(\epsilon,\mathbf{m}_{2p}))^2$. The last relation implies the equalities

$$\sum_{i=1}^{p}\Big([2l_{i,2p}+2]\big([l_{i,2p}]_+[l_{i,2p}+1]_+\big)^{p-\nu-2}\tau_i'(\epsilon,\mathbf{m}_{2p}^{+i})\rho_i'(\epsilon,\mathbf{m}_{2p}) \\ -[2l_{i,2p}-2]\big([l_{i,2p}]_+[l_{i,2p}-1]_+\big)^{p-\nu-2}\rho_i'(\epsilon,\mathbf{m}_{2p}^{-i})\tau_i'(\epsilon,\mathbf{m}_{2p})\Big)=0, \\ \nu=1,2,\dots,p-1, \tag{7.21}$$

$$\sum_{i=1}^{p}\Big([2l_{i,2p}+2]\big([l_{i,2p}]_+[l_{i,2p}+1]_+\big)^{p-2}\tau_i'(\epsilon,\mathbf{m}_{2p}^{+i})\rho_i'(\epsilon,\mathbf{m}_{2p}) \\ -[2l_{i,2p}-2]\big([l_{i,2p}]_+[l_{i,2p}-1]_+\big)^{p-2}\rho_i'(\epsilon,\mathbf{m}_{2p}^{-i})\tau_i'(\epsilon,\mathbf{m}_{2p})\Big)=E. \tag{7.22}$$

Theorem 7.2. *The restriction of a nonclassical type irreducible representation T of $U'_q(\mathrm{so}_n)$ to the subalgebra $U'_q(\mathrm{so}_{n-1})$ contains each irreducible representation of this subalgebra not more than once.*

This theorem is proved (by using relations (7.8), (7.9), (7.10), (7.11), (7.12), (7.13), (7.14),(7.15), (7.16), (7.17), (7.18), (7.19), (7.20), (7.21), and (7.22)) in the same way as Theorem 6.4 and we omit this proof.

According to this theorem, the operators $\rho_j(\epsilon,\mathbf{m}_{2p+1})$, $\rho'_j(\epsilon,\mathbf{m}_{2p})$, $\tau_j(\epsilon,\mathbf{m}_{2p+1})$, $\tau'_j(\epsilon,\mathbf{m}_{2p})$, and $\sigma(\epsilon,\mathbf{m}_{2p+1})$ are numerical functions. We have to find possible expressions for these functions.

First we consider the case of $U'_q(\mathrm{so}_{2p+2})$. We obtain from (7.11) that

$$\sigma(\mathbf{m}_{2p+1}) = \prod_{j=1}^{p}\Big([l_{j,2p+1}]_+[l_{j,2p+1}-1]_+\Big)^{-1}\cdot\sigma, \tag{7.23}$$

where σ is a constant. As in the case of the representations of the classical type, from relations (7.8), (7.9), and (7.10) we derive that the expression

$$\begin{aligned}\beta_j(l_{j,2p+1}) &= \rho_j(\epsilon,\mathbf{m}_{2p+1})\tau_j(\epsilon,\mathbf{m}^{+j}_{2p+1})[l_{j,2p+1}]_+^2[2l_{j,2p+1}-1][2l_{j,2p+1}+1]\\ &\quad\times\prod_{r\neq j}\Big([l_{r,2p+1}]^2-[l_{j,2p+1}]^2\Big)\Big([l_{r,2p+1}-1]^2-[l_{j,2p+1}]^2\Big)\end{aligned} \tag{7.24}$$

depends only on $l_{j,2p+1}$.

We rewrite the relations (7.14), (7.15), and (7.16) for $\beta_j(l_{j,2p+1})$ and introduce the notations

$$l_{r+1,2p+2} = l^{\min}_{r,2p+1}-1, \quad r=1,2,\ldots,p. \tag{7.25}$$

Then we represent σ (without loss of a generality) in the form

$$\sigma = \epsilon_{2p+2}\prod_{r=1}^{p+1}[l_{r,2p+2}]_+, \tag{7.26}$$

where $l_{1,2p+2}$ is a number, which is determined by σ.

Proposition 7.3. *Solutions of the system of equations for $\beta_j(l_{j,2p+1})$ are given by the expressions*

$$\begin{aligned}\beta_i(l_{i,2p+1}) &= \prod_{r=1}^{p+1}\Big([l_{i,2p+1}]^2-[l_{r,2p+2}]^2\Big)\\ &= \prod_{r=1}^{p+1}\Big([l_{i,2p+1}]_+^2-[l_{r,2p+2}]_+^2\Big)\\ &= \sum_{j=0}^{p+1}(-1)^j e_{p-j+1}\Big([l_{1,2p+2}]_+^2,\ldots,[l_{p+1,2p+2}]_+^2\Big)\Big([l_{j,2p+1}]_+^2\Big)^j,\end{aligned} \tag{7.27}$$

where $e_r(x_1,\ldots,x_{p+1})$ are elementary symmetric polynomials in $x_1,\ldots,x_{p+1}$.

This proposition is proved in the same way as Proposition 6.6 by using relations (6.41), (6.42), (6.43), and (6.44).

Separation of $\rho_j(\epsilon,\mathbf{m}_{2p+1})$ and $\tau_j(\epsilon,\mathbf{m}_{2p+1}^{+j})$ from $\beta_j(l_{j,2p+1})$ are fulfilled in the same way as in the case of formula (6.32) and we obtain the following formula for $T(I_{2p+2,2p+1})$:

$$
\begin{aligned}
T(I_{2p+2,2p+1})\,|\,\epsilon,\mathbf{m}_{2p+1},\alpha\rangle ={}& \sum_{j=1}^{p}\frac{B_{2p+1}^{j}(\mathbf{m}_{2p+1})}{b(l_{j,2p+1})[l_{j,2p+1}]_+}\,|\,\epsilon,\mathbf{m}_{2p+1}^{+j},\alpha\rangle\\
&-\sum_{j=1}^{p}\frac{B_{2p+1}^{j}(\mathbf{m}_{2p+1}^{-j})}{b(l_{j,2p+1}-1)[l_{j,2p+1}-1]_+}\,|\,\epsilon,\mathbf{m}_{2p+1}^{-j},\alpha\rangle\\
&+\epsilon_{2p}\widehat{C}_{2p+1}(\mathbf{m}_{2p+1})\,|\,\mathbf{m}_{2p+1},\alpha\rangle,
\end{aligned}
\tag{7.28}
$$

where $B_{2p+1}^{j}(\mathbf{m}_{2p+1})$ and $b(l_{j,2p+1})$ are given by the same expressions as in (6.53) and

$$
\widehat{C}_{2p+1}(\mathbf{m}_{2p+1}) = \frac{\prod_{s=1}^{p+1}[l_{s,2p+2}]_+\prod_{s=1}^{p}[l_{s,2p}]_+}{\prod_{s=1}^{p}[l_{s,2p+1}]_+[l_{s,2p+1}-1]_+}.
\tag{7.29}
$$

This formula coincides with (3.15) if we replace $p+1$ by p.

Now we consider the case of $U_q'(\mathrm{so}_{2p+1})$. We derive from the relations (7.17), (7.18), and (7.19) that

$$
\begin{aligned}
\beta_j'(l_{j,2p}) ={}& \rho_j'(\epsilon,\mathbf{m}_{2p})\tau_j'(\epsilon,\mathbf{m}_{2p}^{+j})\left(q^{l_{j,2p}}-q^{-l_{j,2p}}\right)\left(q^{l_{j,2p}+1}-q^{-l_{j,2p}-1}\right)\\
&\times\prod_{r\neq j}\Big([l_{r,2p}]_+[l_{r,2p}-1]_+-[l_{j,2p}]_+[l_{j,2p}+1]_+\Big)\\
&\times\Big([l_{r,2p}+1]_+[l_{r,2p}]_+-[l_{j,2p}]_+[l_{j,2p}+1]_+\Big)
\end{aligned}
\tag{7.30}
$$

depends only on $l_{j,2p}$ (we used here the relation $[x][x-1]-[y][y-1]=[x]_+[x-1]_+-[y]_+[y-1]_+$). Then we rewrite the relations (7.21) and (7.22) for $\beta_j'(l_{j,2p})$ and, using the equalities (6.41) and (6.43), derive the following proposition.

Proposition 7.4. *Solutions of the system of equations for $\beta_j'(l_{j,2p})$ are given by the expression*

$$
\beta_j'(l_{j,2p}) = \prod_{r=1}^{p}\Big([l_{j,2p}]_+[l_{j,2p}+1]_+-[l_{r,2p+1}]_+[l_{r,2p+1}-1]_+\Big),
\tag{7.31}
$$

where $l_{i,2p+1}=l_{i,2p}^{\max}+1$, $i=1,2,\dots,p$.

We separate $\rho'_j(\mathbf{m}_{2p})$ and $\tau'_j(\mathbf{m}_{2p}^{+j})$ from $\beta'_j(l_{j,2p})$ and obtain for the operator $T(I_{2p+1,2p})$ of an irreducible representation T of $U'_q(\mathrm{so}_{2p+1})$ the expression

$$T(I_{2p+1,2p})\,|\,\epsilon,\mathbf{m}_{2p},\alpha\rangle = \delta_{m_{p,2p},1/2}\frac{\epsilon_{2p+1}}{q^{1/2}-q^{-1/2}}D_{2p}(\alpha_n)\,|\,\epsilon,\mathbf{m}_{2p},\alpha\rangle \\ + \sum_{j=1}^{p}\frac{A_{2p}^{j}(\mathbf{m}_{2p})}{a'(l_{j,2p})}\,|\,\epsilon,\mathbf{m}_{2p}^{+j}\alpha\rangle - \sum_{j=1}^{p}\frac{A_{2p}^{j}(\mathbf{m}_{2p}^{-j})}{a'(l_{j,2p}-1)}\,|\,\epsilon,\mathbf{m}_{2p}^{-j},\alpha\rangle, \tag{7.32}$$

where ϵ_{2p+1} takes one of the values ±1, $A_{2p}^{j}(\mathbf{m}_{2p})$ is given by the same expression as in the case of the formula (6.59), $a'(l_{j,2p})$ is such as in (3.14) and

$$D_{2p}(\mathbf{m}_{2p}) = \frac{\prod_{i=1}^{p}[l_{i,2p+1}-1/2]\prod_{i=1}^{p-1}[l_{i,2p-1}-1/2]}{\prod_{i=1}^{p-1}[l_{i,2p}+1/2][l_{i,2p}-1/2]}. \tag{7.33}$$

8. Complete reducibility

In this section, we prove complete reducibility of finite-dimensional representations of $U'_q(\mathrm{so}_n)$ if Assumption 6.1 of Section 6 is true. For the algebras $U'_q(\mathrm{so}_3)$ and $U'_q(\mathrm{so}_4)$, this assumption is fulfilled (see [10, 12]).

Theorem 8.1. *If Assumption 6.1 of Section 6 is true, then each finite-dimensional representation of $U'_q(\mathrm{so}_n)$ is completely reducible.*

Proof. To prove the theorem, it is enough to show that every finite-dimensional representation T of $U'_q(\mathrm{so}_n)$, containing two irreducible constituents, is completely reducible. We represent the space $\mathcal{H}$ of the representation T in the form $H = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant with respect to $U'_q(\mathrm{so}_{n-1})$ and on $\mathcal{H}_1$ and $\mathcal{H}/\mathcal{H}_1$ irreducible representations of $U'_q(\mathrm{so}_n)$ are realized (we denote them by T_1 and T_2, resp.). We have to consider three cases.
Case 1. One irreducible constituent of T is of the classical type and another of the nonclassical type.
Case 2. Both irreducible constituents of T are of the classical type.
Case 3. Both irreducible constituents of T are of the nonclassical type.

Proof of Case 1. We restrict the representation T onto $U'_q(\mathrm{so}_{n-1})$ and decompose it into a direct sum of irreducible representations of $U'_q(\mathrm{so}_{n-1})$. Then $\mathcal{H}$ is the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are sums of the linear subspaces on which irreducible representations of $U'_q(\mathrm{so}_{n-1})$ are realized, which belong to the classical type and to the nonclassical type, respectively. Let $\xi_1 \in \mathcal{H}_1$ transform under an irreducible representation of $U'_q(\mathrm{so}_{n-1})$. Then due to Proposition 5.4 and the statements of Section 4 on decomposition of tensor products of irreducible representations, $T(I_{n,n-1})\xi_1 \in \mathcal{H}_1$. Similarly, if $\xi_2 \in \mathcal{H}_2$ transforms under an irreducible representation of $U'_q(\mathrm{so}_{n-1})$, then by the same reason $T(I_{n,n-1})\xi_2 \in \mathcal{H}_2$. Therefore, $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant (with respect to $U'_q(\mathrm{so}_n)$) subspaces of $\mathcal{H}$. This means that the representation T is completely irreducible. $\square$

Proof of Case 2. Under restriction of the representation T upon $U_q'(\mathrm{so}_{n-1})$, its irreducible constituents T_1 and T_2 decompose into a direct sum of irreducible representations of this subalgebra. We denote the corresponding collections of numbers, characterizing these representations of $U_q'(\mathrm{so}_{n-1})$, by $\mathbf{m}_{n-1}$ and $\tilde{\mathbf{m}}_{n-1}$, respectively. The corresponding sets of $\mathbf{m}_{n-1}$ and of $\tilde{\mathbf{m}}_{n-1}$ will be denoted by Ω_1 and Ω_2, respectively. Since for $\mathbf{m}_{n-1} \in \Omega_1$ each $m_{i,n-1}$ runs over values independent of values of $m_{j,n-1}$, $j \neq i$, then in Ω_1 there exists a single maximal $\mathbf{m}_{n-1}$ denoted by $\mathbf{m}_{n-1}^{\max}$. Similarly, in Ω_2 there exists a single $\tilde{\mathbf{m}}_{n-1}^{\max}$. We divide Case 2 into four subcase.

Subcase 1. There exists no irreducible representation $T_{\mathbf{m}_{n-1}}$ of $U_q'(\mathrm{so}_{n-1})$ with $\mathbf{m}_{n-1} \in \Omega_1$ such that $\tilde{\mathbf{m}}_{n-1}^{\max} = \mathbf{m}_{n-1}$.

Subcase 2. The representation $T_{\tilde{\mathbf{m}}_{n-1}^{\max}}$ is equivalent to some irreducible representation $T_{\mathbf{m}_{n-1}}$, $\mathbf{m}_{n-1} \in \Omega_1$ and $\tilde{\mathbf{m}}_{n-1}^{\max} \neq \mathbf{m}_{n-1}^{\max}$.

Subcase 3. $\tilde{\mathbf{m}}_{n-1}^{\max} = \mathbf{m}_{n-1}^{\max}$ and T_1 is not equivalent to T_2.

Subcase 4. T_1 is equivalent to T_2.

We conduct the proof for the representations of the algebra $U_q'(\mathrm{so}_{2p+2})$. For the algebra $U_q'(\mathrm{so}_{2p+1})$, the proof is similar and we omit it.

Let ξ be a vector of the subspace $\mathcal{V}^{\mathrm{irr}}_{\tilde{\mathbf{m}}_{2p+1}^{\max}}$ on which the irreducible representation $T_{\tilde{\mathbf{m}}_{2p+1}^{\max}}$ of $U_q'(\mathrm{so}_{2p+1})$ is realized. A multiplicity of $T_{\tilde{\mathbf{m}}_{2p+1}^{\max}}$ in the representation $T\downarrow_{U_q'(\mathrm{so}_{2p+1})}$ is one. Therefore, ξ is an eigenvector of the operator $\sigma(\tilde{\mathbf{m}}_{2p+1}^{\max})$. We follow the reasoning of the proof of Theorem 6.4 acting successively upon ξ by the operators ρ_i and τ_j of Section 6 (corresponding to the appropriate values of $\tilde{\mathbf{m}}_{2p+1}$). As a result, we obtain an invariant (with respect to $U_q'(\mathrm{so}_{2p+2})$) subspace $\tilde{\mathcal{H}}$ of $\mathcal{H}$, which is a direct sum of nonequivalent irreducible (with respect to the subalgebra $U_q'(\mathrm{so}_{2p+1})$) subspaces $\mathcal{V}^{\mathrm{irr}}_{\tilde{\mathbf{m}}_{2p+1}}$. On $\tilde{\mathcal{H}}$ the irreducible representation T_2 of $U_q'(\mathrm{so}_{2p+2})$ is realized. Therefore, T is a direct sum of its subrepresentations T_1 and T_2.

In Subcase 2, $\tilde{\mathbf{m}}_{2p+1}^{\max}$ is not a maximal set of $(m_{1,2p+1},\dots,m_{p,2p+1})$ for the representation T. Therefore, there exists j, $1 \le j \le p$, such that $\rho_j(\tilde{\mathbf{m}}_{2p+1}^{\max}) \neq 0$. This operator has one-dimensional kernel $\mathcal{K}$. We take a vector $\xi \in \mathcal{K}$. Thus, $\rho_j(\tilde{\mathbf{m}}_{2p+1}^{\max})\xi = 0$. Due to relation (6.11), ξ is an eigenvector of the operator $\sigma(\tilde{\mathbf{m}}_{2p+1}^{\max})$, and due to (6.8) $\rho_i(\tilde{\mathbf{m}}_{2p+1}^{\max})\xi = 0$, $1 \le i \le p$. Now a proof is conducted in the same way as in the previous subcase (by using the reasoning of the proof of Theorem 6.4).

Since T_1 is not equivalent to T_2 in Subcase 3, we easily derive from the results of Section 6 that for irreducible representations T_1 and T_2, the corresponding values $\sigma(\mathbf{m}_{2p+1}^{\max})$ and $\sigma(\tilde{\mathbf{m}}_{2p+1}^{\max})$ are different. Therefore, the operator $\sigma(\mathbf{m}_{2p+1}^{\max})$ for the whole representation T is diagonalizable. We take eigenvectors ξ_1 and ξ_2 belonging to different eigenvalues. Then $\rho_j(\mathbf{m}_{2p+1}^{\max})\xi_s = 0$, $s = 1, 2$, for all values of j. We act upon ξ_1 and ξ_2 by the operators ρ_i and τ_j and then, in the same way as in the proof of Theorem 6.4, obtain two linear invariant (with respect to $U_q'(\mathrm{so}_{2p+2})$) subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$. This proves the theorem for Subcase 3.

For simplicity of notations, in Subcase 4, we set

$$\begin{gathered}\mathbf{m}_{2p+1} = (m_{1,2p+1},\dots,m_{p,2p+1}) \equiv \mathbf{m} = (m_1,\dots,m_p), \\ (l_{1,2p+1},\dots,l_{p,2p+1}) \equiv (l_1,\dots,l_p).\end{gathered} \tag{8.1}$$

The operators $\sigma(\mathbf{m})$, $\rho_j(\mathbf{m})$, and $\tau_j(\mathbf{m})$ for the representation T of $U_q'(\mathrm{so}_{2p+2})$ will be denoted by $\sigma^{(T)}(\mathbf{m})$, $\rho_j^{(T)}(\mathbf{m})$, and $\tau_j^{(T)}(\mathbf{m})$, respectively. In Subcase 4, these operators are of the form

$$\sigma^{(T)}(\mathbf{m}) = \begin{pmatrix} \sigma(\mathbf{m}) & \tilde{\sigma}(\mathbf{m}) \\ 0 & \sigma(\mathbf{m}) \end{pmatrix}, \qquad \rho_j^{(T)}(\mathbf{m}) = \begin{pmatrix} \rho_j(\mathbf{m}) & \tilde{\rho}_j(\mathbf{m}) \\ 0 & \rho_j(\mathbf{m}) \end{pmatrix},$$
$$\tau_j^{(T)}(\mathbf{m}) = \begin{pmatrix} \tau_j(\mathbf{m}) & \tilde{\tau}_j(\mathbf{m}) \\ 0 & \tau_j(\mathbf{m}) \end{pmatrix}, \tag{8.2}$$

where $\sigma(\mathbf{m})$, $\rho_j(\mathbf{m})$, $\tau_j(\mathbf{m})$ $\tilde{\sigma}(\mathbf{m})$, $\tilde{\rho}_j(\mathbf{m})$, and $\tilde{\tau}_j(\mathbf{m})$ are usual functions. Moreover, $\sigma(\mathbf{m})$, $\rho_j(\mathbf{m})$, and $\tau_j(\mathbf{m})$ are functions from Section 6, corresponding to the irreducible representation T_1. Substituting these expressions for $\sigma^{(T)}(\mathbf{m})$ and $\rho_j^{(T)}(\mathbf{m})$ into (6.11), we obtain identities for elements $\sigma(\mathbf{m})$ and $\rho_j(\mathbf{m})$, coinciding with (6.11), and the identities

$$[l_j+1]\big(\sigma(\mathbf{m}^{+j})\tilde{\rho}_j(\mathbf{m})+\tilde{\sigma}(\mathbf{m}^{+j})\rho_j(\mathbf{m})\big) = [l_j-1]\big(\tilde{\rho}_j(\mathbf{m})\sigma(\mathbf{m})+\rho_j(\mathbf{m})\tilde{\sigma}(\mathbf{m})\big). \tag{8.3}$$

The function $\sigma(\mathbf{m})$ corresponds to an irreducible representation of the algebra $U_q'(\mathrm{so}_{2p+2})$ and is given by (6.29) and (6.38). Using the relation $[l_j+1]\sigma(\mathbf{m}^{+j}) = [l_j-1]\sigma(\mathbf{m})$, following from (6.11), we derive from (8.3) that $[l_j+1]\tilde{\sigma}(\mathbf{m}^{+j}) = [l_j-1]\tilde{\sigma}(\mathbf{m})$. Thus, similarly to the case of $\sigma(\mathbf{m})$ in Section 6, we derive

$$\tilde{\sigma}(\mathbf{m}) = \tilde{\sigma}\prod_{j=1}^{p}\big([l_j][l_j-1]\big)^{-1}, \tag{8.4}$$

where $\tilde{\sigma}$ is a constant. We state that $\tilde{\sigma}=0$. In order to show this, we remark that if $\tilde{\sigma}(\mathbf{m}) = 0$ for some $\mathbf{m}$, then $\tilde{\sigma}=0$, and then $\tilde{\sigma}(\mathbf{m}) = 0$ for all $\mathbf{m}$.

In the case when $l_{p+1,2p+2}=0$, the representation $T_1 \sim T_2$ contains representations of $U_q'(\mathrm{so}_{2p+1})$ with $l_p = 1$. In this case, $\sigma = \tilde{\sigma} = 0$.

Let $l_{p+1,2p+2} > 0$. It this case, $\sigma \neq 0$. From the relation (6.13), written for the representation T of $U_q'(\mathrm{so}_{2p+2})$, we derive that

$$\sum_{i=1}^{p}\Big(-[2l_i+1]\prod_{r=1}^{p-1}\big([l_i]^2-[l_{r,2p}]^2\big)F_i(\mathbf{m})+[2l_i-3]\prod_{r=1}^{p-1}\big([l_i-1]^2-[l_{r,2p}]^2\big)F_i(\mathbf{m}^{-i})\Big)$$
$$+\prod_{r=1}^{p-1}[l_{r,2p}]^2\cdot 2\sigma(\mathbf{m})\tilde{\sigma}(\mathbf{m}) = 0, \tag{8.5}$$

where

$$F_i(\mathbf{m}) := \tau_i(\mathbf{m}^{+i})\tilde{\rho}_i(\mathbf{m})+\tilde{\tau}_i(\mathbf{m}^{+i})\rho_i(\mathbf{m}). \tag{8.6}$$

We consider representations $T_{\mathbf{m}}$ of $U_q'(\mathrm{so}_{2p+1})$ from $T{\downarrow}_{U_q'(\mathrm{so}_{2p+1})}$ with $m_2,\dots,m_p$ taking their minimal values. If all $l_{s,2p}$, $s=1,2,\dots,p$, are not fixed for these representations, we have

$$
\begin{aligned}
&[2l_1+1]F_1(m_1,m_2^{\min},\ldots,m_p^{\min})-[2l_1-3]F_1(m_1-1,m_2^{\min},\ldots,m_p^{\min})\\
&\qquad+\sum_{i=2}^{p}[2l_i+1]F_i(m_1,m_2^{\min},\ldots,m_p^{\min})\\
&\quad=(-1)^{p+1}2\sigma(m_1,m_2^{\min},\ldots,m_p^{\min})\tilde{\sigma}(m_1,m_2^{\min},\ldots,m_p^{\min}),\\
&[2l_1+1][l_1]^{2\nu}F_1(m_1,m_2^{\min},\ldots,m_p^{\min})-[2l_1-3][l_1-1]^{2\nu}F_1(m_1-1,m_2^{\min},\ldots,m_p^{\min})\\
&\qquad+\sum_{i=2}^{p}[2l_i+1][l_i]^{2\nu}F_i(m_1,m_2^{\min},\ldots,m_p^{\min})=0,\quad \nu=1,2,\ldots,p-1.
\end{aligned}
\tag{8.7}
$$

We sum each equation in (8.7) over l_1 from $l_1^{\min}=l_{2,2p+2}+1$ to $l_1^{\max}$ with weight coefficients $[2l_1-1]$ and obtain

$$
\sum_{i=2}^{p}G_i=2(-1)^{p+1}\sum_{l_1=l_1^{\min}}^{l_1^{\max}}[2l_1-1]\sigma(m_1,m_2^{\min},\ldots,m_p^{\min})\tilde{\sigma}(m_1,m_2^{\min},\ldots,m_p^{\min}),\tag{8.8}
$$

$$
\sum_{i=2}^{p}[l_i]^{2\nu}G_i=0,\quad \nu=1,2,\ldots,p-1,\tag{8.9}
$$

where

$$
G_i=\sum_{l_1=l_1^{\min}}^{l_1^{\max}}[2l_1-1][2l_i+1]F_i(m_1,m_2^{\min},\ldots,m_p^{\min}).\tag{8.10}
$$

Since the system of homogeneous equations (8.9) for G_i, $i=2,3,\ldots,p$, has nonvanishing determinant, we get $G_i=0$ and, therefore, (8.8) gives

$$
\sum_{l_1=l_1^{\min}}^{l_1^{\max}}[2l_1-1]\sigma(m_1,m_2^{\min},\ldots,m_p^{\min})\tilde{\sigma}(m_1,m_2^{\min},\ldots,m_p^{\min})=0.\tag{8.11}
$$

Taking into account (6.29) and (8.4), we get

$$
\begin{aligned}
0&=\sigma\tilde{\sigma}\sum_{l_1=l_1^{\min}}^{l_1^{\max}}\frac{[2l_1-1]}{[l_1]^2[l_1-1]^2}\\
&=\sigma\tilde{\sigma}\sum_{l_1=l_1^{\min}}^{l_1^{\max}}\left(\frac{1}{[l_1-1]^2}-\frac{1}{[l_1]^2}\right)\\
&=\sigma\tilde{\sigma}\left(\frac{1}{[l_{2,2p+2}]^2}-\frac{1}{[l_1^{\max}]^2}\right).
\end{aligned}
\tag{8.12}
$$

Since $[l_1^{\max}]^2\neq[l_{2,2p+2}]^2$ and $\sigma\neq 0$, we obtain $\bar{\sigma}=0$.

If the values of $l_{s,2p}$ are fixed in the considered representations of $U'_q(\mathrm{so}_{2p+1})$, then the number of relations, which follow from (8.5) and the number of G_i are decreased by the number of fixed $l_{s,2p}$. Thus, as before, we get $G_i = 0$, $i = 2,3,\ldots,p$ and, therefore, $\tilde{\sigma} = 0$.

We have proved that $\tilde{\sigma}(\mathbf{m}) = 0$ for all irreducible representations $T_{\mathbf{m}}$ of $U'_q(\mathrm{so}_{2p+1})$, contained in the representation $T{\downarrow}_{U'_q(\mathrm{so}_n)}$. This means that all operators $\sigma^{(T)}(\mathbf{m})$ are diagonal and the further proofs of complete reducibility are conducted in the same way as in the previous subcase. □

Case 3 is proved in the same way as Case 2 and we omit this proof. The theorem is proved. □

Corollary 8.2. *If irreducible finite-dimensional representations of $U'_q(\mathrm{so}_{n-1})$ are exhausted by irreducible representations of Section 3, then each finite-dimensional representation of $U'_q(\mathrm{so}_n)$ is completely reducible.*

9. Classification theorems

Suppose that Assumption 6.1 of Section 6 is acting.

Proposition 9.1. *If Assumption 6.1 of Section 6 is true, then irreducible finite-dimensional representations T of $U'_q(\mathrm{so}_n)$ such that the restriction $T{\downarrow}_{U'_q(\mathrm{so}_{n-1})}$ contains in the decomposition into irreducible components only representations of the classical type of $U'_q(\mathrm{so}_{n-1})$ are exhausted by the representations of the classical type from Section 3.*

Proof. We prove the proposition when $n = 2p+2$. For $n = 2p+1$, the proof is similar.

Let T be a representation of $U'_q(\mathrm{so}_{2p+2})$ from the formulation of the proposition. Then the functions $\beta_j(l_{i,2p+1})$, defined by the formula (6.32), are given by (6.40). It was shown above that $T{\downarrow}_{U'_q(\mathrm{so}_{2p+1})} = \bigoplus_{\mathbf{m}_{2p+1}} T_{\mathbf{m}_{2p+1}}$ and in this decomposition each $m_{r,2p+1}$ runs over the values $m^{\min}_{r,2p+1}, m^{\min}_{r,2p+1}+1,\ldots,m^{\max}_{r,2p+1}$, where $l^{\min}_{r,2p+1} = l_{r+1,2p+2}+1$. Due to the properties of the functions ρ_j, $\beta_r(l^{\min}_{r,2p+1}+s) \neq 0$ for $s = 0,1,\ldots,l^{\max}_{r,2p+1} - l^{\min}_{r,2p+1} - 1$ and $\beta_r(l^{\max}_{r,2p+1}) = 0$. Then it follows from (6.40) that $l^{\max}_{r,2p+1} = l_{r,2p+2}$, $r \neq 1$. Since $\beta_r(l^{\max}_{1,2p+1}) = 0$, we find from (6.40) that $l^{\max}_{1,2p+1}$ coincides with $l_{1,2p+2}$ or with $-l_{1,2p+2}$. Therefore, $l_{1,2p+2}$ is an integer (a half-integer) if $l_{i,2p+2}$, $i = 2,3,\ldots,p+1$, are integers (half-integers). Moreover, $l_{1,2p+2}$ may be positive or negative. We see that the formula for the operator $T(I_{2p+2,2p+1})$ does not change if we replace $l_{1,2p+2}$ and $l_{p+1,2p+2}$ by $-l_{1,2p+2}$ and $-l_{p+1,2p+2}$, respectively. Therefore, we may consider that $l_{1,2p+2}$ is positive and $l_{p+1,2p+2}$ takes positive and negative values. Now taking into account admissible values for $l_{i,2p+2}$, $i = 1,2,\ldots,p+1$, and formula (6.53) for $T(I_{2p+2,2p+1})$, we see that the representation T coincides with one of the irreducible representations of the classical type from Section 3.

In order to prove the proposition for representations of the algebra $U'_q(\mathrm{so}_{2p+1})$, we use the formula of Proposition 6.7 and formula (6.59) instead of formulas (6.40) and (6.53). The proposition is proved. □

Proposition 9.2. *If Assumption 6.1 of Section 6 is true, then irreducible finite-dimensional representations T of $U'_q(\mathrm{so}_n)$ such that the restriction $T{\downarrow}_{U'_q(\mathrm{so}_{n-1})}$ contains in the decomposition into irreducible components only representations of the nonclassical type of $U'_q(\mathrm{so}_{n-1})$ are exhausted by the representations of the nonclassical type of Section 3.*

The proof of this proposition is the same as that of Proposition 9.1.

THEOREM 9.3. *Irreducible finite-dimensional representations of the algebra $U_q'(\mathrm{so}_n)$ are exhausted by representations of the classical type and of the nonclassical type from Section 3.*

Proof. For the algebra $U_q'(\mathrm{so}_{n-1}) \equiv U_q'(\mathrm{so}_4)$, Assumption 6.1 of Section 6 is true (see [10]). Now the theorem is easily proved by induction taking into account Theorem 8.1 and Propositions 9.1 and 9.2. The theorem is proved. □

COROLLARY 9.4. *Each finite-dimensional representation of $U_q'(\mathrm{so}_n)$ is completely reducible.*

Proof. This assertion follows from Corollary 8.2 of Section 8 and from Theorem 9.3. □

Acknowledgment

The research of N. Z. Iorgov was partially supported by the INTAS Grant no. 03-51-3350.

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