LIMIT THEOREMS FOR RANDOMLY SELECTED ADJACENT ORDER STATISTICS FROM A PARETO DISTRIBUTION

ANDRÉ ADLER

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Consider independent and identically distributed random variables $\{X_{nk},\ 1 \le k \le m,\ n \ge 1\}$ from the Pareto distribution. We randomly select two adjacent order statistics from each row, $X_{n(i)}$ and $X_{n(i+1)}$, where $1 \le i \le m-1$. Then, we test to see whether or not strong and weak laws of large numbers with nonzero limits for weighted sums of the random variables $X_{n(i+1)}/X_{n(i)}$ exist, where we place a prior distribution on the selection of each of these possible pairs of order statistics.

1. Introduction

In this paper, we observe weighted sums of ratios of order statistics taken from small samples. We look at m observations from the Pareto distribution, that is, $f(x) = px^{-p-1}I(x \ge 1)$, where $p > 0$. Then, we observe two adjacent order statistics from our sample, that is, $X_{(i)} \le X_{(i+1)}$ for $1 \le i \le m-1$. Next, we obtain the random variable $R_i = X_{(i+1)}/X_{(i)}$, $i = 1,\dots,m-1$, which is the ratio of our adjacent order statistics. The density of R_i is

$$f(r) = p(m-i)r^{-p(m-i)-1}I(r \ge 1). \tag{1.1}$$

We will derive this and show how the distributions of these random variables are related.

The joint density of the original i.i.d. Pareto random variables $X_1,\dots,X_m$ is

$$f(x_1,\dots,x_m) = p^m x_1^{-p-1} \cdots x_m^{-p-1} I(x_1 \ge 1) \cdots I(x_m \ge 1), \tag{1.2}$$

hence the density of the corresponding order statistics $X_{(1)},\dots,X_{(m)}$ is

$$f(x_{(1)},\dots,x_{(m)}) = p^m m! x_{(1)}^{-p-1} \cdots x_{(m)}^{-p-1} I(1 \le x_{(1)} \le x_{(2)} \le \cdots \le x_{(m)}). \tag{1.3}$$

Next, we obtain the joint density of $X_{(1)}, R_1,\dots,R_{m-1}$. In order to do that, we need the

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inverse transformation, which is

$$\begin{aligned} X_{(1)} &= X_{(1)}, \\ X_{(2)} &= X_{(1)}R_1, \\ X_{(3)} &= X_{(1)}R_1R_2, \end{aligned} \tag{1.4}$$

through

$$X_{(m)} = X_{(1)}R_1R_2\cdots R_{m-1}. \tag{1.5}$$

So, in order to obtain this density, we need the Jacobian, which is the determinant of the matrix

$$\begin{pmatrix} \frac{\partial x_{(1)}}{\partial x_{(1)}} & \frac{\partial x_{(1)}}{\partial r_1} & \frac{\partial x_{(1)}}{\partial r_2} & \cdots & \frac{\partial x_{(1)}}{\partial r_{m-1}} \\ \frac{\partial x_{(2)}}{\partial x_{(1)}} & \frac{\partial x_{(2)}}{\partial r_1} & \frac{\partial x_{(2)}}{\partial r_2} & \cdots & \frac{\partial x_{(2)}}{\partial r_{m-1}} \\ \frac{\partial x_{(3)}}{\partial x_{(1)}} & \frac{\partial x_{(3)}}{\partial r_1} & \frac{\partial x_{(3)}}{\partial r_2} & \cdots & \frac{\partial x_{(3)}}{\partial r_{m-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_{(m)}}{\partial x_{(1)}} & \frac{\partial x_{(m)}}{\partial r_1} & \frac{\partial x_{(m)}}{\partial r_2} & \cdots & \frac{\partial x_{(m)}}{\partial r_{m-1}} \end{pmatrix}, \tag{1.6}$$

which is the lower triangular matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ r_1 & x_{(1)} & 0 & \cdots & 0 \\ r_1r_2 & x_{(1)}r_2 & x_{(1)}r_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_1\cdots r_{m-1} & x_{(1)}r_2\cdots r_{m-1} & x_{(1)}r_1r_3\cdots r_{m-1} & \cdots & x_{(1)}r_1\cdots r_{m-2} \end{pmatrix}. \tag{1.7}$$

Thus the Jacobian is $x_{(1)}^{m-1}r_1^{m-2}r_2^{m-3}r_3^{m-4}\cdots r_{m-2}$.

So, the joint density of $X_{(1)},R_1,\ldots,R_{m-1}$ is

$$\begin{aligned} f\left(x_{(1)},r_1,\ldots,r_{m-1}\right) &= p^m m! x_{(1)}^{-p-1}\left(x_{(1)}r_1\right)^{-p-1}\left(x_{(1)}r_1r_2\right)^{-p-1}\cdots\left(x_{(1)}r_1\cdots r_{m-1}\right)^{-p-1} \\ &\quad\cdot x_{(1)}^{m-1}r_1^{m-2}r_2^{m-3}\cdots r_{m-2} \\ &\quad\cdot I\left(1 \le x_{(1)} \le x_{(1)}r_1 \le x_{(1)}r_1r_2 \le \cdots \le x_{(1)}r_1\cdots r_{m-1}\right) \\ &= p^m m! x_{(1)}^{-pm-1} r_1^{-p(m-1)-1} r_2^{-p(m-2)-1}\cdots r_{m-2}^{-2p-1} r_{m-1}^{-p-1} \\ &\quad\cdot I\left(x_{(1)} \ge 1\right)I\left(r_1 \ge 1\right)I\left(r_2 \ge 1\right)\cdots I\left(r_{m-1} \ge 1\right). \end{aligned} \tag{1.8}$$

This shows that the random variables $X_{(1)}, R_1, \ldots, R_{m-1}$ are independent and that the density of our smallest order statistic is

$$f_{X_{(1)}}(x_{(1)}) = pm x_{(1)}^{-pm-1} I(x_{(1)} \geq 1), \tag{1.9}$$

while the density of the ratio of the ith adjacent order statistic R_i, $i = 1, \ldots, m-1$ is

$$f_{R_i}(r) = p(m-i) r^{-p(m-i)-1} I(r \geq 1). \tag{1.10}$$

We repeat this procedure n times, assuming independence between sets of data, obtaining the sequence $\{R_n = R_{ni},\ n \geq 1\}$. Notice that we have dropped the subscript i, but the density of R_{ni} does depend on i. Hence, we first start out with n independent sets of m i.i.d. Pareto random variables. We then order these m Pareto random variables within each set. Next, we obtain the $m-1$ ratios of the adjacent order statistics. Finally, we select one of these as our random variable Y. Repeating this n times, we obtain the sequence $\{Y_n,\ n \geq 1\}$. We do that via our preset prior distribution $\{\Pi_1, \ldots, \Pi_{m-1}\}$, where $\Pi_i \geq 0$ and $\sum_{i=1}^{m-1} \Pi_i = 1$. The random variable Y_n is one of the R_{ni}, $i = 1, \ldots, m-1$, chosen via this prior distribution. In other words, $P\{Y_n = R_{ni}\} = \Pi_i$ for $i = 1, 2, \ldots, m-1$. It is very important to identify which is our largest acceptable pair of order statistics since the largest order statistic does dominate the partial sums. Hence, we define $\nu = \max\{k : \Pi_k > 0\}$. We need to do this in case $\Pi_{m-1} = 0$.

Our goal is to determine whether or not there exist positive constants a_n and b_N such that $\sum_{n=1}^{N} a_n Y_n / b_N$ converges to a nonzero constant in some sense, where $\{Y_n,\ n \geq 1\}$ are i.i.d. copies of Y. Another important observation is that when $p(m-\nu) = 1$, we have $EY = \infty$. These are called exact laws of large numbers since they create a fair game situation, where the $a_n Y_n$ represents the amount a player wins on the nth play of some game and $b_N - b_{N-1}$ represents the corresponding fair entrance fee for the participant.

In Adler [1], just one order statistic from the Pareto was observed, while in Adler [2], ratios of order statistics were examined. Here we look at the case of randomly selecting one of these adjacent ratios. As usual, we define $\lg x = \log(\max\{e, x\})$ and $\lg_2 x = \lg(\lg x)$. We use throughout the paper the constant C as a generic real number that is not necessarily the same in each appearance.

2. Exact strong laws when $p(m-\nu) = 1$

In this situation, we can get an exact strong law, but only if we select our coefficients and norming sequences properly. We use as our weights $a_n = (\lg n)^{\beta-2}/n$, but we could set $a_n = S(n)/n$, where $S(\cdot)$ is any slowly varying function. Note that if we do change a_n, then we must also revise b_n, and consequently $c_n = b_n/a_n$.

Theorem 2.1. *If $p(m-\nu)=1$, then for all $\beta>0$,*

$$\lim_{N\to\infty}\frac{\sum_{n=1}^{N}\left((\lg n)^{\beta-2}/n\right)Y_n}{(\lg N)^{\beta}}=\frac{\Pi_\nu}{\beta}\quad \textit{almost surely.} \tag{2.1}$$

Proof. Let $a_n=(\lg n)^{\beta-2}/n$, $b_n=(\lg n)^{\beta}$, and $c_n=b_n/a_n=n(\lg n)^2$. We use the usual partition

$$\begin{aligned}\frac{1}{b_N}\sum_{n=1}^{N}a_nY_n&=\frac{1}{b_N}\sum_{n=1}^{N}a_n\left[Y_nI(1\le Y_n\le c_n)-EY_nI(1\le Y_n\le c_n)\right]\\&\quad+\frac{1}{b_N}\sum_{n=1}^{N}a_nY_nI(Y_n>c_n)+\frac{1}{b_N}\sum_{n=1}^{N}a_nEY_nI(1\le Y_n\le c_n).\end{aligned} \tag{2.2}$$

The first term vanishes almost surely by the Khintchine-Kolmogorov convergence theorem, see [3, page 113], and Kronecker's lemma since

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{c_n^2}EY_n^2I(1\le Y_n\le c_n)&=\sum_{i=1}^{m-1}\Pi_i\sum_{n=1}^{\infty}\frac{1}{c_n^2}ER_n^2I(1\le R_n\le c_n)\\&=\sum_{i=1}^{\nu}\Pi_i\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_1^{c_n}p(m-i)r^{-p(m-i)+1}dr\\&=\sum_{i=1}^{\nu}\Pi_i\sum_{n=1}^{\infty}\frac{p(m-i)}{c_n^2}\int_1^{c_n}r^{-p(m-\nu)-p(\nu-i)+1}dr\\&=\sum_{i=1}^{\nu}\Pi_i\sum_{n=1}^{\infty}\frac{p(m-i)}{c_n^2}\int_1^{c_n}r^{-p(\nu-i)}dr\\&\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_1^{c_n}dr\\&\le C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_1^{c_n}dr\\&\le C\sum_{n=1}^{\infty}\frac{1}{c_n}\\&=C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2}<\infty.\end{aligned} \tag{2.3}$$

The second term vanishes, with probability one, by the Borel-Cantelli lemma since

$$\begin{aligned}
\sum_{n=1}^{\infty} P\{Y_n > c_n\} &= \sum_{i=1}^{m-1} \Pi_i \sum_{n=1}^{\infty} P\{R_n > c_n\} \\
&= \sum_{i=1}^{\nu} \Pi_i \sum_{n=1}^{\infty} \int_{c_n}^{\infty} p(m-i) r^{-p(m-i)-1} dr \\
&\leq C \sum_{i=1}^{\nu} \sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-p(m-\nu)-p(\nu-i)-1} dr \\
&= C \sum_{i=1}^{\nu} \sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-p(\nu-i)-2} dr \\
&\leq C \sum_{i=1}^{\nu} \sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-2} dr \\
&\leq C \sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-2} dr \\
&= C \sum_{n=1}^{\infty} \frac{1}{c_n} < \infty.
\end{aligned} \tag{2.4}$$

The limit of our normalized partial sums is realized via the third term in our partition

$$\begin{aligned}
EY_n I(1 \leq Y_n \leq c_n) &= \sum_{i=1}^{\nu} \Pi_i E R_n I(1 \leq R_n \leq c_n) \\
&= \sum_{i=1}^{\nu} \Pi_i \int_1^{c_n} p(m-i) r^{-p(m-i)} dr \\
&= \sum_{i=1}^{\nu} \Pi_i \int_1^{c_n} p(m-i) r^{-p(m-\nu)-p(\nu-i)} dr \\
&= \sum_{i=1}^{\nu} \Pi_i \int_1^{c_n} p(m-i) r^{-p(\nu-i)-1} dr \\
&= \sum_{i=1}^{\nu-1} \Pi_i \int_1^{c_n} p(m-i) r^{-p(\nu-i)-1} dr + \Pi_\nu \int_1^{c_n} p(m-\nu) r^{-1} dr \\
&\sim \Pi_\nu p(m-\nu) \lg c_n \sim \Pi_\nu \lg n
\end{aligned} \tag{2.5}$$

since

$$\sum_{i=1}^{\nu-1} \Pi_i \int_1^{c_n} p(m-i) r^{-p(\nu-i)-1} dr \leq C \sum_{i=1}^{\nu-1} \int_1^{c_n} r^{-p-1} dr \leq C \int_1^{c_n} r^{-p-1} dr = O(1). \tag{2.6}$$

Thus

$$\frac{\sum_{n=1}^{N} a_n E Y_n I(1 \le Y_n \le c_n)}{b_N} \sim \frac{\Pi_\nu \sum_{n=1}^{N} (\lg n)^{\beta-1}/n}{(\lg N)^\beta} \longrightarrow \frac{\Pi_\nu}{\beta}, \tag{2.7}$$

which completes the proof. □

3. Exact weak laws when $p(m-\nu)=1$

We investigate the behavior of our random variables $\{Y_n,\ n \ge 1\}$, where we slightly increase the coefficient of Y_n. Instead of a_n being a power of logarithm times n^{-1}, we now allow a_n to be n to any power larger than negative one. In this case, there is no way to obtain an exact strong law (see Section 4), but we are able to obtain exact weak laws.

Theorem 3.1. *If $p(m-\nu)=1$ and $\alpha > -1$, then*

$$\frac{\sum_{n=1}^{N} n^\alpha L(n) Y_n}{N^{\alpha+1} L(N) \lg N} \xrightarrow{P} \frac{\Pi_\nu}{\alpha+1} \tag{3.1}$$

for any slowly varying function $L(\cdot)$.

Proof. This proof is a consequence of the degenerate convergence theorem, see [3, page 356]. Here, we set $a_n = n^\alpha L(n)$ and $b_N = N^{\alpha+1} L(N) \lg N$. Thus, for all $\epsilon > 0$, we have

$$\begin{aligned}
\sum_{n=1}^{N} P\left\{Y_n \ge \frac{\epsilon b_N}{a_n}\right\} &= \sum_{i=1}^{\nu} \Pi_i \sum_{n=1}^{N} P\left\{R_n \ge \frac{\epsilon b_N}{a_n}\right\} \\
&= \sum_{i=1}^{\nu} \Pi_i p(m-i) \sum_{n=1}^{N} \int_{\epsilon b_N/a_n}^{\infty} r^{-p(m-i)-1} dr \\
&= p \sum_{i=1}^{\nu} \Pi_i (m-i) \sum_{n=1}^{N} \int_{\epsilon b_N/a_n}^{\infty} r^{-p(m-\nu)-p(\nu-i)-1} dr \\
&= p \sum_{i=1}^{\nu} \Pi_i (m-i) \sum_{n=1}^{N} \int_{\epsilon b_N/a_n}^{\infty} r^{-p(\nu-i)-2} dr \\
&< \sum_{i=1}^{\nu} \sum_{n=1}^{N} \int_{\epsilon b_N/a_n}^{\infty} r^{-2} dr \\
&< C \sum_{n=1}^{N} \frac{a_n}{b_N} \\
&= C \sum_{n=1}^{N} \frac{n^\alpha L(n)}{N^{\alpha+1} L(N) \lg N} \\
&< \frac{C}{\lg N} \longrightarrow 0.
\end{aligned} \tag{3.2}$$

Similarly,

$$
\begin{aligned}
\sum_{n=1}^{N}\operatorname{Var}\left(\frac{a_n}{b_N}Y_nI\left(1\le Y_n\le\frac{b_N}{a_n}\right)\right) &= \sum_{i=1}^{\nu}\Pi_i\sum_{n=1}^{N}\operatorname{Var}\left(\frac{a_n}{b_N}R_nI\left(1\le R_n\le\frac{b_N}{a_n}\right)\right)\\
&< C\sum_{i=1}^{\nu}\sum_{n=1}^{N}\frac{a_n^2}{b_N^2}\int_1^{b_N/a_n} r^{-p(m-i)+1}dr\\
&= C\sum_{i=1}^{\nu}\sum_{n=1}^{N}\frac{a_n^2}{b_N^2}\int_1^{b_N/a_n} r^{-p(m-\nu)-p(\nu-i)+1}dr\\
&= C\sum_{i=1}^{\nu}\sum_{n=1}^{N}\frac{a_n^2}{b_N^2}\int_1^{b_N/a_n} r^{-p(\nu-i)}dr\\
&< C\sum_{n=1}^{N}\frac{a_n^2}{b_N^2}\int_1^{b_N/a_n} dr < C\sum_{n=1}^{N}\frac{a_n}{b_N}\\
&= C\sum_{n=1}^{N}\frac{n^{\alpha}L(n)}{N^{\alpha+1}L(N)\lg N}\le\frac{C}{\lg N}\longrightarrow 0.
\end{aligned}
\tag{3.3}
$$

As for our truncated expectation, we have

$$
\begin{aligned}
EY_nI\left(1\le Y_n\le\frac{b_N}{a_n}\right) &= \sum_{i=1}^{\nu}\Pi_iER_nI\left(1\le R_n\le\frac{b_N}{a_n}\right)\\
&= \sum_{i=1}^{\nu}\Pi_ip(m-i)\int_1^{b_N/a_n} r^{-p(m-i)}dr\\
&= p\sum_{i=1}^{\nu}\Pi_i(m-i)\int_1^{b_N/a_n} r^{-p(m-\nu)-p(\nu-i)}dr\\
&= p\sum_{i=1}^{\nu}\Pi_i(m-i)\int_1^{b_N/a_n} r^{-p(\nu-i)-1}dr\\
&= p\sum_{i=1}^{\nu-1}\Pi_i(m-i)\int_1^{b_N/a_n} r^{-p(\nu-i)-1}dr+\Pi_\nu\int_1^{b_N/a_n} r^{-1}dr.
\end{aligned}
\tag{3.4}
$$

The last term is the dominant term since

$$
\sum_{n=1}^{N}\frac{a_n}{b_N}p\sum_{i=1}^{\nu-1}\Pi_i(m-i)\int_1^{b_N/a_n} r^{-p(\nu-i)-1}dr < C\sum_{n=1}^{N}\frac{a_n}{b_N}\int_1^{b_N/a_n} r^{-p-1}dr < C\sum_{n=1}^{N}\frac{a_n}{b_N}\longrightarrow 0,
\tag{3.5}
$$

while

$$\sum_{n=1}^{N} \frac{a_n}{b_N} \Pi_\nu \int_1^{b_N/a_n} r^{-1} dr$$
$$= \Pi_\nu \sum_{n=1}^{N} \frac{a_n}{b_N} \lg\left(\frac{b_N}{a_n}\right)$$
$$= \frac{\Pi_\nu \sum_{n=1}^{N} n^\alpha L(n) \lg\left[N^{\alpha+1} L(N) \lg N / (n^\alpha L(n))\right]}{N^{\alpha+1} L(N) \lg N}$$
$$= \frac{\Pi_\nu \sum_{n=1}^{N} n^\alpha L(n) \left[(\alpha+1) \lg N + \lg L(N) + \lg_2 N - \alpha \lg n - \lg L(n)\right]}{N^{\alpha+1} L(N) \lg N}. \tag{3.6}$$

The important terms are

$$\frac{\sum_{n=1}^{N} n^\alpha L(n)(\alpha+1) \lg N}{N^{\alpha+1} L(N) \lg N} = \frac{(\alpha+1) \sum_{n=1}^{N} n^\alpha L(n)}{N^{\alpha+1} L(N)} \longrightarrow 1,$$
$$\frac{\sum_{n=1}^{N} n^\alpha L(n)(-\alpha \lg n)}{N^{\alpha+1} L(N) \lg N} = -\frac{\alpha \sum_{n=1}^{N} n^\alpha L(n) \lg n}{N^{\alpha+1} L(N) \lg N} \longrightarrow -\frac{\alpha}{\alpha+1}, \tag{3.7}$$

while the other three terms vanish as $N \to \infty$. For completeness, we will verify these claims:

$$\frac{\sum_{n=1}^{N} n^\alpha L(n) \lg L(N)}{N^{\alpha+1} L(N) \lg N} < \frac{C \lg L(N)}{\lg N} \longrightarrow 0,$$
$$\frac{\sum_{n=1}^{N} n^\alpha L(n) \lg_2 N}{N^{\alpha+1} L(N) \lg N} < \frac{C \lg_2 N}{\lg N} \longrightarrow 0, \tag{3.8}$$
$$\frac{\sum_{n=1}^{N} n^\alpha L(n) \lg L(n)}{N^{\alpha+1} L(N) \lg N} < \frac{C N^{\alpha+1} L(N) \lg L(N)}{N^{\alpha+1} L(N) \lg N} = \frac{C \lg L(N)}{\lg N} \longrightarrow 0.$$

Therefore,

$$\frac{\sum_{n=1}^{N} a_n E Y_n I(1 \le Y_n \le b_N/a_n)}{b_N} \longrightarrow \Pi_\nu \left(1 - \frac{\alpha}{\alpha+1}\right) = \frac{\Pi_\nu}{\alpha+1}, \tag{3.9}$$

which completes this proof. □

4. Further almost sure behavior when $p(m-\nu) = 1$

Using our exact weak law, we are able to obtain a generalized law of the iterated logarithm. This shows that under the hypotheses of Theorem 4.1, exact strong laws do not exist when $a_n = n^\alpha L(n)$, $\alpha > -1$, where $L(\cdot)$ is a slowly varying function. Hence, the coefficients selected in Theorem 2.1 are the only permissible ones that will allow us to obtain an exact strong law, that is, $a_n = S(n)/n$ for some slowly varying function $S(\cdot)$, where we used logarithms as our function $S(\cdot)$.

Theorem 4.1. *If $p(m-\nu)=1$ and $\alpha>-1$, then*

$$\liminf_{N\to\infty}\frac{\sum_{n=1}^{N} n^{\alpha}L(n)Y_n}{N^{\alpha+1}L(N)\lg N}=\frac{\Pi_\nu}{\alpha+1}\quad \textit{almost surely},$$
$$\limsup_{N\to\infty}\frac{\sum_{n=1}^{N} n^{\alpha}L(n)Y_n}{N^{\alpha+1}L(N)\lg N}=\infty\quad \textit{almost surely}, \tag{4.1}$$

for any slowly varying function $L(\cdot)$.

Proof. From Theorem 3.1, we have

$$\liminf_{N\to\infty}\frac{\sum_{n=1}^{N} n^{\alpha}L(n)Y_n}{N^{\alpha+1}L(N)\lg N}\le\frac{\Pi_\nu}{\alpha+1}\quad \text{almost surely}. \tag{4.2}$$

Set $a_n=n^{\alpha}L(n)$, $b_n=n^{\alpha+1}L(n)\lg n$, and $c_n=b_n/a_n=n\lg n$. In order to obtain the opposite inequality, we use the following partition:

$$\begin{aligned}\frac{1}{b_N}\sum_{n=1}^{N}a_nY_n&\ge\frac{1}{b_N}\sum_{n=1}^{N}a_nY_nI(1\le Y_n\le n)\\&=\frac{1}{b_N}\sum_{n=1}^{N}a_n\left[Y_nI(1\le Y_n\le n)-EY_nI(1\le Y_n\le n)\right]\\&\quad+\frac{1}{b_N}\sum_{n=1}^{N}a_nEY_nI(1\le Y_n\le n).\end{aligned} \tag{4.3}$$

The first term goes to zero, almost surely, since b_n is essentially increasing and

$$\begin{aligned}\sum_{n=1}^{\infty}c_n^{-2}EY_n^2I(1\le Y_n\le n)&\le\sum_{i=1}^{\nu}\sum_{n=1}^{\infty}c_n^{-2}ER_n^2I(1\le R_n\le n)\\&\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty}c_n^{-2}\int_1^n r^{-p(\nu-i)}dr\\&\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty}c_n^{-2}\int_1^n dr\\&\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty}\frac{n}{c_n^2}\\&\le C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2}<\infty.\end{aligned} \tag{4.4}$$

As for the second term, we once again focus on the last term, our two largest permissible order statistics,

$$
\begin{aligned}
EY_nI(1 \le Y_n \le n) &= \sum_{i=1}^{\nu} \Pi_i EY_nI(1 \le Y_n \le n) \\
&= \sum_{i=1}^{\nu} \Pi_i \int_1^n p(m-i) r^{-p(m-i)} dr \\
&= p \sum_{i=1}^{\nu} \Pi_i (m-i) \int_1^n r^{-p(m-\nu)-p(\nu-i)} dr \\
&= p \sum_{i=1}^{\nu} \Pi_i (m-i) \int_1^n r^{-p(\nu-i)-1} dr \\
&= p \sum_{i=1}^{\nu-1} \Pi_i (m-i) \int_1^n r^{-p(\nu-i)-1} dr + \Pi_\nu p(m-\nu) \int_1^n r^{-1} dr \\
&\sim \Pi_\nu \lg n
\end{aligned}
\tag{4.5}
$$

since

$$
p \sum_{i=1}^{\nu-1} \Pi_i (m-i) \int_1^n r^{-p(\nu-i)-1} dr < C \sum_{i=1}^{\nu-1} \int_1^n r^{-p-1} dr < C \int_1^n r^{-p-1} dr = O(1). \tag{4.6}
$$

Thus,

$$
\begin{aligned}
\liminf_{N\to\infty} \frac{\sum_{n=1}^N a_n Y_n}{b_N} &\ge \liminf_{N\to\infty} \frac{\sum_{n=1}^N a_n EY_nI(1 \le Y_n \le n)}{b_N} \\
&= \lim_{N\to\infty} \frac{\Pi_\nu \sum_{n=1}^N n^\alpha L(n) \lg n}{N^{\alpha+1} L(N) \lg N} \\
&= \frac{\Pi_\nu}{\alpha+1},
\end{aligned}
\tag{4.7}
$$

establishing our almost sure lower limit.

As for the upper limit, let $M > 0$, then

$$
\begin{aligned}
\sum_{n=1}^{\infty} P\{Y_n > Mc_n\} &= \sum_{i=1}^{\nu} \Pi_i \sum_{n=1}^{\infty} P\{R_n > Mc_n\} \\
&= \sum_{i=1}^{\nu} \Pi_i \sum_{n=1}^{\infty} p(m-i) \int_{Mc_n}^{\infty} r^{-p(m-i)-1} dr \\
&\ge \sum_{i=\nu}^{\nu} \Pi_i \sum_{n=1}^{\infty} p(m-i) \int_{Mc_n}^{\infty} r^{-p(m-i)-1} dr \\
&= \Pi_\nu \sum_{n=1}^{\infty} p(m-\nu) \int_{Mc_n}^{\infty} r^{-p(m-\nu)-1} dr
\end{aligned}
$$

$$
\begin{aligned}
&= \Pi_\nu \sum_{n=1}^{\infty} \int_{Mc_n}^{\infty} r^{-2} dr \\
&= \frac{\Pi_\nu}{M} \sum_{n=1}^{\infty} \frac{1}{c_n} \\
&= \frac{\Pi_\nu}{M} \sum_{n=1}^{\infty} \frac{1}{n \lg n} \\
&= \infty.
\end{aligned}
\tag{4.8}
$$

This implies that

$$
\limsup_{n\to\infty} \frac{a_n Y_n}{b_n} = \infty \quad \text{almost surely,} \tag{4.9}
$$

which in turn allows us to conclude that

$$
\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} a_n Y_n}{b_N} = \infty \quad \text{almost surely,} \tag{4.10}
$$

which completes this proof. □

5. Typical strong laws when $p(m-\nu) > 1$

When $p(m-\nu) > 1$, we have $EY < \infty$, hence all kinds of strong laws exist. In this case, $\{a_n,\ n \geq 1\}$ and $\{b_n,\ n \geq 1\}$ can be any pair of positive sequences as long as $b_n \uparrow \infty$, $\sum_{n=1}^{N} a_n/b_N \to L$, where $L \neq 0$, and the condition involving $c_n = b_n/a_n$ in each theorem is satisfied. If $L = 0$, then these limit theorems still hold, however the limit is zero, which is not that interesting.

This section is broken down into three cases, each has different conditions as to whether the strong law exists. The calculation of EY follows in the ensuing lemma.

Lemma 5.1. *If $p(m-\nu) > 1$, then*

$$
EY = \sum_{i=1}^{\nu} \frac{p\Pi_i (m-i)}{p(m-i)-1}. \tag{5.1}
$$

Proof. The proof is rather trivial, since $p(m-\nu) > 1$, we have

$$
EY = \sum_{i=1}^{\nu} \Pi_i E R_n = \sum_{i=1}^{\nu} p\Pi_i (m-i) \int_1^{\infty} r^{-p(m-i)} dr = \sum_{i=1}^{\nu} \frac{p\Pi_i (m-i)}{p(m-i)-1}, \tag{5.2}
$$

which completes the proof of the lemma. □

In all three ensuing theorems, we use the partition

$$\begin{aligned}\frac{1}{b_N}\sum_{n=1}^{N}a_nY_n &= \frac{1}{b_N}\sum_{n=1}^{N}a_n\left[Y_nI(1\le Y_n\le c_n)-EY_nI(1\le Y_n\le c_n)\right]\\ &\quad+\frac{1}{b_N}\sum_{n=1}^{N}a_nY_nI(Y_n>c_n)\\ &\quad+\frac{1}{b_N}\sum_{n=1}^{N}a_nEY_nI(1\le Y_n\le c_n),\end{aligned} \tag{5.3}$$

where the selection of a_n, b_n, and $c_n=b_n/a_n$ must satisfy the assumption of each theorem. These three hypotheses are slightly different and are dependent on how large a first moment the random variable Y possesses. The difference in the these theorems is the condition involving the sequence $\{c_n,\ n\ge 1\}$.

Theorem 5.2. *If* $1<p(m-\nu)<2$ *and* $\sum_{n=1}^{\infty}c_n^{-p(m-\nu)}<\infty$, *then*

$$\lim_{N\to\infty}\frac{\sum_{n=1}^{N}a_nY_n}{b_N}=L\sum_{i=1}^{\nu}\frac{p\Pi_i(m-i)}{p(m-i)-1}\quad \textit{almost surely.} \tag{5.4}$$

Proof. The first term in our partition goes to zero, with probability one, since

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{c_n^2}EY_n^2I(1\le Y_n\le c_n) &= \sum_{i=1}^{\nu}\Pi_i\sum_{n=1}^{\infty}\frac{1}{c_n^2}ER_n^2I(1\le R_n\le c_n)\\ &\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_1^{c_n}r^{-p(m-i)+1}dr\\ &\le C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\int_1^{c_n}r^{-p(m-\nu)+1}dr\\ &\le C\sum_{n=1}^{\infty}\frac{c_n^{-p(m-\nu)+2}}{c_n^2}\\ &= C\sum_{n=1}^{\infty}c_n^{-p(m-\nu)}<\infty.\end{aligned} \tag{5.5}$$

As for the second term,

$$\begin{aligned}\sum_{n=1}^{\infty}P\{Y_n>c_n\} &= \sum_{i=1}^{\nu}\Pi_i\sum_{n=1}^{\infty}P\{R_n>c_n\}\\ &\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty}\int_{c_n}^{\infty}r^{-p(m-i)-1}dr\\ &\le C\sum_{n=1}^{\infty}\int_{c_n}^{\infty}r^{-p(m-\nu)-1}dr\\ &\le C\sum_{n=1}^{\infty}c_n^{-p(m-\nu)}<\infty.\end{aligned} \tag{5.6}$$

Then, from our lemma and $\sum_{n=1}^{N} a_n \sim Lb_N$, we have

$$\frac{\sum_{n=1}^{N} a_n EY_n I(1 \le Y_n \le c_n)}{b_N} \longrightarrow L\sum_{i=1}^{\nu} \frac{p\Pi_i(m-i)}{p(m-i)-1}, \tag{5.7}$$

which completes this proof. □

THEOREM 5.3. *If* $p(m-\nu) = 2$ *and* $\sum_{n=1}^{\infty} \lg(c_n)/c_n^2 < \infty$, *then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} a_n Y_n}{b_N} = L\sum_{i=1}^{\nu} \frac{p\Pi_i(m-i)}{p(m-i)-1} \quad \textit{almost surely.} \tag{5.8}$$

Proof. The first term goes to zero, almost surely, since

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{c_n^2} EY_n^2 I(1 \le Y_n \le c_n) &\le \sum_{i=1}^{\nu}\sum_{n=1}^{\infty} \frac{1}{c_n^2} ER_n^2 I(1 \le R_n \le c_n) \\
&\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_1^{c_n} r^{-p(m-i)+1}\,dr \\
&\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_1^{c_n} r^{-p(m-\nu)+1}\,dr \\
&\le C\sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_1^{c_n} r^{-1}\,dr \\
&= C\sum_{n=1}^{\infty} \frac{\lg c_n}{c_n^2} < \infty.
\end{aligned} \tag{5.9}$$

Likewise, the second term disappears, with probability one, since

$$\begin{aligned}
\sum_{n=1}^{\infty} P\{Y_n > c_n\} &\le \sum_{i=1}^{\nu}\sum_{n=1}^{\infty} P\{R_n > c_n\} \\
&\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-p(m-i)-1}\,dr \\
&\le C\sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-p(m-\nu)-1}\,dr \\
&= C\sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-3}\,dr \\
&\le C\sum_{n=1}^{\infty} \frac{1}{c_n^2} \\
&\le C\sum_{n=1}^{\infty} \frac{\lg c_n}{c_n^2} < \infty.
\end{aligned} \tag{5.10}$$

As in the last proof, the calculation for the truncated mean is exactly the same, which leads us to the same limit. □

Theorem 5.4. *If* $p(m-\nu) > 2$ *and* $\sum_{n=1}^{\infty} c_n^{-2} < \infty$, *then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} a_n Y_n}{b_N} = L\sum_{i=1}^{\nu} \frac{p\Pi_i(m-i)}{p(m-i)-1} \quad \text{almost surely.} \tag{5.11}$$

Proof. The first term goes to zero, with probability one, since

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{c_n^2} EY_n^2 I(1 \le Y_n \le c_n) &\le \sum_{i=1}^{\nu}\sum_{n=1}^{\infty} \frac{1}{c_n^2} ER_n^2 I(1 \le R_n \le c_n) \\
&\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_1^{c_n} r^{-p(m-i)+1}\,dr \\
&\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_1^{c_n} r^{-p(m-\nu)+1}\,dr \\
&\le C\sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_1^{c_n} r^{-p(m-\nu)+1}\,dr \\
&\le C\sum_{n=1}^{\infty} \frac{1}{c_n^2} < \infty.
\end{aligned} \tag{5.12}$$

As for the second term,

$$\begin{aligned}
\sum_{n=1}^{\infty} P\{Y_n > c_n\} &= \sum_{i=1}^{\nu} \Pi_i \sum_{n=1}^{\infty} P\{R_n > c_n\} \\
&\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-p(m-i)-1}\,dr \\
&\le C\sum_{i=1}^{\nu}\sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-p(m-\nu)-1}\,dr \\
&\le C\sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-p(m-\nu)-1}\,dr \\
&\le C\sum_{n=1}^{\infty} \int_{c_n}^{\infty} r^{-3}\,dr \\
&\le C\sum_{n=1}^{\infty} \frac{1}{c_n^2} < \infty.
\end{aligned} \tag{5.13}$$

Then as in the last two theorems,

$$\frac{\sum_{n=1}^{N} a_n E Y_n I(1 \le Y_n \le c_n)}{b_N} \longrightarrow L \sum_{i=1}^{\nu} \frac{p\Pi_i(m-i)}{p(m-i)-1}, \tag{5.14}$$

which completes this proof. □

Clearly, in all of these three theorems, the situation of $a_n = 1$ and $b_n = n = c_n$ is easily satisfied. Whenever $p(m-\nu) > 1$, we have tremendous freedom in selecting our constants. That is certainly not true when $p(m-\nu) = 1$.

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André Adler: Department Applied of Mathematics, College of Science and Letters, Illinois Institute of Technology, Chicago, IL 60616, USA

E-mail address: adler@iit.edu