By using the concept of $\mathcal{I}$-convergence defined by Kostyrko et al. in 2001, the $\mathcal{I}$-limit superior of real sequences was introduced and the inequality $\mathcal{I} - \limsup(Ax) \leq \mathcal{I} - \limsup(x)$ for all $x \in \ell_\infty$ was studied by Demirci in 2001. In this paper, we have characterized a class of $\mathcal{I}$-conservative matrices by studying some new inequalities related to the $\mathcal{I}$-limit superior.

1. Introduction

Let $\ell_\infty$ and $c$ be the Banach spaces of bounded and convergent sequence $x = (x_k)$ with the usual supremum norm. Let $\sigma$ be a one-to-one mapping of $\mathbb{N}$, the set of positive integers, into itself and $T : \ell_\infty \to \ell_\infty$ a linear operator defined by $Tx = (Tx_k) = (x_{\sigma(k)})$. An element $\phi \in \ell'_\infty$, the conjugate space of $\ell_\infty$, is called an invariant mean or a $\sigma$-mean if and only if (i) $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all $k$, (ii) $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$, and (iii) $\phi(Tx) = \phi(x)$ for all $x \in \ell_\infty$. Let $M$ be the set of all $\sigma$-means on $\ell_\infty$. A sublinear functional $P$ on $\ell_\infty$ is said to generate $\sigma$-means if $\phi \in \ell'_\infty$ and $\phi \leq P \Rightarrow \phi$ is a $\sigma$-mean, and to dominate $\sigma$-means if $\phi \leq P$ for all $\phi \in M$, where $\phi \leq P$ means that $\phi(x) \leq P(x)$ for all $x \in \ell_\infty$.

It is shown [8] that the sublinear functional

$$V(x) = \sup_n \limsup_p t_{pn}(x)$$

(1.1)

both generates and dominates $\sigma$-means, where

$$t_{pn}(x) = \frac{1}{p+1} (x_n + x_{\sigma(n)} + \cdots + x_{\sigma^p(n)}), \quad t_{-1,n}(x) = 0.$$ (1.2)

A bounded sequence $x$ is called $\sigma$-convergent to $s$ if $V(x) = -V(-x) = s$. In this case, we write $\sigma - \lim x = s$. Let $V_\sigma$ be the set of all $\sigma$-convergent sequences. We assume throughout this paper that $\sigma^p(n) \neq n$ for all $n \geq 0$ and $p \geq 1$, where $\sigma^p(n)$ is the $p$th iterate of
The matrix $A = (a_{nk})$ be an infinite matrix of real numbers and $x = (x_k)$ a real sequence such that $Ax = (A_n(x)) = (\sum k a_{nk}x_k)$ exists for each $n$. Then, the sequence $Ax = (A_n(x))$ is called an $A$-transform of $x$. For two sequence spaces $E$ and $F$, we say that the matrix $A$ maps $E$ into $F$ if $Ax$ exits and belongs to $F$ for each $x \in E$. By $(E,F)$, we denote the set of all matrices which map $E$ into $F$.

A matrix $A \in (c,c)$ is said to be conservative. It is known [1, page 21] that $A$ is conservative if and only if $||Ax|| = \sup_n \sum_k |a_{nk}| < \infty$, $a_k = \lim_n a_{nk}$ for each $k$, and $a = \lim_n \sum_k b_{nk}$. If $A$ is conservative, the number $\chi = \chi(A) = a - \sum_k a_k$ called the characteristic of $A$ is of importance in summability [1, page 46].

Let $E$ be a subset of $\mathbb{N}$. Natural density $\delta$ of $E$ is defined by

$$\delta(E) = \lim_{n} \frac{1}{n} \left| \{k \leq n : k \in E\} \right|,$$

where the vertical bars indicate the number of elements in the enclosed set. The number sequence $x = (x_k)$ is said to be statistically convergent to the number $l$ if for every $\varepsilon$, $\delta \{k : |x_k - l| \geq \varepsilon\} = 0$ [4]. In this case, we write $st - \lim x = l$.

A matrix $A \in (c,c)_{reg}$ is said to be regular and it is known [1, page 21] that $A$ is regular if and only if $||A|| \leq \infty$, $\lim_n a_{nk} = 0$ for each $k$, and $\lim_n \sum_k a_{nk} = 1$. For a given nonnegative regular matrix $A$, the number

$$\delta_A(E) = \lim_{n} \sum_{k \in E} a_{nk}$$

is said to be the $A$-density of $E \subseteq \mathbb{N}$ [5]. A sequence $x = (x_k)$ is said to be $A$-statistical convergent to a number $s$ if for every $\varepsilon > 0$, the set $\{k : |x_k - s| \geq \varepsilon\}$ has $A$-density zero [5]. In this case, we write $st_A - \lim x = s$. By $st_A$, we denote the set of all $A$-statistically convergent sequences.

Let $B = (B_i) = (b_{nk}(i))$ be a sequence of infinite matrices. Then, a bounded sequence $x$ is said to be $B$ summable to the value $l$ if

$$\lim_n Bx = \lim_n \sum_{k} b_{nk}(i)x_k = l \quad \text{uniformly in } i.$$

The matrix $B$ is regular [11] if and only if $||B|| < \infty$, $\lim_i b_{nk}(i) = 0$ for all $k$, uniformly in $i$, and $\lim_n \sum_k b_{nk}(i) = 1$ uniformly in $i$, where $||B|| = \sup_{n,i} \sum_k |b_{nk}(i)|$. For a given nonnegative regular matrix sequence $B$, Kolk [6] introduced the $B$-density of a subset of $\mathbb{N}$ as follows.
The number

\[ \delta_E (E) = \lim_n \sum_{k \in E} b_{nk} (i) = d \quad \text{uniformly in } i \]  

(1.7)

is said to be \( B \)-density of \( E \) if it exists. In the cases \( B = (A) \) and \( B = (C, 1) \), the Cesàro matrix, the \( B \)-density reduces to the \( A \)-density and natural density, respectively. A sequence \( x = (x_k) \) is said to be \( B \)-statistically convergent [6] to a number \( s \) if for every \( \epsilon > 0 \), the set \( \{ k : |x_k - s| \geq \epsilon \} \) has \( B \)-density zero. The set of all \( B \)-statistically convergent sequences is denoted by \( s_B \).

Let \( X \neq \emptyset \). A class \( S \subset 2^X \) of subsets of \( X \) is said to be an ideal in \( X \) if \( S \) satisfies the conditions (i) \( \emptyset \in S \), (ii) \( Y \cup Z \in S \) whenever \( Y, Z \in S \), (iii) \( Y \in S \) and \( Z \subseteq Y \) implies that \( Z \in S \). An ideal is called nontrivial if \( X \notin S \). A nontrivial ideal is called admissible if \( \{ x \} \in S \) for each \( x \in X \) [7].

Let \( \mathcal{I} \) be a nontrivial ideal in \( \mathbb{N} \). A sequence \( x = (x_k) \) is said to be \( \mathcal{I} \)-convergent to a number \( l \) if for every \( \epsilon > 0 \), \( \{ k : |x_k - l| > \epsilon \} \in \mathcal{I} \) [7]. In this case, we write \( \mathcal{I} \)-lim \( x = l \). It is clear that a \( \mathcal{I} \)-convergent sequence need not be bounded. Let \( F_\mathcal{I} (b) \) be the set of all \( \mathcal{I} \)-convergent and bounded sequences.

Note that in the cases \( \mathcal{I}_c = \{ E \subseteq \mathbb{N} : \delta (E) = 0 \} \), \( \mathcal{I}_{\delta A} = \{ E \subseteq \mathbb{N} : \delta_A (E) = 0 \} \), and \( \mathcal{I}_{\delta B} = \{ E \subseteq \mathbb{N} : \delta_B (E) = 0 \} \), the \( \mathcal{I} \)-convergence is reduced to the statistically convergence, \( A \)-statistically convergence, and \( B \)-statistically convergence, respectively.

An admissible ideal \( \mathcal{I} \) in \( \mathbb{N} \) is said to satisfy the additive property if for every countable system \( \{ Y_1, Y_2, \ldots \} \) of mutually disjoint sets in \( \mathcal{I} \), there exist sets \( Z_j \subseteq \mathbb{N} (j = 1, 2, \ldots) \) such that the symmetric differences \( Y_j \Delta Z_j \), \( j = 1, 2, \ldots \) are finite and \( \bigcup_j Z_j \in \mathcal{I} \) [7].

Demirci [3] has introduced the concepts \( \mathcal{I} \)-limit superior and inferior. For a real number sequence \( x \), let \( B_x \) and \( A_x \) denote the sets \( \{ b \in \mathbb{R} : \{ k : x_k > b \} \in \mathcal{I} \} \) and \( \{ a \in \mathbb{R} : \{ k : x_k < a \} \in \mathcal{I} \} \), respectively, and also let \( \mathcal{I} \) be admissible. Then,

\[
\mathcal{I} \text{-lim sup } x = \begin{cases} 
\sup B_x & \text{if } B_x \neq \emptyset, \\
-\infty & \text{if } B_x = \emptyset,
\end{cases}
\]

\[
\mathcal{I} \text{-lim inf } x = \begin{cases} 
\inf A_x & \text{if } A_x \neq \emptyset, \\
\infty & \text{if } A_x = \emptyset.
\end{cases}
\]

(1.8)

It is shown [3] that \( \mathcal{I} \)-lim sup \( x = \beta \) if and only if for every \( \epsilon > 0 \), \( \{ k : x_k < \beta - \epsilon \} \notin \mathcal{I} \) and \( \{ k : x_k > \beta + \epsilon \} \in \mathcal{I} \). Also, \( \mathcal{I} \)-lim inf \( x = \alpha \) if and only if for every \( \epsilon > 0 \), \( \{ k : x_k < \alpha + \epsilon \} \notin \mathcal{I} \) and \( \{ k : x_k < \alpha - \epsilon \} \in \mathcal{I} \). Recall that a sequence \( x = (x_k) \) is said to be \( \mathcal{I} \)-bounded if there exists an \( N > 0 \) such that \( \{ k : |x_k| > N \} \in \mathcal{I} \). It is proved in [3] that a \( \mathcal{I} \)-bounded sequence \( x \) is \( \mathcal{I} \)-convergent if and only if \( \mathcal{I} \)-lim sup \( x = \mathcal{I} \)-lim inf \( x \).

For all \( x \in \ell_\infty \), the inequality

\[
\mathcal{I} \text{-lim sup } A(x) \leq \mathcal{I} \text{-lim sup } (x)
\]

(1.9)

has been studied in [3].

In this paper, we have characterized a class of matrices \( A \in (c, F_\mathcal{I} (b)) \) by studying some new inequalities related to the \( \mathcal{I} \)-limit superior and limit inferior.
2. The main results

Firstly, we will begin with the following lemma.

**Lemma 2.1.** \( A \in (c, F_\beta(b)) \) if and only if

\[
\sup_n \sum_k |a_{nk}| < \infty, \quad (2.1)
\]

\( F - \lim_n a_{nk} = t_k \) for every \( k \), \( (2.2) \)

\( F - \lim_n \sum_k a_{nk} = t. \) \( (2.3) \)

**Proof.** Assume that \( A \in (c, F_\beta(b)) \). Then, (2.1) follows from the fact that \((c, F_\beta(b)) \subset (\ell_\infty, \ell_\infty)\). For the necessity of the other conditions it is enough to consider the sequences \((e_k)\) and \(e\), respectively, where \((e_k)\) is the sequence whose \(k\)th place is 1 and the others are all zero.

Conversely, suppose that the conditions (2.1)–(2.3) hold. Let \( x \in c \) and \( \lim x = l \). Then, for any given \( \varepsilon > 0 \), there exists a \( k_0 \in \mathbb{N} \) such that \( |x_k - l| \leq \varepsilon \) whenever \( k \geq k_0 \). Now, we can write

\[
Ax = \sum_k a_{nk}(x_k - l) + l \sum_k a_{nk}. \quad (2.4)
\]

By an easy calculation, one can see that

\[
F - \lim_n \sum_k a_{nk}(x_k - l) = \sum_k t_k(x_k - l). \quad (2.5)
\]

So, by applying \( F - \lim_n \) in (2.4), we get that

\[
F - \lim Ax = lt + \sum_k t_k(x_k - l). \quad (2.6)
\]

This completes the proof. \( \square \)

In what follows, a matrix \( A \in (c, F_\beta(b)) \) is said to be \( F_\beta \)-conservative. In the case \( A \) is \( F_\beta \)-conservative, the number

\[
K_\beta = K_\beta(A) = t - \sum_k t_k \quad (2.7)
\]

is said to be \( F_\beta \)-characteristic of \( A \).

To the proof of our main results, we need two lemmas which can be proved by the same technique used in [2, Lemmas 2.3–2.4], respectively.

**Lemma 2.2.** Let \( A \) be \( F_\beta \)-conservative and \( \lambda > 0 \). Then,

\[
F - \limsup_n \sum_k |a_{nk} - t_k| \leq \lambda \quad (2.8)
\]
if and only if

\[
\mathcal{J} - \limsup_n \sum_k (a_{nk} - t_k)^+ \leq \frac{\lambda + K\delta}{2},
\]
\[
\mathcal{J} - \limsup_n \sum_k (a_{nk} - t_k)^- \leq \frac{\lambda - K\delta}{2}.
\]  

(2.9)

**Lemma 2.3.** Let \(\|A\| < \infty\) and \(\mathcal{J} - \lim_n |a_{nk}| = 0\). Then there exists a \(y \in \ell_\infty\) such that

\[
\mathcal{J} - \limsup \sum_k a_{nk} y_k = \mathcal{J} - \limsup \sum_k |a_{nk}|.
\]  

(2.10)

**Theorem 2.4.** Let \(A\) be \(\mathcal{J}\)-conservative. Then, for some constant \(\lambda \geq |K\delta|\) and for all \(x \in \ell_\infty\),  

\[
\mathcal{J} - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K\delta}{2} L(x) - \frac{\lambda - K\delta}{2} l(x)
\]  

(2.11)

if and only if  

\[
\mathcal{J} - \limsup_n \sum_k |a_{nk} - t_k| \leq \lambda.
\]  

(2.12)

**Proof.** Let (2.11) hold. Define \(B = (b_{nk})\) by \(b_{nk} = (a_{nk} - t_k)\) for all \(n, k\). Then, since \(A\) is \(\mathcal{J}\)-conservative, the matrix \(B\) satisfies the hypothesis of Lemma 2.3. Hence, we have from (2.11) for a \(y \in \ell_\infty\) with \(\|y\| \leq 1\) that

\[
\mathcal{J} - \limsup_n \sum_k |b_{nk}| = \mathcal{J} - \limsup_n \sum_k b_{nk} y_k
\]  

\[
\leq \frac{\lambda + K\delta}{2} L(y) - \frac{\lambda - K\delta}{2} l(y)
\]

(2.13)

\[
\leq \left( \frac{\lambda + K\delta}{2} + \frac{\lambda - K\delta}{2} \right) \|y\| = \lambda,
\]

which yields (2.12).

Conversely, let (2.12) hold and \(x \in \ell_\infty\). Then, for any \(\varepsilon > 0\), there exits a \(k_0 \in \mathbb{N}\) such that \(l(x) - \varepsilon < x_k < L(x) + \varepsilon\) whenever \(k > k_0\). Now, we can write

\[
\sum_k (a_{nk} - t_k) x_k = \sum_{k \leq k_0} (a_{nk} - t_k) x_k + \sum_{k > k_0} (a_{nk} - t_k)^+ x_k - \sum_{k > k_0} (a_{nk} - t_k)^- x_k.
\]  

(2.14)

Since \(A\) is \(\mathcal{J}\)-conservative and by Lemma 2.2, we obtain

\[
\mathcal{J} - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq (L(x) + \varepsilon) \left( \frac{\lambda + K\delta}{2} \right) - (l(x) - \varepsilon) \left( \frac{\lambda - K\delta}{2} \right)
\]

\[
= \frac{\lambda + K\delta}{2} L(x) - \frac{\lambda - K\delta}{2} l(x) + \lambda \varepsilon,
\]

which yields (2.11), since \(\varepsilon\) is arbitrary.
When \( K_\delta > 0 \) and \( \lambda = K_\delta \), we can conclude from Theorem 2.4 the following result.

**Theorem 2.5.** Let \( A \) be \( \mathcal{I} \)-conservative. Then, for all \( x \in \ell_\infty \),

\[
\mathcal{I} - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq K_\delta L(x) \tag{2.16}
\]

if and only if

\[
\mathcal{I} - \lim_n \sum_k |a_{nk} - t_k| \leq K_\delta. \tag{2.17}
\]

In the cases \( \mathcal{I} = \mathcal{I}_{\delta_a} \) and \( \mathcal{I} = \mathcal{I}_{\delta_A} \), we respectively have the following results from Theorem 2.4.

**Theorem 2.6.** (a) Let \( A \in (c, st_{\delta_a} \cap \ell_\infty) \). Then, for some constant \( \lambda \geq |K_{\delta_a}| \) and for all \( x \in \ell_\infty \),

\[
st_{\delta_a} - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K_{\delta_a}}{2} L(x) - \frac{\lambda - K_{\delta_a}}{2} l(x) \tag{2.18}
\]

if and only if

\[
st_{\delta_a} - \limsup_n \sum_k |a_{nk} - t_k| \leq \lambda. \tag{2.19}
\]

(b) Let \( A \in (c, st_A \cap \ell_\infty) \). Then, for some constant \( \lambda \geq |K_A| \) and for all \( x \in \ell_\infty \),

\[
st_A - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K_A}{2} L(x) - \frac{\lambda - K_A}{2} l(x) \tag{2.20}
\]

if and only if

\[
st_A - \limsup_n \sum_k |a_{nk} - t_k| \leq \lambda. \tag{2.21}
\]

Also, if \( \mathcal{I} = \mathcal{I}_\delta \), Theorem 2.4 appears as in [2, Theorem 2.5].

**Theorem 2.7.** Let \( A \) and \( \lambda \) be as in Theorem 2.4. Then, for all \( x \in \ell_\infty \),

\[
\mathcal{I} - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K_\delta}{2} V(x) + \frac{\lambda - K_\delta}{2} V(-x) \tag{2.22}
\]

if and only if (2.12) holds and

\[
\mathcal{I} - \lim_n \sum_k |a_{nk} - a_{n,\sigma(k)} - t_k + t_{\sigma(k)}| = 0. \tag{2.23}
\]

**Proof.** Let (2.22) hold. Then, since \( V(x) \leq L(x) \) and \( V(-x) \leq -l(x) \) for all \( x \in \ell_\infty \), (2.12) follows from Theorem 2.4.
Define a matrix $C = (c_{nk})$ by $c_{nk} = (b_{nk} - b_{n,\sigma(k)})$ for all $n, k$, where $b_{nk}$ is defined as in Theorem 2.4. Then, we have the hypothesis of Lemma 2.3. Now, choose the sequence $y$ such that $y_k = 0$ for $k \notin \sigma(\mathbb{N})$. Then, $(y_k - y_{\sigma(k)}) \in Z$ and also, by the same argument used in [10, Theorem 23], one can easily see that

$$\sum_k b_{nk} (y_k - y_{\sigma(k)}) = \sum_k c_{nk} y_{\sigma(k)}. \quad (2.24)$$

Hence, (2.22) implies that

$$\mathcal{J} - \limsup_n \sum_k |c_{nk}| = \mathcal{J} - \limsup_n \sum_k c_{nk} y_{\sigma(k)}$$

$$= \mathcal{J} - \limsup_n \sum_k b_{nk} (y_k - y_{\sigma(k)})$$

$$\leq \frac{\lambda + K_\mathcal{J}}{2} V(y_k - y_{\sigma(k)}) + \frac{\lambda - K_\mathcal{J}}{2} V(y_{\sigma(k)} - y_k) = 0. \quad (2.25)$$

This yields (2.23).

Conversely, suppose that (2.12) and (2.23) hold. Then, for any $x \in \ell_\infty$, we have (2.24). Hence, since $(x_k - x_{\sigma(k)}) \in Z$, (2.23) implies that $B \in (Z, F_j(b))$ with $\mathcal{J} - \lim Bz = 0, (z \in Z)$. We also see from the assumption that (2.11) holds. Thus, by taking infimum over $z \in Z$ in (2.11), we observe that

$$\inf_{z \in Z} \left( \mathcal{J} - \limsup_n \sum_k b_{nk} (x_k + z_k) \right) \leq \frac{\lambda + K_\mathcal{J}}{2} L(x + z) - \frac{\lambda - K_\mathcal{J}}{2} l(x + z)$$

$$= \frac{\lambda + K_\mathcal{J}}{2} W(x) + \frac{\lambda - K_\mathcal{J}}{2} W(-x). \quad (2.26)$$

On the other hand, since $\mathcal{J} - \lim Bz = 0$,

$$\inf_{z \in Z} \left( \mathcal{J} - \limsup_n \sum_k b_{nk} (x_k + z_k) \right) \geq \mathcal{J} - \limsup_n \sum_k b_{nk} x_k + \inf_{z \in Z} \left( \mathcal{J} - \limsup_n \sum_k b_{nk} z_k \right)$$

$$= \mathcal{J} - \limsup_n \sum_k b_{nk} x_k. \quad (2.27)$$

Since $W(x) = V(x)$ for all $x \in \ell_\infty$, we conclude that (2.22) holds and the proof is completed. \hfill \Box

When $K_\mathcal{J} > 0$ and $\lambda = K_\mathcal{J}$, we have the following result.

**Theorem 2.8.** Let $A$ be $\mathcal{J}$-conservative. Then, for all $x \in \ell_\infty$,

$$\mathcal{J} - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq K_\mathcal{J} V(x) \quad (2.28)$$

if and only if (2.17) and (2.23) hold.
A class of $\mathcal{H}$-conservative matrices

The following results can be derived from Theorem 2.7 for the special cases $\mathcal{H} = \mathcal{H}_{\delta_A}$ and $\mathcal{H} = \mathcal{H}_{\delta_A}$.

**Theorem 2.9.** (a) Let $A \in (c, st_A \cap \ell_\infty)$. Then, for some constant $\lambda \geq |K_A|$ and for all $x \in \ell_\infty$,

$$st_{\mathcal{H}} - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K_A}{2} V(x) + \frac{\lambda - K_A}{2} V(-x)$$

if and only if (2.19) holds and

$$st_{\mathcal{H}} - \lim_n \sum_k |a_{nk} - a_{n,\sigma(k)} - t_k + t_{\sigma(k)}| = 0.$$  (2.30)

(b) Let $A \in (c, st_A \cap \ell_\infty)$. Then, for some constant $\lambda \geq |K_A|$ and for all $x \in \ell_\infty$,

$$st_A - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K_A}{2} V(x) + \frac{\lambda - K_A}{2} V(-x)$$

if and only if (2.21) holds and

$$st_A - \lim_n \sum_k |a_{nk} - a_{n,\sigma(k)} - t_k + t_{\sigma(k)}| = 0.$$  (2.32)

Further, for $\mathcal{H} = \mathcal{H}_{\delta}$, Theorem 2.7 is reduced to [2, Theorem 2.7].

**Theorem 2.10.** Let $A$ and $\lambda$ be as in Theorem 2.4. Then, for all $x \in \ell_\infty$,

$$\mathcal{H} - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K_{\mathcal{H}}}{2} y(x) + \frac{\lambda - K_{\mathcal{H}}}{2} y(-x)$$

if and only if (2.12) holds and

$$\mathcal{H} - \lim_n \sum_{k \in E} |a_{nk} - t_k| = 0.$$  (2.34)

for every $E \in \mathcal{H}$, where $y(x) = \mathcal{H} - \limsup_k x_k$.

**Proof.** If (2.33) holds, since $y(x) \leq L(x)$ and $y(-x) \leq -l(x)$, (2.12) follows from Theorem 2.4. To show the necessity of (2.34), for any $E \in \mathcal{H}$, let us define a matrix $D = (d_{nk})$ by $d_{nk} = a_{nk} - t_k, k \in E$; otherwise, it equals zero for all $n$. Then, clearly, $D$ satisfies the conditions of Lemma 2.2, and therefore there exists a $y \in \ell_\infty$ such that $\|y\| \leq 1$ and

$$\mathcal{H} - \limsup_n \sum_k d_{nk} y_k = \mathcal{H} - \limsup_n \sum_k |d_{nk}|.$$  (2.35)

Now, for the same $E$, we choose the sequence $y$ as

$$y_k = \begin{cases} 1, & k \in E, \\ 0, & k \notin E. \end{cases}$$  (2.36)
Then, since \( \lim y = y(y) = y(-y) = 0 \), (2.33) implies that
\[
\mathcal{F} - \lim sup_n \sum_{k \in E} |d_{nk}| \leq \frac{\lambda + K_{\mathcal{F}}}{2} y(y) + \frac{\lambda - K_{\mathcal{F}}}{2} y(-y) = 0,
\] (2.37)
which yields (2.34).

Conversely, suppose that the conditions of the theorem hold and \( x \in \ell_\infty \). Let \( E_1 = \{ k : x_k > y(x) + \varepsilon \} \) and \( E_2 = \{ k : x_k < y(x) - \varepsilon \} \). Then, since \( E_1, E_2 \subseteq \mathcal{F} \), \( E = E_1 \cap E_2 \subseteq \mathcal{F} \). Now, we can write
\[
\sum_k (a_{nk} - t_k) x_k = \sum_{k \in E} (a_{nk} - t_k) x_k + \sum_{k \notin E} (a_{nk} - t_k)^+ x_k - \sum_{k \notin E} (a_{nk} - t_k)^- x_k.
\] (2.38)
Thus, by (2.34) and Lemma 2.2, (2.33) is obtained since
\[
\mathcal{F} - \lim sup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K_{\mathcal{F}}}{2} y(x) + \frac{\lambda - K_{\mathcal{F}}}{2} y(-x) + \lambda \varepsilon
\] (2.39)
and \( \varepsilon \) is arbitrary. \( \square \)

When \( K_{\mathcal{F}} > 0 \) and \( \lambda = K_{\mathcal{F}} \), we have the following result.

**Theorem 2.11.** Let \( A \) be \( \mathcal{F} \)-conservative. Then, for all \( x \in \ell_\infty \),
\[
\mathcal{F} - \lim sup_n \sum_k (a_{nk} - t_k) x_k \leq K_{\mathcal{F}} y(x)
\] (2.40)
if and only if (2.17) and (2.34) hold.

We can choose \( \mathcal{F} = \mathcal{F}_{\delta_{\mathbb{A}}} \) and \( \mathcal{F} = \mathcal{F}_{\delta_{\mathbb{A}}} \) in Theorem 2.10 to obtain the following results.

**Theorem 2.12.** (a) Let \( A \in (c, st_{\mathbb{A}} \cap \ell_\infty) \). Then, for some constant \( \lambda \geq |K_{\mathbb{A}}| \) and for all \( x \in \ell_\infty \),
\[
st_{\mathbb{A}} - \lim sup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K_{\mathbb{A}}}{2} y(x) + \frac{\lambda - K_{\mathbb{A}}}{2} y(-x)
\] (2.41)
if and only if (2.19) holds and
\[
st_{\mathbb{A}} - \lim_n \sum_{k \in E} |a_{nk} - t_k| = 0,
\] (2.42)
for every \( E \in \mathcal{F} \).

(b) Let \( A \in (c, st_{\mathbb{A}} \cap \ell_\infty) \). Then, for some constant \( \lambda \geq |K_{\mathbb{A}}| \) and for all \( x \in \ell_\infty \),
\[
st_{\mathbb{A}} - \lim sup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K_{\mathbb{A}}}{2} y(x) + \frac{\lambda - K_{\mathbb{A}}}{2} y(-x)
\] (2.43)
A class of $\mathcal{I}$-conservative matrices

if and only if (2.21) holds and

$$
st_A - \lim_{n \to \infty} \sum_{k \in E} |a_{nk} - t_k| = 0,
$$

for every $E \in \mathcal{I}$.

Moreover, Theorem 2.10 is a dual case of [2, Theorem 2.6] for $\mathcal{I} = \mathcal{I}_{\delta}$.

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References


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<th>May 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
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<tr>
<td>Publication Date</td>
<td>November 1, 2009</td>
</tr>
</tbody>
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