GENERALIZED g-QUASIVARIATIONAL INEQUALITY

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Suppose that X is a nonempty subset of a metric space E and Y is a nonempty subset of a topological vector space F. Let $g : X \to Y$ and $\psi : X \times Y \to \mathbb{R}$ be two functions and let $S : X \to 2^Y$ and $T : Y \to 2^{F^*}$ be two maps. Then the generalized g-quasivariational inequality problem (GgQVI) is to find a point $\overline{x} \in X$ and a point $f \in T(g(\overline{x}))$ such that $g(\overline{x}) \in S(\overline{x})$ and $\sup_{y\in S(\overline{x})}\{\mathrm{Re}\langle f, y - g(\overline{x})\rangle + \psi(\overline{x}, y)\} = \psi(\overline{x},g(\overline{x}))$. In this paper, we prove the existence of a solution of (GgQVI).

1. Introduction and preliminaries

The quasivariational inequality has proven to be useful in different areas such as mathematical physics, nonlinear optimization, optimal control theory, and mathematical economics (see Arrow and Debreu [2], Aubin [3], Aubin and Ekeland [6], Mosco [17], and Shafer and Sonnenschein [21]). Many researchers attempted to generalize this inequality by weakening the conditions of existence of a solution. Among these researchers, we can mention Shih and Tan [22], Tian and Zhou [23, 24], Zhou and Chen [26], and Nessah and Chu [19]. Our work follows this direction of reseach. In this paper, we introduce the generalized g-quasivariational inequality (GgQVI) and provide sufficient conditions for the existence of its solution.

Let E be a metric space and let F be a topological vector space. Let X and Y be nonempty subsets of E and F, respectively, and let 2^X be the family of all nonempty subsets of X. We will denote by F^* the continuous dual of F, by $\mathrm{Re}\langle f, y\rangle$ the real part of pairing between F^* and F for $f \in F^*$ and $y \in F$. Given the functions $g : X \to Y$ and $\psi : X \times Y \to \mathbb{R}$ and the maps $S : X \to 2^Y$ and $T : Y \to 2^{F^*}$, the generalized g-quasivariational inequality problem (GgQVI) is to find a point $\overline{x} \in X$, $g(\overline{x}) \in S(\overline{x})$, and a point $f \in T(g(\overline{x}))$ such that $\sup_{y\in S(\overline{x})}\{\mathrm{Re}\langle f, y - g(\overline{x})\rangle + \psi(\overline{x}, y)\} = \psi(\overline{x},g(\overline{x}))$.

Some particular cases of the (GgQVI) were introduced before: by Chan and Pang [9] in 1982 in the case where $E = F = \mathbb{R}^n$, $g = \mathrm{id}_X$, and $\psi = 0$, by Shih and Tan [22] in 1985 in the case where $E = F$ is infinite dimensional, $g = \mathrm{id}_X$, $\psi = 0$, and by Chowdhury and Tarafdar [10] in the case where $E = F$, $g = \mathrm{id}_X$, and $\psi = 0$.

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Gwinner [14], Ansari et al. [1], Ding et al. [12], and Nessah [18] introduced and studied the following nonlinear inequality problem of finding $\overline{x} \in X$ such that

$$g(\overline{x}) \in C(\overline{x}), \quad \phi(\overline{x}, y) \leq \operatorname{Re}\langle f, y - g(\overline{x})\rangle, \quad \forall y \in C(\overline{x}), \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the pairing between F^* and F, in the case where $E = F$, $X = Y$, $g = \mathrm{id}_X$ and $C(x) = Y$, for all $x \in X$. This problem is equivalent to the problem of solving the GgQVI, where $T(y) = 0$, for all $y \in Y$ and $\psi(x, y) = \phi(x, y) \leq \operatorname{Re}\langle f, y - g(x)\rangle$.

It is to be noted that in all the previous works, it is assumed that the function $\phi(x, y)$ is defined on the cartesian product $X \times X$ of the same set X. In contrast, in GgQVI, the function $\phi(x, y)$ is defined on the cartesian product of two different sets $X \times Y$. This generalization opens more possibilities for applications of the quasivariational inequalities. One of the potential areas of application of the GgQVI is game theory. Indeed, the existence of some equilibria like the strong Berge equilibrium [16] requires a function $\phi(x, y)$ defined on the product of two different sets.

Let us consider the following notations. Let Y be a subset of a topological vector space. Let K be a subset of Y and $x \in K$.

(1) The tangent cone of K in x is defined by

$$T_K(x) = \overline{\bigcup_{h>0} \frac{[K - x]}{h}}. \tag{1.2}$$

(2) The normal cone of K in x is defined by

$$\begin{gathered} N_K(x) = \{p \in X^* \text{ such that } \operatorname{Re}\langle p, v\rangle \leq 0,\ \forall v \in T_K(x)\}, \\ Z_K(x) = [T_K(x) + x] \cap Y. \end{gathered} \tag{1.3}$$

Note that $\overline{A}$ is the closure of the subset A and ∂A is its boundary.

Consider X a nonempty subset of a metrical space E, Y a nonempty subset of a locally convex space F. Let 2^Y be the set of all the parts of Y.

A map $C : X \to 2^Y$ is said to be upper semicontinuous if the set $\{x \in X \text{ such that } C(x) \cap A \neq \emptyset\}$ is closed in X, for all closed set A in Y [25]; it is said to be closed if the corresponding graph is closed in $X \times Y$, that is, the set $\{(x, y) \in X \times Y \text{ such that } y \in C(x)\}$ is closed in $X \times Y$ [5].

A function $f : Y \to \mathbb{R}$ is said to be upper semicontinuous if for all $y_0 \in Y$, for all $\lambda > f(y_0)$, there is a neighborhood v of y_0 such that for all $y \in v$, $\lambda \geq f(y)$; f is said to be continuous if f and $-f$ are upper semicontinuous. We say that f is quasiconcave if for any y_1, y_2 in Y and for any $\theta \in [0, 1]$, we have $\min\{f(y_1), f(y_2)\} \leq f(\theta y_1 + (1 - \theta) y_2)$; f is said to be quasiconvex if $-f$ is quasiconcave.

A function $f : Y \to F^*$ is said to be upper hemicontinuous along line segments in Y if for all y_1, $y_2 \in Y$, the function $z \mapsto \langle f(z), y_2 - y_1\rangle$ is upper semicontinuous on the line segment $[y_1, y_2]$.

We say that the map $C : Y \to 2^Y$ is upper hemicontinuous if for any $p \in Y^*$, function $x \mapsto \sigma(C(x), p) = \sup_{y \in C(x)} \operatorname{Re}\langle p, y\rangle$ is upper semicontinuous on Y.

We say that the map $C : X \to 2^E$ satisfies [4]

(1) the tangential condition if

$$\forall x \in X, \quad C(x) \cap T_X(x) \neq \emptyset, \tag{1.4}$$

where X is assumed to be convex,

(2) the dual tangential condition if

$$\forall x \in X, \quad \forall p \in N_X(x), \quad \text{then } \sigma(C(x), -p) \geq 0. \tag{1.5}$$

We will use the following results.

Lemma 1.1 [4]. *The tangential condition (1.4) implies the dual tangential condition (1.5).*

Lemma 1.2 [15]. *Let X be a nonempty convex subset of a vector space and let Y be a nonempty compact convex subset of a Hausdorff topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that for each $x \in X$, the map $y \mapsto f(x,y)$ is lower semicontinuous and convex on Y and for each fixed $y \in Y$, the map $x \mapsto f(x,y)$ is concave on X. Then,*

$$\min_{y \in Y} \sup_{x \in X} f(x,y) = \sup_{x \in X} \min_{y \in Y} f(x,y). \tag{1.6}$$

Lemma 1.3 [10]. *Let E be a topological vector space, let X be a nonempty convex subset of E, let $h : X \to \mathbb{R}$ be convex, and let $T : X \to 2^{E^*}$ be an upper hemicontinuous along line segments in X. Suppose $\overline{y} \in X$ is such that $\inf_{u \in T(x)} \operatorname{Re}\langle u, \overline{y} - x\rangle \leq h(x) - h(\overline{y})$ for all $x \in X$. Then, $\inf_{u \in T(\overline{y})} \operatorname{Re}\langle u, \overline{y} - x\rangle \leq h(x) - h(\overline{y})$ for all $x \in X$.*

Lemma 1.4 [8]. *Let $C : E \to 2^F$ be a map, where E and F are metric spaces. If the graph of C is compact, then C is upper semicontinuous.*

Lemma 1.5. *Let X be a nonempty, compact set in a metric space E, let Y be a nonempty convex, compact set in a Hausdorff locally convex space F, let g be a continuous function from X into Y, and let C be an upper hemicontinuous set-valued function from X into Y, with $C(x)$ nonempty, closed, and convex. Suppose that the following conditions are met.*

(1) $g(X)$ is convex in Y.

(2) For each $g(x) \in \partial g(X)$, $C(x) \cap Z_{g(X)}(g(x)) \neq \emptyset$.

Then, there exists $\overline{x} \in X$ such that $g(\overline{x}) \in C(\overline{x})$.

Proof. Consider the map Υ defined as follows:

$$\begin{gathered} \Upsilon : g(X) \longrightarrow 2^Y, \\ g(x) \longmapsto \Upsilon(g(x)) = C(x) - g(x). \end{gathered} \tag{1.7}$$

Let us prove that Υ is upper hemicontinuous.

Indeed, let $g(x) \in g(X)$ and $p \in Y^*$, we have

$$\sigma(\Upsilon(g(x)), p) = \sup_{y \in \Upsilon(g(x))} \operatorname{Re}\langle p, y\rangle = \sup_{y \in C(x) - g(x)} \operatorname{Re}\langle p, y\rangle = \sup_{y + g(x) \in C(x)} \operatorname{Re}\langle p, y\rangle. \tag{1.8}$$

Let $z = y + g(x)$, then we obtain $y = z - g(x)$ and

$$\sigma(\Upsilon(g(x)), p) = \sup_{z \in C(x)} \operatorname{Re}\langle p, z - g(x)\rangle = \sup_{z \in C(x)} \operatorname{Re}\langle p, z\rangle - \operatorname{Re}\langle p, g(x)\rangle. \tag{1.9}$$

Then

$$\sigma(\Upsilon(g(x)), p) = \sigma(C(x), p) - \operatorname{Re}\langle p, g(x)\rangle. \tag{1.10}$$

Since C is upper hemicontinuous and p, g are continuous functions, then we conclude that F is upper hemicontinuous. Thus, the map Υ is upper hemicontinuous with nonempty, closed, and convex values. Since g is continuous on the compact X, then Weierstrass theorem implies that $g(X)$ is compact. Taking into account condition (2) of Lemma 1.5 and the fact that for $g(x) \in \operatorname{int} g(X)$, we have $T_{g(X)}(g(x)) = Y$, we obtain $T_{g(X)}(g(x)) \cap \Upsilon(g(x)) \neq \emptyset$, for all $g(x) \in g(X)$. Since $g(X)$ is convex in a Hausdorff locally convex space, then all the conditions of the zero-map theorem [7] are verified for Υ. From this theorem, we deduce that there exists $\bar{x} \in X$ such that $0 \in \Upsilon(g(\bar{x}))$, that is, $g(\bar{x}) \in C(\bar{x})$. $\square$

2. Existence of solution

In the following theorem, we establish a sufficient condition for the existence of a solution of the GgQVI.

Theorem 2.1. *Let*

(1) *X be a nonempty compact subset of a metrical space E,*
(2) *Y a nonempty convex and compact subset of a locally convex Hausdorff topological vector space F,*
(3) *$g : X \to Y$ a continuous function such that $g(X)$ is a compact and convex subset of Y,*
(4) *S an upper hemicontinuous map from X into 2^Y with nonempty, convex, and closed values such that for any $g(x) \in \partial g(X)$, $[S(x) - g(x)] \cap T_{g(X)}(g(x)) \neq \emptyset$,*
(5) *$T : Y \to 2^{F^*}$ an upper hemicontinuous along line segments in X with respect to the weak*-topology on F^* such that each $T(y)$ is weak*-compact convex and the function $y \mapsto \inf_{f \in T(y)} \operatorname{Re}\langle f, y\rangle$ is continuous and quasiconcave on Y,*
(6) *$\psi : X \times Y \to \mathbb{R}$ a function satisfying that*
  (6.1) *ψ is continuous;*
  (6.2) *for any $x \in X$, the function $y \mapsto \psi(x, y)$ is quasiconcave on Y;*
  (6.3) *for any $g(x) \in \partial g(X)$, for any $y \in Y$, and for any $q \in F^*$, there is a $w \in Z_{g(X)}(g(x))$ such that $\inf_{f \in T(y)} \operatorname{Re}\langle f, y - g(x)\rangle + \psi(x, y) \leq \inf_{f \in T(w)} \operatorname{Re}\langle f, g(x) - w\rangle + \psi(x, w)$ and $\operatorname{Re}\langle q, y\rangle \leq \operatorname{Re}\langle q, w\rangle$,*
(7) *V_0 the set*

$$V_0 = \left\{ x \in X \text{ such that } \alpha(x) = \sup_{y \in S(x)} \left\{ \inf_{f \in T(y)} \operatorname{Re}\langle f, y - g(x)\rangle + \psi(x, y)\right\} > \psi(x, g(x))\right\}, \tag{2.1}$$

which must be open.

Then, there exists an $\overline{x} \in X$ such that

$$g(\overline{x}) \in S(\overline{x}), \quad f \in T(g(\overline{x})) \quad \textit{such that } \max_{y\in S(\overline{x})} \{\operatorname{Re}\langle f, y - g(\overline{x})\rangle + \psi(\overline{x}, y)\} = \psi(\overline{x}, g(\overline{x})). \tag{2.2}$$

Proof. We divide the proof into three steps.

Step 1. There exists a point $\overline{x} \in X$ such that $g(\overline{x}) \in S(\overline{x})$ and

$$\sup_{y\in S(\overline{x})} \left\{ \inf_{f\in T(y)} \operatorname{Re}\langle f, y - g(\overline{x})\rangle + \psi(\overline{x}, y)\right\} = \psi(\overline{x}, g(\overline{x})). \tag{2.3}$$

Suppose that (2.3) is not true. Then for each $x \in X$, either $g(x) \notin S(x)$ or $\sup_{y\in S(x)} \{\inf_{f\in T(y)} \operatorname{Re}\langle f, y - g(x)\rangle + \psi(x, y)\} > \psi(x, g(x))$; that is, for each $x \in X$, either $g(x) \notin S(x)$ or $x \in V_0$.

According to separation theorem and considering the fact that $S(x)$ is nonempty, convex, and closed, $g(x) \notin S(x)$ implies that for all $x \in X$, there exists $q \in F^*$ such that

$$\operatorname{Re}\langle -q, g(x)\rangle - \sigma(S(x), -q) > 0, \tag{2.4}$$

where $\sigma(S(x), q) = \sup_{y\in S(x)} \operatorname{Re}\langle -q, y\rangle$ is the support function of $S(x)$.

Let

$$V_q = \{x \in X \text{ such that } \operatorname{Re}\langle -q, g(x)\rangle > \sigma(S(x), -q)\}. \tag{2.5}$$

Assumptions (3), (4), and (7) of Theorem 2.1 imply that the sets V_0, V_q, and $q \in F^*$ are open in E.

The equality (2.3) implies that $X \subset V_0 \cup \bigcup_{q\in F^*} V_q$. Since X is compact, it is possible to cover it by a finite number n of its subsets $\{V_0, V_{q_1}, \ldots, V_{q_n}\}$. Let $\{h_i\}_{i=0,\ldots,n}$ be a continuous partition of unity associated with the subcover $\{V_0, V_{q_1}, \ldots, V_{q_n}\}$.

Let us introduce the function $\Phi : X \times Y \to \mathbb{R}$ defined by

$$(x, y) \longmapsto \Phi(x, y) = h_0(x)\left\{ \inf_{f\in T(y)} \operatorname{Re}\langle f, y - g(x)\rangle + \psi(x, y)\right\} + \sum_{i=1}^{n} h_i(x) \operatorname{Re}\langle q_i, y - g(x)\rangle. \tag{2.6}$$

We now show that there is an $\overline{x} \in X$ such that

$$\sup_{y\in Y} \Phi(\overline{x}, y) = \Phi(\overline{x}, g(\overline{x})). \tag{2.7}$$

Assume that

$$\forall x \in X, \quad \exists y \in Y \quad \text{such that } \Phi(x, y) > \Phi(x, g(x)). \tag{2.8}$$

Consider the following set:

$$\theta_y = \{x \in X \text{ such that } \Phi(x, y) > \Phi(x, g(x))\}, \quad y \in Y. \tag{2.9}$$

Then, for all $y \in Y$, θ_y is open and $X \subset \bigcup_{y \in Y} \theta_y$. Since X is compact, it can be covered by a finite number r of its subsets $\{\theta_{y_1}, \ldots, \theta_{y_r}\}$. Let $\{l_j\}_{j=\overline{1,r}}$ be a continuous partition of unity associated with the subcover $\{\theta_{y_1}, \ldots, \theta_{y_r}\}$; that is, we have for all $x \in X$, $\sum_{j=1}^{r} l_j(x) = 1$ and for all $j = \overline{1,r}$, supp $l_j \subset \theta_{y_j}$.

Consider the map

$$M : X \longrightarrow 2^Y \tag{2.10}$$

defined by

$$x \longmapsto M(x) = \left\{ y \in Y \text{ such that } \max_{\lambda \in S} \sum_{i=1}^{r} \lambda_i \Phi(x, y_i) \le \Phi(x, y) \right\}, \tag{2.11}$$

where

$$S = \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^r \text{ such that } \sum_{i=1}^{r} \lambda_i = 1,\ \lambda_i \ge 0,\ \forall i = \overline{1,r} \right\}. \tag{2.12}$$

We now show that the map M is upper semicontinuous on X, with nonempty, convex, and closed values in Y and satisfying that for all $g(x) \in \partial g(X)$, there exists $u \in X$, there exists $\alpha > 0$ such that $\alpha g(u) + (1-\alpha) g(x) \in M(x)$.

(1) Let us prove that for all $x \in X$, $M(x) \neq \emptyset$.

Consider a point $x \in X$, the function $\lambda \mapsto \sum_{i=1}^{r} \lambda_i \Phi(x, y_i)$ is linear on $\mathbb{R}^r$. Therefore, it is continuous over the compact set S and according to the theorem of Weierstrass [5], there exists $\overline{\lambda} \in S$ such that

$$\max_{\lambda \in S} \sum_{i=1}^{r} \lambda_i \Phi(x, y_i) = \sum_{i=1}^{r} \overline{\lambda}_i \Phi(x, y_i) \le \sum_{i=1}^{r} \overline{\lambda}_i \max_{i=\overline{1,r}} \Phi(x, y_i) = \Phi(x, y_{i_0}). \tag{2.13}$$

Therefore, $y_{i_0} \in M(x)$, which implies that $M(x) \neq \emptyset$.

(2) For all $x \in X$, $M(x)$ is closed in Y.

Consider $x \in X$ and $z \in \overline{M(x)}$. There is a sequence $\{z_k\}_{k \ge 1}$ of elements of $M(x)$ which converges to z.

As a consequence of the fact that for all $k \ge 1$, $z_k \in M(x)$, we get

$$\forall k \ge 1, \quad \max_{\lambda \in S} \sum_{i=1}^{r} \lambda_i \Phi(x, y_i) \le \Phi(x, z_k). \tag{2.14}$$

Taking into account condition (6.1) of Theorem 2.1 and the fact that $p_i \in Y^*$, $i = \overline{1,r}$ with $k \to +\infty$, we obtain

$$\max_{\lambda \in S} \sum_{i=1}^{r} \lambda_i \Phi(x, y_i) \le \Phi(x, z). \tag{2.15}$$

Therefore, $z \in M(x)$, that is, $M(x)$ is closed.

(3) For all $x \in X$, $M(x)$ is convex in Y.

Let $x \in X$ and let $\overline{z}, \overline{\overline{z}}$ be two elements of $M(x)$ and $\theta \in [0,1]$.

We now show that $\theta \overline{z} + (1-\theta) \overline{\overline{z}} \in M(x)$.

Since $\overline{z}$ and $\overline{\overline{z}}$ are two elements of $M(x)$, we have $\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i)\leq\Phi(x,\overline{z})$ and $\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i)\leq\Phi(x,\overline{\overline{z}})$. Therefore,

$$\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i)\leq\min\{\Phi(x,\overline{z}),\ \Phi(x,\overline{\overline{z}})\}. \tag{2.16}$$

Taking into account condition (6.2) of Theorem 2.1, the fact that $p_i\in Y^*$, $i=\overline{1,r}$, and inequality (2.16), we obtain

$$\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i)\leq\Phi(x,\theta\overline{z}+(1-\theta)\overline{\overline{z}}),\quad\forall\theta\in[0,1], \tag{2.17}$$

that is, $\theta\overline{z}+(1-\theta)\overline{\overline{z}}\in M(x)$.

(4) M is upper semicontinuous.

According to Lemma 1.2, it is sufficient to show that the graph of M is closed in the compact set $X\times Y$.

Let $(x,z)\in\overline{\text{Graph}(M)}$. There is a sequence $\{(x_k,z_k)\}_{k\geq1}$ of elements of Graph(M) which converges to (x,z). Therefore, for all $k\geq1$, $z_k\in M(x_k)$; that is, for all $k\geq1$,

$$\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x_k,y_i)\leq\Phi(x_k,z_k). \tag{2.18}$$

Taking into account condition (6.1) of Theorem 2.1 and the fact that $p_i\in Y^*$, $i=\overline{1,r}$, when $k\to\infty$, we obtain $\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i)\leq\Phi(x,z)$; that is, $z\in M(x)$. Hence, $(x,z)\in\text{Graph}(M)$. In other words, Graph(M) is closed.

(5) For all $g(x)\in\partial g(X)$, there exists $\alpha>0$, there exists $u\in X$ such that $\alpha g(u)+(1-\alpha)g(x)\in M(x)$.

Let $g(x)\in\partial g(X)$. It is shown in (1) that for all $x\in X$, there exists $y_{i_0}\in Y$ such that

$$\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i)\leq\Phi(x,y_{i_0}). \tag{2.19}$$

(In particular, (2.19) remains true for any $x\in X$ such that $g(x)\in\partial g(X)$.)

Condition (6.3) of Theorem 2.1 implies that there exists $\alpha>0$, there exists $u\in X$ such that $\Phi(x,y_{i_0})\leq\Phi(x,\alpha g(u)+(1-\alpha)g(x))$ with $\alpha g(u)+(1-\alpha)g(x)\in Y$.Therefore,

$$\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i)\leq\Phi(x,\alpha g(u)+(1-\alpha)g(x)), \tag{2.20}$$

that is, $\alpha g(u)+(1-\alpha)g(x)\in M(x)$.

From (1)–(5), we deduce that M satisfies all conditions of Lemma 1.5. Hence, there exists a point $\overline{x}\in X$ such that $g(\widetilde{x})\in M(\widetilde{x})$; that is,

$$\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(\widetilde{x},y_i)\leq\Phi(\widetilde{x},g(\widetilde{x})). \tag{2.21}$$

Thus, for all $\lambda\in S$, $\sum_{i=1}^{r}\lambda_i\Phi(\widetilde{x},y_i)\leq\Phi(\widetilde{x},g(\widetilde{x}))$.

Let $\tilde{\lambda} = (l_1(\tilde{x}),\dots,l_r(\tilde{x}))$. We have $\tilde{\lambda} \in S$ since $l_i(\tilde{x}) \geq 0$ and $\sum_{i=1}^{r} l_i(\tilde{x}) = 1$, then

$$\sum_{i=1}^{r} l_i(\tilde{x})\Phi(\tilde{x}, y_i) \leq \Phi(\tilde{x}, g(\tilde{x})). \tag{2.22}$$

Consider the set $J = \{i \in \{1,\dots,r\}$ such that $l_i(\tilde{x}) > 0\}$. By construction, $J \neq \emptyset$.

Note that $\sum_{i=1}^{r} l_i(\tilde{x})\Phi(\tilde{x}, y_i) = \sum_{i\in J} l_i(\tilde{x})\Phi(\tilde{x}, y_i)$.

We have for all $i \in J$, $l_i(\tilde{x}) > 0$. Therefore, $\tilde{x} \in \operatorname{supp} l_i \subset \theta_{y_i}$, for all $i \in J$, that is, for all $i \in J$, $\Phi(\tilde{x}, y_i) > \Phi(\tilde{x}, g(\tilde{x}))$.

Then, we have $\sum_{i\in J} l_i(\tilde{x})\Phi(\tilde{x}, y_i) > \sum_{i\in J} l_i(\tilde{x})\Phi(\tilde{x}, g(\tilde{x})) = \Phi(\tilde{x}, g(\tilde{x}))$, that is, $\Phi(\tilde{x}, g(\tilde{x})) < \Phi(\tilde{x}, g(\tilde{x}))$, which is impossible.

Thus, we conclude that there exists $\bar{x} \in X$ such that $\sup_{y\in Y} \Phi(\bar{x}, y) = \Phi(\bar{x}, g(\bar{x}))$, that is, for all $y \in Y$, we have

$$h_0(\bar{x})\left\{\inf_{f\in T(y)} \operatorname{Re}\langle f, y - g(\bar{x})\rangle + \psi(\bar{x}, y)\right\} + \sum_{i=1}^{n} h_i(\bar{x}) \operatorname{Re}\langle q_i, y - g(\bar{x})\rangle \leq h_0(\bar{x})\psi(\bar{x}, g(\bar{x})). \tag{2.23}$$

If $h_0(\bar{x}) = 0$, we have $\sum_{i=1}^{n} h_i(\bar{x}) = 1$. Therefore, (2.23) becomes

$$\sum_{i=1}^{n} h_i(\bar{x}) \operatorname{Re}\langle q_i, y - g(\bar{x})\rangle \leq 0, \quad \forall y \in Y. \tag{2.24}$$

Inequality (2.24) implies that $\bar{q} = \sum_{i=1}^{n} h_i(\bar{x}) q_i$ belongs to the *normal cone* $N_{g(X)}(g(\bar{x}))$. According to Lemma 1.1 and condition (4) of Theorem 2.1, we have

$$\sigma(S(\bar{x}), -\bar{q}) \geq \operatorname{Re}\langle -\bar{q}, g(\bar{x})\rangle. \tag{2.25}$$

The fact that $h_i(\bar{x}) > 0$, $i = 1,\dots,n$, implies that $\bar{x} \in \operatorname{supp} h_i \subset V_{q_i}$, that is,

$$\operatorname{Re}\langle -q_i, g(\bar{x})\rangle > \sigma(S(\bar{x}), -q_i). \tag{2.26}$$

Then,

$$\begin{aligned}\sigma(S(\bar{x}), -\bar{q}) = \sigma\left(S(\bar{x}), -\sum_{i=1}^{n} h_i(\bar{x}) q_i\right) &\leq \sum_{i=1}^{n} h_i(\bar{x})\sigma(S(\bar{x}), -q_i)\\ &< \sum_{i=1}^{n} h_i(\bar{x}) \operatorname{Re}\langle -q_i, g(\bar{x})\rangle = \operatorname{Re}\langle -\bar{q}, g(\bar{x})\rangle,\end{aligned} \tag{2.27}$$

which contradicts inequality (2.25). We then conclude that $h_0(\bar{x}) > 0$.

The inequality $h_0(\bar{x}) > 0$ implies that $\bar{x} \in \operatorname{supp} h_0 \subset V_0$. Therefore,

$$h_0(\bar{x}) \sup_{y\in S(\bar{x})} \left\{\inf_{f\in T(y)} \operatorname{Re}\langle f, y - g(\bar{x})\rangle + \psi(\bar{x}, y)\right\} > h_0(\bar{x})\psi(\bar{x}, g(\bar{x})). \tag{2.28}$$

Since the function $y \mapsto \inf_{f\in T(y)} \operatorname{Re}\langle f, y - g(\bar{x})\rangle + \psi(\bar{x}, y)$ is continuous on the compact $S(\bar{x})$, it follows that according to Weierstrass theorem [5], there exists $\bar{y} \in S(\bar{x})$ such that $\sup_{y\in S(\overline{x})}\{\inf_{f\in T(y)}\operatorname{Re}\langle f, y-g(\overline{x})\rangle+\psi(\overline{x},y)\} = \inf_{f\in T(\overline{y})}\operatorname{Re}\langle f,\overline{y}-g(\overline{x})\rangle+\psi(\overline{x},\overline{y})$. Therefore,

$$h_0(\overline{x})\left\{\inf_{f\in T(\overline{y})}\operatorname{Re}\langle f,\overline{y}-g(\overline{x})\rangle+\psi(\overline{x},\overline{y})\right\} > h_0(\overline{x})\psi(\overline{x},g(\overline{x})). \tag{2.29}$$

If $\sum_{i=1}^n h_i(\overline{x})=0$, (2.23) becomes $h_0(\overline{x})\{\inf_{f\in T(y)}\operatorname{Re}\langle f,y-g(\overline{x})\rangle+\psi(\overline{x},y)\}\le h_0(\overline{x})\psi(\overline{x}, g(\overline{x}))$, for all $y\in Y$, which contradicts inequality (2.29). Therefore, $\sum_{i=1}^n h_i(\overline{x})>0$. Let $K=\{i\in\{1,\dots,n\}/h_i(\overline{x})>0\}$, then $K\neq\emptyset$. If $i\in K$, then $\overline{x}\in\operatorname{supp}h_i\subset V_{q_i}$; that is,

$$\operatorname{Re}\langle -q_i,g(\overline{x})\rangle > \sigma(S(\overline{x}),-q_i). \tag{2.30}$$

We have

$$\begin{aligned}\operatorname{Re}\langle-\overline{q},\overline{y}\rangle &\le \sigma(S(\overline{x}),-\overline{q}) = \sigma\left(S(\overline{x}),-\sum_{i=1}^n h_i(\overline{x})q_i\right)\\ &\le \sum_{i=1}^n h_i(\overline{x})\sigma(S(\overline{x}),-q_i)\\ &< \sum_{i\in K} h_i(\overline{x})\operatorname{Re}\langle -q_i,g(\overline{x})\rangle\\ &= \operatorname{Re}\langle -\overline{q},g(\overline{x})\rangle.\end{aligned} \tag{2.31}$$

Thus,

$$\sum_{i=1}^n h_i(\overline{x})\operatorname{Re}\langle q_i,\overline{y}-g(\overline{x})\rangle > 0. \tag{2.32}$$

Inequalities (2.29) and (2.32) imply that

$$h_0(\overline{x})\left\{\inf_{f\in T(\overline{y})}\operatorname{Re}\langle f,\overline{y}-g(\overline{x})\rangle+\psi(\overline{x},\overline{y})\right\}+\sum_{i=1}^n h_i(\overline{x})\operatorname{Re}\langle q_i,\overline{y}-g(\overline{x})\rangle > h_0(\overline{x})\psi(\overline{x},g(\overline{x})), \tag{2.33}$$

which contradicts (2.23). This contradiction proves the statement of Step 1.

Step 2. We have

$$\inf_{f\in T(g(\overline{x}))}\operatorname{Re}\langle f,y-g(\overline{x})\rangle+\psi(\overline{x},y)\le\psi(\overline{x},g(\overline{x})),\quad \forall y\in S(\overline{x}). \tag{2.34}$$

Indeed, from Step 1, $g(\overline{x})\in S(\overline{x})$ and $S(\overline{x})$ is a convex subset of X. We have also

$$\inf_{f\in T(y)}\operatorname{Re}\langle f,y-g(\overline{x})\rangle+\psi(\overline{x},y)\le\psi(\overline{x},g(\overline{x})),\quad \forall y\in S(\overline{x}). \tag{2.35}$$

Hence, by assumption (6.2) of Theorem 2.1 and Lemma 1.3, we have

$$\inf_{f\in T(g(\overline{x}))}\operatorname{Re}\langle f,y-g(\overline{x})\rangle+\psi(\overline{x},y)\le\psi(\overline{x},g(\overline{x})),\quad \forall y\in S(\overline{x}). \tag{2.36}$$

Step 3. There exists a function $f \in T(\overline{x})$ such that $\mathrm{Re}\langle f, y - g(\overline{x})\rangle + \psi(\overline{x}, y) \leq \psi(\overline{x}, g(\overline{x}))$, for all $y \in S(\overline{x})$.

From Step 2, we have $\sup_{y \in S(\overline{x})}\{\inf_{f \in T(g(\overline{x}))} \mathrm{Re}\langle f, y - g(\overline{x})\rangle + \psi(\overline{x}, y)\} = \psi(\overline{x}, g(\overline{x}))$, where $T(g(\overline{x}))$ is a weak*-compact convex subset of the Hausdorff topological vector space F^* and $S(\overline{x})$ is a compact convex subset of X.

Indeed, define $\digamma : S(\overline{x}) \times T(g(\overline{x})) \to \mathbb{R}$ by $\digamma(y, f) = \mathrm{Re}\langle f, y - g(\overline{x})\rangle + \psi(\overline{x}, y)$ for all $y \in S(\overline{x})$ and for all $f \in T(g(\overline{x}))$. For each $y \in S(\overline{x})$, the function $f \mapsto \digamma(f, y)$ is linear and continuous on $T(g(\overline{x}))$ and for each $f \in T(g(\overline{x}))$, the function $y \mapsto \digamma(f, y)$ is quasiconcave on $S(\overline{x})$. Thus by Lemma 1.2, we have

$$\min_{f \in T(g(\overline{x}))} \max_{y \in S(\overline{x})} \digamma(f, y) = \max_{y \in S(\overline{x})} \min_{f \in T(g(\overline{x}))} \digamma(f, y). \tag{2.37}$$

Hence,

$$\min_{f \in T(g(\overline{x}))} \max_{y \in S(\overline{x})} \{\mathrm{Re}\langle f, y - g(\overline{x})\rangle + \psi(\overline{x}, y)\} = \psi(\overline{x}, g(\overline{x})). \tag{2.38}$$

Since $T(g(\overline{x}))$ is compact, there exists $f \in T(g(\overline{x}))$ such that $\mathrm{Re}\langle f, y - g(\overline{x})\rangle + \psi(\overline{x}, y) \leq \psi(\overline{x}, g(\overline{x}))$, for all $y \in S(\overline{x})$. □

Remark 2.2. If we consider $X = Y$, and $g = \mathrm{id}_X$, then [10, Theorem 3.1] becomes a particular case of Theorem 2.1.

From Theorem 2.1, we deduce the following quasivariational equation theorem [19].

Corollary 2.3. *Assume that*

(1) *X is a nonempty compact subset of a metric space E,*
(2) *Y is a nonempty convex and compact subset of a locally convex Hausdorff topological vector space F,*
(3) *$g : X \to Y$ is a continuous function such that $g(X)$ is convex,*
(4) *C is an upper hemicontinuous map from X into 2^Y with nonempty, convex, and closed values such that for any $g(x) \in \partial g(X)$, $[C(x) - g(x)] \cap T_{g(X)}(g(x)) \neq \emptyset$,*
(5) *$\Psi : X \times Y \to \mathbb{R}$ is a function satisfying that*
  (5.1) *Ψ is continuous;*
  (5.2) *for any $x \in X$, the function $y \mapsto \Psi(x, y)$ is quasiconcave on Y;*
  (5.3) *for any $g(x) \in \partial g(X)$, for any $y \in Y$, and for any $p \in Y^*$, there exists $w \in Z_{g(X)}(g(x))$ such that*
    (5.3.1) $\Psi(x, y) \leq \Psi(x, w)$,
    (5.3.2) $\mathrm{Re}\langle p, y\rangle \leq \mathrm{Re}\langle p, w\rangle$,
(6) *the set $\{x \in X : \alpha(x) = \sup_{y \in C(x)} \Psi(x, y) \leq \Psi(x, g(x))\}$ is closed.*

Then there exists $\overline{x} \in X$ such that

$$g(\overline{x}) \in C(\overline{x}), \qquad \sup_{y \in C(\overline{x})} \Psi(\overline{x}, y) = \Psi(\overline{x}, g(\overline{x})). \tag{2.39}$$

Proof. It is sufficient to consider $T : Y \to 2^{F^*}$ such that $T(y) = 0$, for all $y \in Y$, where $0(z) = \langle 0, z\rangle = 0$, for all $z \in F$. □

From Theorem 2.1, we deduce the following theorem [18].

Corollary 2.4. *Let X be a nonempty compact subset of a metric space E, let Y be nonempty convex and compact subset of a locally convex separated space F, and let f be a nonzero continuous linear functional on F. Assume that*

(1) *$g : X \to Y$ is a continuous function such that $g(X)$ is convex over Y,*

(2) *C is an upper hemicontinuous set-valued function from X into 2^Y with nonempty, convex, and closed values such that for any $g(x) \in \partial g(X)$, $[C(x) - g(x)] \cap T_{g(X)}(g(x)) \neq \emptyset$,*

(3) *$\phi : X \times Y \to \mathbb{R}$ is a function satisfying that*

(3.1) *ϕ is continuous over $X \times Y$ and $\phi(x, g(x)) = 0$, for all $x \in X$;*

(3.2) *for all $x \in X$, the function $y \mapsto \phi(x, y)$ is quasiconcave on Y;*

(3.3) *for any $g(x) \in \partial g(X)$, for all $y \in Y$, and for all $p \in Y^*$, there exists $w \in Z_{g(X)}(g(x))$ such that*

(3.3.1) $\phi(x, y) - \operatorname{Re}\langle f, y - g(x)\rangle \leq \phi(x, w) - \operatorname{Re}\langle f, w - g(x)\rangle$,

(3.3.2) $\operatorname{Re}\langle p, y\rangle \leq \operatorname{Re}\langle p, w\rangle$,

(4) *the set $\{x \in X \text{ such that } \alpha(x) = \sup_{y \in C(x)} \phi(x, y) - \operatorname{Re}\langle f, y - g(x)\rangle \leq \phi(x, g(x))\}$ is closed.*

Then there exists $\overline{x} \in X$ such that

$$g(\overline{x}) \in C(\overline{x}), \quad \phi(\overline{x}, y) \leq \operatorname{Re}\langle f, y - g(\overline{x})\rangle, \quad \forall y \in C(\overline{x}). \tag{2.40}$$

Proof. Assume that in Theorem 2.1 we have $\psi(x, y) = \phi(x, y) - \operatorname{Re}\langle f, y - g(x)\rangle$ and $T : Y \to 2^{F^*}$ such that $T(y) = 0$, for all $y \in Y$. Then Corollary 2.4 follows immediately from Theorem 2.1. □

From Theorem 2.1, we deduce the following g-maximum equality theorem [20].

Corollary 2.5 [20] (g-maximum equality theorem). *Assume that*

(1) *X is a nonempty, compact subset of a metric space E,*

(2) *Y is a nonempty, convex, and compact subset of a separated locally convex space F,*

(3) *$g : X \to Y$ is a continuous function such that $g(X)$ is compact and convex in Y,*

(4) *$\Psi : X \times Y \to \mathbb{R}$ is a function satisfying*

(4.1) *Ψ is continuous;*

(4.2) *for any $x \in X$, the function $y \mapsto \Psi(x, y)$ is quasiconcave on Y;*

(4.3) *for all $g(x) \in \partial g(X)$ and for all $y \in Y$, there exists $z \in Z_{g(X)}(g(x))$ such that $\Psi(x, y) \leq \Psi(x, z)$.*

Then there exists $\overline{x} \in X$ such that

$$\sup_{y \in Y} \Psi(\overline{x}, y) = \Psi(\overline{x}, g(\overline{x})). \tag{2.41}$$

Remark 2.6. Corollary 2.5 (g-maximum equality theorem) is a generalization of the minimax inequality (see Fan [13]).

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