ON RIEMANNIAN MANIFOLDS ENDOWED WITH A LOCALLY CONFORMAL COSYMPLECTIC STRUCTURE

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We deal with a locally conformal cosymplectic manifold $M(\phi,\Omega,\xi,\eta,g)$ admitting a conformal contact quasi-torse-forming vector field T. The presymplectic 2-form Ω is a locally conformal cosymplectic 2-form. It is shown that T is a 3-exterior concurrent vector field. Infinitesimal transformations of the Lie algebra of $\wedge M$ are investigated. The Gauss map of the hypersurface M_ξ normal to ξ is conformal and $M_\xi \times M_\xi$ is a Chen submanifold of $M \times M$.

1. Introduction

Locally conformal cosymplectic manifolds have been investigated by Olszak and Rosca [7] (see also [6]).

In the present paper, we consider a $(2m+1)$-dimensional Riemannian manifold $M(\phi,\Omega,\xi,\eta,g)$ endowed with a locally conformal cosymplectic structure. We assume that M admits a principal vector field (or a conformal contact quasi-torse-forming), that is,

$$\nabla T = s\,dp + T \wedge \xi = s\,dp + \eta \otimes T - T^\flat \otimes \xi, \tag{1.1}$$

with $ds = s\eta$.

First, we prove certain geometrical properties of the vector fields T and ϕT. The existence of T and ϕT is determined by an exterior differential system in involution (in the sense of Cartan [3]).

The principal vector field T is 3-exterior concurrent (see also [8]), it defines a Lie relative contact transformation of the co-Reeb form η, and the Lie differential of $T^\flat$ with respect to T is conformal to $T^\flat$. The vector field ϕT is an infinitesimal transformation of generators T and ξ. The vector fields ξ, T, and ϕT commute and the distribution $D_T = \{T,\phi T,\xi\}$ is involutive. The divergence and the Ricci curvature of T are computed.

Next, we investigate infinitesimal transformations on the Lie algebra of $\wedge M$.

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In the last section, we study the hypersurface M_ξ normal to ξ. We prove that M_ξ is Einsteinian, its Gauss map is conformal, and the product submanifold $M_\xi \times M_\xi$ in $M \times M$ is a $\mathcal{U}$-submanifold in the sense of Chen.

2. Preliminaries

Let (M,g) be an n-dimensional Riemannian manifold endowed with a metric tensor g. Let ΓTM and $\flat : TM \to T^*M, Z \mapsto Z^\flat$ be the set of sections of the tangent bundle TM and the musical isomorphism defined by g, respectively. Following a standard notation, we set $A^q(M,TM) = \mathrm{Hom}(\Lambda^q TM, TM)$ and notice that the elements of $A^q(M,TM)$ are the vector-valued q-forms ($q \le n$) (see also [9]). Denote by $d^\nabla : A^q(M,TM) \to A^{q+1}(M,TM)$ the exterior covariant derivative operator with respect to the Levi-Civita connection ∇. It should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$. If $dp \in A^1(M,TM)$ denotes the soldering form on M, one has $d^\nabla(dp) = 0$.

The cohomology operator d^ω acting on ΛM is defined by $d^\omega \gamma = d\gamma + \omega \wedge \gamma$, where ω is a closed 1-form. If $d^\omega \gamma = 0$, γ is said to be d^ω-closed.

Let R be the curvature operator on M. Then, for any vector field Z on M, the second covariant differential is defined as

$$\nabla^2 Z = d^\nabla(\nabla Z) \in A^2(M,TM) \tag{2.1}$$

and satisfies

$$\nabla^2 Z(V,W) = R(V,W)Z, \quad Z,V,W \in \Gamma TM. \tag{2.2}$$

Let $O = \mathrm{vect}\{e_A \mid A = 1,\dots,n\}$ be an adapted local field of orthonormal frames over M and let $O^* = \mathrm{covect}\{\omega^A\}$ be its associated coframe. With respect to O and O^*, É. Cartan's structure equation can be written, in the indexless manner, as

$$\begin{aligned} \nabla e &= \theta \otimes e \in A^1(M,TM), \\ d\omega &= -\theta \wedge \omega, \\ d\theta &= -\theta \wedge \theta + \Theta. \end{aligned} \tag{2.3}$$

In the above equations, θ, respectively, Θ are the local connection forms in the bundle $\mathbb{O}(M)$, respectively, the curvature forms on M.

3. Locally conformal cosymplectic structure

Let $M(\phi,\Omega,\xi,\eta,g)$ be a $(2m+1)$-dimensional Riemannian manifold carrying a quintuple of structure tensor fields, where ϕ is an automorphism of the tangent bundle TM, Ω a presymplectic form of rank $2m$, ξ the Reeb vector field, and $\eta = \xi^\flat$ the associated Reeb covector, g the metric tensor.

We assume in the present paper that η is closed and λ is a scalar ($\lambda \in \Lambda^0 M$) such that $d\lambda = \lambda'\eta$, with $\lambda' \in \Lambda^0 M$.

We agree to denominate the manifold M a *locally conformal cosymplectic manifold* if it satisfies

$$
\begin{gathered}
\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1, \\
\nabla \xi = \lambda(dp - \eta \otimes \xi), \\
d\lambda = \lambda' \eta, \\
\Omega(Z, Z') = g(\phi Z, Z'), \qquad \Omega^m \wedge \eta \neq 0,
\end{gathered} \tag{3.1}
$$

where $dp \in A^1(M, TM)$ denotes the canonical vector-valued 1-form (or the soldering form [5]) on M. Then Ω is called the fundamental 2-form on M and is expressed by

$$
\Omega = \sum_{i=1}^{m} \omega^i \wedge \omega^{i^*}, \quad i^* = i + m. \tag{3.2}
$$

By the well-known relations

$$
\theta_j^i = \theta_{j^*}^{i^*}, \quad \theta_j^{i^*} = \theta_i^{j^*}, \quad i^* = i + m, \tag{3.3}
$$

one derives by differentiation of Ω

$$
d^{-2\lambda\eta}\Omega = 0 \quad (d\Omega = 2\lambda\eta \wedge \Omega), \tag{3.4}
$$

which shows that the presymplectic 2-form Ω is a locally conformal cosymplectic form. Operating on ϕdp by d^∇, it follows that

$$
d^\nabla(\phi\, dp) = 2\lambda\Omega \otimes \xi + 2\eta \wedge \phi\, dp. \tag{3.5}
$$

On the other hand, we agree to call a vector field T, such that

$$
\nabla T = s\, dp + T \wedge \xi = s\, dp + \eta \otimes T - T^\flat \otimes \xi, \tag{3.6}
$$

a *principal vector field* on M, or a *conformal contact quasi-torse-forming* if

$$
ds = s\eta. \tag{3.7}
$$

In these conditions, since the qth covariant differential ∇^q of a vector field $Z \in \Gamma TM$ is defined inductively, that is, $\nabla^q Z = d^\nabla(\nabla^{q-1} Z)$, one derives from (3.6)

$$
\nabla^4 T = -\lambda^3 \eta \wedge T^\flat \otimes dp. \tag{3.8}
$$

As a natural concept of concurrent vector fields and by reference to [8], this proves that T is a 3-exterior concurrent vector field.

Since, as it is known, the divergence of a vector field Z is defined by

$$
\operatorname{div} Z = \sum_A g(\nabla_{e_A} Z, e_A), \tag{3.9}
$$

one derives from (3.2) and (3.6)

$$\operatorname{div}\xi = 2m\lambda, \qquad \operatorname{div}T = T^0 + (2m+1)s, \tag{3.10}$$

where $T^0 = \eta(T)$. On the other hand, from (3.6), we derive

$$dT^a + T^b\theta_b^a + \lambda T^0\omega^a = s\omega^a + T^a\eta, \quad a,b \in \{1,\dots,2m\}, \qquad dT^0 = -(1+\lambda)T^\flat + [s+(1+\lambda)T^0]\eta. \tag{3.11}$$

After some calculations, one gets

$$dT^\flat = \lambda\, dT^0 \wedge \eta = \lambda(1+\lambda)\eta \wedge T^\flat, \tag{3.12}$$

which proves that $T^\flat$ is an exterior recurrent form [1].

Taking the Lie differential of η with respect of T, one gets

$$\mathcal{L}_T\eta = dT^0, \tag{3.13}$$

and so it turns out that

$$d(\mathcal{L}_T\eta) = 0. \tag{3.14}$$

Following a known terminology, T defines a relative contact transformation of the co-Reeb form η.

Next, we will point out some properties of the vector field ϕT.

By virtue of (3.11), one derives

$$\nabla\phi T = (s - \lambda T^0)\phi\, dp + \phi T \otimes \eta, \tag{3.15}$$

and so, by (3.6) and (3.2), one gets

$$[\phi T, T] = -\lambda T^0\phi T, \qquad [\phi T, \xi] = (1-\lambda)\phi T, \qquad [T,\xi] = 0. \tag{3.16}$$

The above relations prove that ϕT admits an infinitesimal transformation of generators T and ξ. In addition, it is seen that ξ and the principal vector field T commute and that the distribution $D_T = \{T, \phi T, \xi\}$ is involutive.

By Orsted lemma [1], if one takes

$$\mathcal{L}_T T^\flat = \rho T^\flat + [T,\xi]^\flat, \tag{3.17}$$

one gets at once by (3.16)

$$\mathcal{L}_T T^\flat = \rho T^\flat, \tag{3.18}$$

which shows that the Lie differential of $T^\flat$ with respect to the principal vector field T is conformal to $T^\flat$.

Moreover, making use of the contact ϕ-Lie derivative operator $(\mathcal{L}_\xi\phi)Z = [\xi,\phi] - \phi[\xi, Z]$, one gets in the case under discussion

$$(\mathcal{L}_\xi\phi)T = (\lambda - 1)\phi T. \tag{3.19}$$

Hence, ξ defines a ϕ-Lie transformation of the principal vector field T.

It is worth to point out that the existence of T and ϕT is determined by an exterior differential system $\sum$ whose characteristic numbers are $r = 3$, $s_0 = 1$, $s_1 = 2$ $(r = s_0 + s_1)$. Consequently, the system $\sum$ is in *involution* (in the sense of Cartan [3]) and so T and ϕT depend on 1 arbitrary function of 2 arguments (É. Cartan's test).

Recall Yano's formula for any vector field Z, that is,

$$\operatorname{div}(\nabla_Z Z) - \operatorname{div}(\operatorname{div} Z)Z = \mathcal{R}(Z,Z) - (\operatorname{div} Z)^2 + \sum_{A,B}(\nabla_{e_A}Z, e_B)(\nabla_{e_B}Z, e_A), \tag{3.20}$$

where $\mathcal{R}$ denotes the Ricci tensor.

Then, since one has

$$\begin{aligned} \operatorname{div} T &= T^0 + (2m+1)s, \\ \nabla_T T &= (s + T^0)T - \|T\|^2\xi, \end{aligned} \tag{3.21}$$

it follows by (3.20) that the Ricci tensor corresponding to T is expressed by

$$\mathcal{R}(T,T) = (s + T^0)(T^0 + (2m+1)s) - 4m^2 - s^2. \tag{3.22}$$

Finally, in the same order of ideas, since one has $i_{\phi T}T^\flat = 0$, then, by the Lie differentiation, one derives $\mathcal{L}_{\phi T}T^\flat = 0$, which shows that ϕT defines a Lie Pfaffian transformation of the dual form of the vector field T.

Besides, by the Ricci identity involving the triple T, ϕT, ξ, that is,

$$(\mathcal{L}_\xi g)(T,\phi T) = g(\nabla_\xi T, \phi T) + g(T, \nabla_\xi \phi T), \tag{3.23}$$

one gets $(\mathcal{L}_\xi g)(T,\phi T) = 0$.

Hence, one may say that the Lie structure vanishes.

Thus, we have the following.

Theorem 3.1. *Let $M(\phi,\Omega,\xi,\eta,g)$ be a $(2m+1)$-dimensional Riemannian manifold endowed with a locally conformal cosymplectic structure and a principal vector field T defined as a conformal contact quasi-torse-forming and structure scalar λ.*

The following properties hold.

(i) *Ω is a locally conformal cosymplectic 2-form.*

(ii) *The principal vector field T is 3-exterior concurrent, that is,*

$$\nabla^4 T = -\lambda^3 \eta \wedge T^\flat \otimes dp. \tag{3.24}$$

(iii) *T defines a Lie relative contact transformation of the co-Reeb form η.*

(iv) *ϕT is an infinitesimal transformation of generators T and ξ. The vector fields ξ, T, and ϕT commute and the distribution $D_T = \{T, \phi T, \xi\}$ is involutive.*
(v) *The Lie differential of $T^\flat$ with respect to T is conformal to $T^\flat$.*
(vi) $\operatorname{div} T = T^0 + (2m+1)s$.
(vii) *The Ricci tensor corresponding to T is expressed by*

$$\mathcal{R}(T,T) = (s+T^0)(T^0+(2m+1)s) - 4m^2 - s^2. \tag{3.25}$$

(viii) *The dual form $T^\flat$ of T is an exterior recurrent form.*

4. Conformal symplectic form

We will point out some problems regarding the conformal symplectic form Ω. Taking the Lie differential of Ω with respect to the Reeb vector field ξ, we quickly get

$$d(\mathcal{L}_\xi \Omega) = 2\lambda\Omega. \tag{4.1}$$

Hence, we may say that ξ defines a *conformal Lie derivative* of Ω.

Next, taking the Lie differential of Ω with respect to the vector field ϕT, one gets in two steps

$$\mathcal{L}_{\phi T}\Omega = d(T^0\eta - T^\flat), \tag{4.2}$$

and, by (3.12), one derives at once

$$d(\mathcal{L}_{\phi T}\Omega) = 0. \tag{4.3}$$

Consequently, from above, we may state that the vector field ϕT defines a relative almost-Pfaffian transformation of the form Ω (see [6]).

In the same order of ideas, one derives after some longer calculations

$$d(\mathcal{L}_T\Omega) = 2\lambda\eta \wedge d(\phi T)^\flat - 2\lambda(1+\lambda)T^\flat \wedge \Omega + [s + (1+s)T^0 + 4\lambda^2 T^0]\eta \wedge \Omega, \tag{4.4}$$

and we may say that the principal vector field T defines a *Lie almost-conformal transformation*of Ω.

Finally, we agree to define the 3-form

$$\psi = T^\flat \wedge \Omega, \tag{4.5}$$

the *principal* 3-*form* on the manifold M under consideration.

Making use of (3.4) and (3.12), one derives

$$d\psi = \lambda(1+\lambda)\eta \wedge \psi. \tag{4.6}$$

This shows that ψ is a recurrent 3-form. Consequently, since one gets

$$i_{\phi T}T^\flat = 0, \qquad i_{\phi T}\Omega = T^0\eta - T^\flat, \tag{4.7}$$

one derives

$$i_{\phi T}\psi = T^0\eta \wedge T^\flat, \tag{4.8}$$

and so one obtains

$$\mathcal{L}_{\phi T}\psi = 0. \tag{4.9}$$

Hence, we may say that the Lie derivative defines ϕT as a *Pfaffian transformation* of ψ. Thus, we may state the following theorem.

THEOREM 4.1. *Let $M(\phi,\Omega,\xi,\eta,g)$ be a locally conformal cosymplectic manifold. Then, the following hold.*

(i) *The Reeb vector field ξ defines a conformal Lie derivative of Ω.*
(ii) *The vector field ϕT defines a relative almost-Pfaffian transformation of the 2-form Ω.*
(iii) *The principal vector field T defines a Lie almost-conformal transformation of Ω.*
(iv) *Let $\psi = T^\flat \wedge \Omega$ be the principal 3-form on the manifold M. Then ψ is a recurrent 2-form and the Lie derivative defines ϕT as a Pfaffian transformation of ψ.*

5. Hypersurface M_ξ normal to ξ

We denote by M_ξ the hypersurface of M normal to ξ. Since $d\eta = 0$ ($\eta = \xi^\flat$), one may consider the $2m$-dimensional manifold M_ξ and the 1-dimensional foliation in the direction of ξ is totally geodesic.

Recall that the Weingarten map

$$A : T_{\overline{p}}(M_\xi) \longrightarrow T_{\overline{p}}(M_\xi), \quad \forall \overline{p} \in M_\xi, \tag{5.1}$$

is a linear and selfadjoint application and Ω_η is symplectic.

Then, if Z^T is any horizontal vector field, one gets by $d\eta = 0$

$$AZ^T = \nabla_{Z^T}\xi = -Z^T, \tag{5.2}$$

and this shows that Z^T is a principal vector field of M_ξ.

Recall that $II = \langle d\overline{p}, d\overline{p}\rangle$ and $III = \langle \nabla\xi, \nabla\xi\rangle$ denote the second and the third fundamental forms associated with the immersion $x : M_\xi \to M$.

Then, by the expression of $\nabla\xi$, one finds that $II = g^T$ and $III = g^T$, where g^T means the horizontal component of g. Hence, we may conclude that the immersion $x : M_\xi \to M$ is horizontally umbilical and has $2m$ principal curvatures equal to 1.

The expression of III proves that the Gauss map is conformal and it can also be seen that M_ξ is Einsteinian.

On the other hand, since obviously the mean curvature field ξ is nowhere zero, by reference to [4], it follows that the product submanifold $M_\xi \times M_\xi$ in $M \times M$ is a $\mathcal{U}$-submanifold (i.e., its allied mean curvature vanishes), or a *Chen* submanifold.

We may summarize the above by the following.

Theorem 5.1. *Let $M(\phi,\Omega,\xi,\eta,g)$ be a locally conformal cosymplectic manifold and $x : M_\xi \to M$ the immersion of one hypersurface normal to ξ. Then, the following hold.*

(i) *The Gauss map associated to the immersion $x : M_\xi \to M$ is conformal.*

(ii) *The product submanifold $M_\xi \times M_\xi$ in $M \times M$ is a $\mathcal{U}$-submanifold.*

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