BIHARMONIC SUBMANIFOLDS IN 3-DIMENSIONAL (κ,μ)-MANIFOLDS

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Received 19 April 2005 and in revised form 11 October 2005

Biharmonic maps between Riemannian manifolds are defined as critical points of the bienergy and generalized harmonic maps. In this paper, we give necessary and sufficient conditions for nonharmonic Legendre curves and anti-invariant surfaces of 3-dimensional (κ,μ)-manifolds to be biharmonic.

1. Introduction

Let $f:(M,g)\to(N,h)$ be a smooth map between two Riemannian manifolds. The bienergy $E_2(f)$ of f over compact domain $\Omega\subset M$ is defined by

$$E_2(f)=\int_\Omega h(\tau(f),\tau(f))\,dv_g, \tag{1.1}$$

where $\tau(f)$ is the tension field of f and dv_g is the volume form of M.

It is clear that $E_2(f\,|_\Omega)=0$ on any compact domain if and only if f is a harmonic map. Thus E_2 provides a measure for the extent to which f fails to be harmonic. If f is a critical point of (1.1) over every compact domain, then f is called a *biharmonic map* or 2-*harmonic maps*. Jiang [10] proved that f is biharmonic if and only if

$$\mathcal{J}_f(\tau(f))=0, \tag{1.2}$$

here $\mathcal{J}_f$ is the Jacobi operator of f.

Clearly, any harmonic map is biharmonic. But the converse is not true. Nonharmonic biharmonic maps are said to be proper. It is well known that proper biharmonic maps into R, that is, biharmonic functions, play an important role in elasticity and hydrodynamics.

Proper biharmonic submanifolds in real space forms have been studied by many geometers during the last two decades. However, in the Euclidean space and the hyperbolic space, such submanifolds have not been found yet. On the other hand, many proper biharmonic submanifolds exist in the unit sphere.

The unit sphere of odd dimension is the typical example of Sasakian space forms. Lately, J. Inoguchi and T. Sasahara initiated the study of proper biharmonic submanifolds

International Journal of Mathematics and Mathematical Sciences 2005:22 (2005) 3575–3586
DOI: 10.1155/IJMMS.2005.3575

in Sasakian space forms. Inoguchi [9] classified proper biharmonic Legendre curves and Hopf cylinders (automatically anti-invariant surfaces) in Sasakian 3-space forms. Sasahara [11] classified proper biharmonic Legendre surfaces in Sasakian 5-space forms. Also, in [1], the authors studied anti-invariant submanifolds in Sasakian 5-space forms.

In [4], Blair et al. introduced a new class of contact metric manifolds (M, ϕ, ξ, η, g): (κ,μ)-*manifolds*, which are defined as manifolds whose curvature tensor $\widetilde{R}$ satisfies

$$\widetilde{R}(X,Y)\xi = (\kappa I + \mu h)\big(\eta(Y)X - \eta(X)Y\big) \tag{1.3}$$

for any vector fields X and Y, where I is the identity and $2h$ is the Lie differentiation of ϕ with respect to ξ, and κ, μ are constant.

Sasakian manifolds are (κ,μ)-manifolds with $\kappa = 1$ and $h = 0$. Also, the unit tangent sphere bundle of a Riemannian manifold of constant sectional curvature c satisfies (1.3), with $\kappa = c(2-c)$ and $\mu = -2c$. The class of (κ,μ)-manifolds has been classified at least locally (see [4, 5]). In particular, in case the dimension is 3, a (κ,μ)-manifold is either Sasakian or locally isometric to one of the unimodular Lie groups $SU(2)$, $SL(2,R)$, $E(2)$, $E(1,1)$ with a left invariant metric.

Proper biharmonic Legendre curves (resp., anti-invariant surfaces) in Sasakian 3-space forms are completely determined by the curvature (resp., the mean curvature) (cf. [1, 9]). Since Sasakian 3-space forms are special examples of (κ,μ)-manifolds, it is natural and interesting to investigate proper biharmonic Legendre curves and anti-invariant surfaces in general 3-dimensional (κ,μ)-manifolds.

In this paper, in terms of the curvature and the torsion, (resp., the mean curvature), we give necessary and sufficient conditions for nonharmonic Legendre curves (resp., anti-invariant surfaces) in 3-dimensional (κ,μ)-manifolds to be biharmonic.

2. (κ,μ)-manifolds

In this section, we collect some basic facts about contact metric manifolds. We refer to [3] for a more detailed treatment. A $(2n+1)$-dimensional differentiable manifold N^{2n+1} is called a *contact manifold* if there exists a globally defined 1-form η such that $(d\eta)^n \wedge \eta \neq 0$. On a contact manifold there exists a unique global vector field ξ satisfying

$$d\eta(\xi,X) = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

for all $X \in TN^{2n+1}$.

Moreover, it is well known that there exist a $(1,1)$-tensor field ϕ and a Riemannian metric g which satisfy

$$\begin{gathered} \phi^2 = -I + \eta \otimes \xi, \\ g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(\xi,X) = \eta(X), \\ d\eta(X,Y) = g(X,\phi Y), \end{gathered} \tag{2.2}$$

for all $X, Y \in TN^{2n+1}$.

The structure (ϕ,ξ,η,g) is called *contact metric structure* and the manifold N^{2n+1} with a contact metric structure is said to be a *contact metric manifold*. Following [3], we define on N^{2n+1} the (1,1)-tensor fields h:

$$h = \frac{1}{2}(\mathscr{L}_\xi \phi), \tag{2.3}$$

where $\mathscr{L}_\xi$ is the Lie differentiation in the direction of ξ. The tensor field h is self-adjoint and satisfies

$$h\xi = 0, \tag{2.4}$$

$$h\phi + \phi h = 0, \tag{2.5}$$

$$\tilde{\nabla}_X \xi = -\phi X - \phi h X, \tag{2.6}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of g.

A (κ,μ)-manifold is defined as a contact metric manifold satisfying

$$\tilde{R}(X,Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y), \tag{2.7}$$

for any vector field X and Y, where κ, μ are constant. We denote an n-dimensional (κ,μ)-manifold by $M^n(\kappa,\mu)$. Due to [4], on $M^n(\kappa,\mu)$ we have the following (cf. [5]):

$$\begin{aligned} \tilde{\nabla}_X hY - h(\tilde{\nabla}_X Y) &= ((1-\kappa)g(X,\phi Y) - g(X,\phi hY))\xi \\ &\quad - \eta(Y)((1-\kappa)\phi Y + \phi hX) - \mu\eta(X)\phi hY, \\ \tilde{\nabla}_X \phi Y - \phi(\tilde{\nabla}_X Y) &= (g(X,Y) + g(X,hY))\xi - \eta(Y)(X + hX). \end{aligned} \tag{2.8}$$

It is well known that the curvature tensor $\tilde{R}$ of 3-dimensional Riemannian manifolds satisfy the following:

$$\begin{aligned} \tilde{R}(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y \\ &\quad - \frac{S}{2}(g(Y,Z)X - g(X,Z)Y), \end{aligned} \tag{2.9}$$

where Q is the Ricci operator and S is the scalar curvature.

Substituting $Y = Z = \xi$ to (2.9) and using (2.7), on $M^3(\kappa,\mu)$ we obtain

$$Q = \frac{1}{2}(S - 2\kappa)I + \frac{1}{2}(6\kappa - S)\eta \otimes \xi + \mu h. \tag{2.10}$$

In general, $\kappa \le 1$ on a (κ,μ)-manifold. If $\kappa = 1$, the manifold is Sasakian. If $\kappa < 1$, the relation (2.7) determines the curvature of (κ,μ)-manifold completely (see [5]). The scalar curvature S of $M^3(\kappa,\mu)$ is equal to

$$2(\kappa - \mu). \tag{2.11}$$

Remark 2.1. A non-Sasakian 3-dimensional (κ,μ)-manifold is locally isometric to one of the unimodular Lie groups $SU(2)$, $SL(2,R)$, $E(2)$, $E(1,1)$ with a left invariant metric. (See [4]).

3. Biharmonic maps

Let M^m and N^n be Riemannian manifolds and $f : M^m \to N^n$ a smooth map. The *tension field* $\tau(f)$ of f is a section of the vector bundle f^*TN^n defined by

$$\tau(f) := \operatorname{tr}(\nabla^f df) = \sum_{i=1}^{m} \{\nabla^f_{e_i} df(e_i) - df(\nabla_{e_i} e_i)\}, \tag{3.1}$$

where ∇^f, ∇, and $\{e_i\}$ denote an induced connection, the Levi-Civita connection of M^m, and a local orthonormal frame field of M^m, respectively.

A smooth map f is said to be a *harmonic map* if its tension field vanishes. It is well known that f is harmonic if and only if f is a critical point of the *energy*:

$$E(f) = \int_\Omega \sum_{i=1}^{m} h(df(e_i), df(e_i))\, dv_g \tag{3.2}$$

over every compact domain Ω of M^m.

Eells and Sampson [8] suggested to study *biharmonic maps* which are critical points of the bienergy E_2:

$$E_2(f) = \int_\Omega h(\tau(f), \tau(f))\, dv_g. \tag{3.3}$$

The Euler-Lagrange equation of the functional E_2 was computed by Jiang [10] as follows:

$$\mathcal{J}_f(\tau(f)) = 0. \tag{3.4}$$

Here the operator $\mathcal{J}_f$ is the *Jacobi operator* defined by

$$\begin{aligned} \mathcal{J}_f(V) &:= \bar{\Delta}_f V - \mathcal{R}_f(V), \quad V \in \Gamma(f^*TN^n), \\ \bar{\Delta}_f &:= -\sum_{i=1}^{m} \left(\nabla^f_{e_i}\nabla^f_{e_i} - \nabla^f_{\nabla_{e_i} e_i}\right), \\ \mathcal{R}_f(V) &:= \sum_{i=1}^{m} R^{N^n}(V, df(e_i))\, df(e_i), \end{aligned} \tag{3.5}$$

where R^{N^n} is the curvature tensor of N^n.

4. Biharmonic Legendre curves

A curve $C = C(s) : I \to M^3(\kappa,\mu)$ parametrized by arclength parameter is said to be a *Legendre curve* if $\eta(C') = 0$. In this section, in terms of the curvature and the torsion, we characterize proper biharmonic Legendre curves in 3-dimensional (κ,μ)-manifolds.

Let C be a Legendre curve in $M^3(\kappa,\mu)$. Then we can take a Frenet field, $F = (T,N,B)$, so that $T = C'$, $N = \phi C'$, and $B = \xi$ (see [2]). Frenet-Serret formula of C is given explicitly by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{4.1}$$

where α (resp., τ) is the curvature (resp., the torsion).

Proposition 4.1. *Let $C : I \to M^3(\kappa,\mu)$ be a nongeodesic Legendre curve. Then C is biharmonic if and only if C is a helix satisfying $\alpha^2 + \tau^2 = (1/2)(C^*S - 4\kappa)$, where C^*S is the pullback of S by C.*

Proof. Frenet-Serret formula implies that the mean curvature vector field H is given by $H = \nabla_T T = \alpha N$. By direct computations, we obtain

$$\mathcal{J}_C(H) = 3\alpha\alpha' T - (\alpha'' - \alpha^3 - \alpha\tau^2)N - (2\alpha'\tau + \alpha\tau')B - \mathcal{R}_C(H). \tag{4.2}$$

Using (2.9) and (2.10), we have

$$\mathcal{R}_C(H) = QH + \langle QT,T\rangle H - \langle QH,T\rangle T - \frac{C^*S}{2}H, \tag{4.3}$$

$$QT = \frac{1}{2}(C^*S - 2\kappa)T + \mu hT, \tag{4.4}$$

$$QH = \frac{1}{2}(C^*S - 2\kappa)H + \alpha\mu\{-\langle hT,T\rangle\phi T + \langle h\phi T,T\rangle T\}. \tag{4.5}$$

Substituting (4.4) and (4.5) into (4.3), we get

$$\mathcal{R}_C(H) = \frac{1}{2}(C^*S - 4\kappa)\alpha N. \tag{4.6}$$

If γ is biharmonic, $\mathcal{J}_C(H) = 0$. Hence it follows from (4.2) and (4.6) that α and τ are constant, and moreover, they satisfy $\alpha^2 + \tau^2 = (1/2)(C^*S - 4\kappa)$. Conversely, if C is a helix with $\alpha^2 + \tau^2 = (1/2)(C^*S - 4\kappa)$, then $\mathcal{J}_C(H) = 0$ on C. Hence C is biharmonic. □

Corollary 4.2. *There exist no proper biharmonic Legendre curves in $M^3(\kappa,\mu)$ with $S \le 4\kappa$.*

Corollary 4.3. *Let $C : I \to M^3(\kappa,\mu)$ be a nongeodesic Legendre curve in 3-dimensional (κ,μ)-manifolds. Assume that $\kappa < 1$. Then C is biharmonic if and only if C is a helix satisfying $\alpha^2 + \tau^2 = -(\kappa + \mu)$.*

Corollary 4.4. *There exist no proper biharmonic Legendre curves in $M^3(\kappa,\mu)$ with $\kappa < 1$ and $\kappa + \mu \ge 0$.*

A contact metric manifold is said to be a *Sasakian manifold* if it satisfies $[\phi,\phi] + 2d\eta \otimes \xi = 0$ on N^{2n+1}, where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ.

The tangent planes in T_pN^{2n+1} which is invariant under ϕ are called *ϕ-section* (see [3]). The sectional curvature of ϕ-section is called ϕ-sectional curvature. If the ϕ-sectional curvature is constant on N^{2n+1}, then N^{2n+1} is said to be of *constant ϕ-sectional curvature*. Complete and connected Sasakian manifolds of constant ϕ-sectional curvature are called

Sasakian space forms. Denote Sasakian space forms of constant ϕ-sectional curvature c by $N^{2n+1}(c)$.

The curvature tensor $\widetilde{R}$ of $N^{2n+1}(c)$ is given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & \frac{c+3}{4}\{g(Y,Z)X - g(Z,X)Y\} \\ & + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ & \qquad - g(Y,Z)\eta(X)\xi + g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z\}.\end{aligned} \tag{4.7}$$

We can easily see that Sasakian space forms are (κ,μ)-manifold, with $\kappa = 1$ and $h = 0$. Legendre curves in Sasakian space forms satisfy $\tau^2 = 1$ (see [2]). Therefore, by applying Proposition 4.1, we have the following (cf. [9]).

Corollary 4.5. *Let $C : I \to M^3(c)$ be a Legendre curve in Sasakian space forms of constant ϕ-sectional curvature c. Then C is proper biharmonic if and only if $c > 1$ and C is a helix satisfying $\alpha^2 = c - 1$.*

5. Biharmonic anti-invariant surfaces

Let M^m be a submanifold tangent to ξ in a contact metric manifold. If ϕX is normal to TM^m for any $X \in TM^m$, then M^m is called an *anti-invariant* submanifold (see [12]).

Let $f : M^2 \to M^3(k,\mu)$ be a nonminimal anti-invariant surface. The formulas of Gauss and Weingarten are given, respectively, by

$$\begin{aligned}\widetilde{\nabla}_X Y &= \nabla_X Y + \sigma(X,Y), \\ \widetilde{\nabla}_X V &= -A_V X + D_X V,\end{aligned} \tag{5.1}$$

where $X, Y \in TM^m$, $V \in T^\perp M^m$, σ, A, and D are the second fundamental form, the shape operator, and the normal connection.

Denote by R the Riemann curvature tensor of M^2. Then the equations of Gauss and Codazzi are given, respectively, by

$$\langle R(X,Y)Z,W\rangle = \langle A_{\sigma(Y,Z)}X,W\rangle - \langle A_{\sigma(X,Z)}Y,W\rangle + \langle \widetilde{R}(X,Y)Z,W\rangle, \tag{5.2}$$

$$(\widetilde{R}(X,Y)Z)^\perp = (\bar{\nabla}_X\sigma)(Y,Z) - (\bar{\nabla}_Y\sigma)(X,Z), \tag{5.3}$$

where X, Y, Z, W are vectors tangent to M^2, $\langle\cdot,\cdot\rangle = g(\cdot,\cdot)$ and $\bar{\nabla}\sigma$ is defined by

$$(\bar{\nabla}_X\sigma)(Y,Z) = D_X\sigma(Y,Z) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z). \tag{5.4}$$

Let $\{e_1,e_2\}$ be orthonormal frame fields along M^2 such that $e_2 = \xi$. We may assume that $H = \alpha\phi e_1$, where $\alpha \in C^\infty(M)$ and $\alpha > 0$. Then from (2.6), we see that the second fundamental form σ takes the following form:

$$\sigma(e_1,e_1) = 2\alpha\phi e_1, \tag{5.5}$$
$$\sigma(e_2,e_2) = 0, \tag{5.6}$$
$$\sigma(e_1,e_2) = -\beta\phi e_1, \tag{5.7}$$

for some function β. Equations (5.5)–(5.7) are equivalent to

$$A_{\phi e_1}e_1 = 2\alpha e_1 - \beta e_2, \tag{5.8}$$
$$A_{\phi e_1}e_2 = -\beta e_1. \tag{5.9}$$

We put $\gamma = \langle he_1, \phi e_1\rangle$. We need the following lemmas for the later use.

Lemma 5.1.

$$\nabla_{e_1}e_1 = -\gamma e_2, \tag{5.10}$$
$$\nabla_{e_1}e_2 = \gamma e_1, \tag{5.11}$$
$$\nabla_{e_2}e_1 = \nabla_{e_2}e_2 = 0. \tag{5.12}$$

Proof. From (2.6), we have (5.11) and (5.12). Equation (5.10) is obtained by (5.11). □

Lemma 5.2.

$$2e_2\alpha = -e_1\beta - 2\alpha\gamma, \tag{5.13}$$
$$e_2\beta = \gamma(\mu - 2\beta), \tag{5.14}$$
$$-e_2\gamma - \gamma^2 = -\beta^2 + \kappa + \mu(\beta - 1). \tag{5.15}$$

Proof. First we put $X = e_2$, $Y = Z = e_1$ in (5.3). Then from (2.9), (2.10), (5.5)–(5.7), and Lemma 5.1, we obtain

$$\begin{aligned} 0 &= D_{e_2}\sigma(e_1,e_1) - \{D_{e_1}\sigma(e_2,e_1) - \sigma(\nabla_{e_1}e_2,e_1) - \sigma(e_2,\nabla_{e_1}e_1)\} \\ &= 2e_2\alpha\phi e_1 + e_1\beta\phi e_1 + 2\alpha\gamma\phi e_1. \end{aligned} \tag{5.16}$$

This implies (5.13).

Next, we substitute $X = e_1$, $Y = Z = e_2$ into (5.3). Similarly, we have

$$\begin{aligned} \langle Qe_1,\phi e_1\rangle\phi e_1 &= -2\sigma(\nabla_{e_1}e_2,e_2) - D_{e_2}\sigma(e_1,e_2) \\ &= 2\beta\gamma\phi e_1 + e_2\beta\phi e_1. \end{aligned} \tag{5.17}$$

Since $\langle Qe_1,\phi e_1\rangle\phi e_1 = \mu\gamma\phi e_1$, we get (5.14).

Finally, we put $X = W = e_1$ and $Y = Z = e_2$ in (5.2). Then it follows from (5.5)–(5.7) and Lemma 5.1 that the left-hand side of (5.2) is

$$\langle -\nabla_{e_2}\nabla_{e_1}e_2 - \nabla_{\nabla_{e_1}e_2}e_2, e_1\rangle = -e_2\gamma - \gamma^2. \tag{5.18}$$

On the other hand, the right-hand side of (5.2) is

$$-\beta^2 + \langle Qe_1, e_1\rangle + \langle Qe_2, e_2\rangle - \frac{f^*S}{2} = -\beta^2 + \frac{1}{2}(f^*S - 2\kappa) + \mu\langle he_1, e_1\rangle \\ + \frac{1}{2}(f^*S - 2\kappa) + \frac{1}{2}(6\kappa - f^*S) - \frac{f^*S}{2}. \tag{5.19}$$

By (2.6) and (5.7), we get

$$\beta - 1 = \langle he_1, e_1\rangle. \tag{5.20}$$

Thus, (5.15) is proved. □

Lemma 5.3.

$$(\beta - 1)(2\beta - \mu) = e_2\gamma, \tag{5.21}$$
$$e_1\beta = 4\alpha\gamma. \tag{5.22}$$

Proof. First, we differentiate both sides of $\gamma = \langle he_1, \phi e_1\rangle$. Then it follows from (2.5), (2.8), (5.7), and (5.12) that

$$\begin{aligned} e_2\gamma &= e_2\langle he_1, \phi e_1\rangle \\ &= \langle \tilde{\nabla}_{e_2} he_1, \phi e_1\rangle + \langle he_1, \tilde{\nabla}_{e_2}\phi e_1\rangle \\ &= \langle h(\tilde{\nabla}_{e_2} e_1), \phi e_1\rangle - \langle \mu\phi he_1, \phi e_1\rangle + \langle he_1, \phi(\tilde{\nabla}_{e_2} e_1)\rangle \\ &= -\beta\langle h\phi e_1, \phi e_1\rangle - \mu(\beta - 1) + \beta(\beta - 1) \\ &= (\beta - 1)(2\beta - \mu). \end{aligned} \tag{5.23}$$

Next, we differentiate both sides of (5.20). Then from (2.5), (2.8), (5.5), and (5.10), we get

$$\begin{aligned} e_1\beta &= e_1\langle he_1, e_1\rangle \\ &= \langle \tilde{\nabla}_{e_1} he_1, e_1\rangle + \langle he_1, \tilde{\nabla}_{e_1} e_1\rangle \\ &= \langle h(\tilde{\nabla}_{e_1} e_1), e_1\rangle + \langle he_1, \tilde{\nabla}_{e_1} e_1\rangle \\ &= 2\alpha\langle h\phi e_1, e_1\rangle + 2\alpha\langle he_1, \phi e_1\rangle \\ &= 4\alpha\gamma. \end{aligned} \tag{5.24}$$

The proof is finished. □

By using the Gauss and Weingarten formulas, we obtain

$$\bar{\Delta}_f H = \operatorname{tr}(\bar{\nabla} A_H) + \Delta^D H + (\operatorname{tr} A^2_{\phi e_1})H, \tag{5.25}$$

where A is the shape operator, $\Delta^D = -\sum_{i=1}^{2}(D_{e_i}D_{e_i} - D_{\nabla_{e_i}e_i})$, and $\operatorname{tr}(\bar{\nabla} A_H) = \sum_{i=1}^{2}(A_{D_{e_i}H}e_i + \nabla_{e_i}(A_H e_i) - A_H(\nabla_{e_i}e_i))$. For detailed computation, we refer to [7].

LEMMA 5.4.

$$\begin{aligned}\operatorname{tr}(\bar{\nabla} A_H) &= (6\alpha e_1\alpha - 2\beta e_2\alpha - \alpha e_2\beta - 2\alpha\beta\gamma)e_1 \\ &\quad - (2\beta e_1\alpha + 2\alpha^2\gamma + \alpha e_1\beta)e_2,\end{aligned} \tag{5.26}$$

$$\Delta^D H = (-e_1e_1\alpha - e_2e_2\alpha - \gamma e_2\alpha)\phi e_1, \tag{5.27}$$

$$\operatorname{tr} A^2_{\phi e_1} H = (4\alpha^2 + 2\beta^2)\alpha\phi e_1, \tag{5.28}$$

$$\mathcal{R}_f(H) = \alpha\mu\gamma e_1 + \left\{\frac{1}{2}(f^*S - 2\kappa) - \mu(\beta - 1)\right\}\alpha\phi e_1. \tag{5.29}$$

Proof. Using (5.8)–(5.12), we obtain the following:

$$\begin{aligned}
&A_{D_{e_1}(\alpha\phi e_1)}e_1 + \nabla_{e_1}(A_He_1) - A_H(\nabla_{e_1}e_1) \\
&\quad = (e_1\alpha)(2\alpha e_1 - \beta e_2) + \nabla_{e_1}(2\alpha^2 e_1 - \alpha\beta e_2) - \alpha A_{\phi e_1}(-\gamma e_2) \\
&\quad = (e_1\alpha)(2\alpha e_1 - \beta e_2) + 4\alpha(e_1\alpha)e_1 - 2\alpha^2\gamma e_2 - e_1(\alpha\beta)e_2 - 2\alpha\beta\gamma e_1 \\
&\quad = \{6\alpha e_1\alpha - 2\alpha\beta\gamma\}e_1 - \{2\beta e_1\alpha + 2\alpha^2\gamma + \alpha e_1\beta\}e_2, \\
&A_{D_{e_2}(\alpha\phi e_2)}e_2 + \nabla_{e_2}(A_He_2) - A_H(\nabla_{e_2}e_2) \\
&\quad = (e_2\alpha)(-\beta e_1) + \nabla_{e_2}(-\alpha\beta e_1) \\
&\quad = -\{2\beta e_2\alpha + \alpha e_2\beta\}e_1.
\end{aligned} \tag{5.30}$$

Combining them, we get (5.26). Equations (5.27) and (5.28) can be proved easily.

Finally, we will prove (5.29):

$$\begin{aligned}
\mathcal{R}_f(H) &= \alpha\widetilde{R}(\phi e_1, e_1)e_1 + \alpha\widetilde{R}(\phi e_1, e_2)e_2 \\
&= \alpha\left\{Q\phi e_1 + \langle Qe_1, e_1\rangle\phi e_1 - \langle Q\phi e_1, e_1\rangle e_1 - \frac{S}{2}\phi e_1\right\} \\
&\quad + \alpha\left\{Q\phi e_1 + \langle Qe_2, e_2\rangle\phi e_1 - \langle Q\phi e_1, e_2\rangle e_2 - \frac{S}{2}\phi e_1\right\} \\
&= \alpha\left\{\left(\frac{f^*S}{2} - 2\kappa\right)\phi e_1 + \mu h\phi e_1 + \mu\langle he_1, e_1\rangle\phi e_1 - \mu\langle h\phi e_1, e_1\rangle\phi e_1\right\} \\
&\quad + \alpha\{\kappa\phi e_1 + \mu h\phi e_1\} \\
&= \frac{f^*S - 2\kappa}{2}\alpha\phi e_1 + \alpha\mu\gamma e_1 - \mu(\beta - 1)\alpha\phi e_1.
\end{aligned} \tag{5.31}$$

The proof is completed. □

Using Lemma 5.4, we obtain the following system of partial differential equations.

LEMMA 5.5. *M^2 is biharmonic if and only if*

$$6\alpha e_1\alpha - 2\beta e_2\alpha - \alpha e_2\beta - 2\alpha\beta\gamma - \alpha\mu\gamma = 0, \tag{5.32}$$

$$2\beta e_1\alpha + 2\alpha^2\gamma + \alpha e_1\beta = 0, \tag{5.33}$$

$$e_1e_1\alpha + e_2e_2\alpha + \gamma e_2\alpha - \alpha(4\alpha^2 + 2\beta^2) - \alpha\mu(\beta - 1) + \frac{1}{2}\alpha(f^*S - 2\kappa) = 0. \tag{5.34}$$

By solving the system of (5.32)–(5.34), we characterize proper biharmonic anti-invariant surfaces in 3-dimensional (κ,μ)-manifolds in terms of the mean curvature.

THEOREM 5.6. *Let* $f : M^2 \to M^3(\kappa,\mu)$ *be a nonminimal anti-invariant surface of a 3-dimensional* (κ,μ)*-manifold. Then* M^2 *is biharmonic if and only if* $\kappa = 1$*; that is,* $M^3(\kappa,\mu)$ *is a Sasakian manifold, and moreover,* $|H|^2 = (1/8)(f^*S - 6) = \text{constant}(\neq 0)$.

Proof. From (5.13) and (5.22), we get

$$e_2\alpha = -3\alpha\gamma. \tag{5.35}$$

Substituting (5.14) and (5.35) into (5.32), we have

$$3e_1\alpha + 3\beta\gamma - \mu\gamma = 0. \tag{5.36}$$

Also, substituting (5.22) into (5.33) gives us

$$\beta e_1\alpha + 3\alpha^2\gamma = 0. \tag{5.37}$$

If $\beta = 0$ at a point p, (5.37) implies $\gamma = 0$ at p. We put $W_1 = \{p \in M^2 \mid \beta \neq 0\}$. Suppose that W_1 is not empty. Then combining (5.36) and (5.37) on W_1, we obtain

$$\gamma(-9\alpha^2 + 3\beta^2 - \beta\mu) = 0. \tag{5.38}$$

We put $W_2 = \{p \in W_1 \mid \gamma \neq 0\}$ and assume that W_2 has a nonempty interior. On W_2, we have

$$-9\alpha^2 + 3\beta^2 - \beta\mu = 0, \tag{5.39}$$

and hence, differentiating (5.39) by e_1, we get

$$-18\alpha e_1\alpha + 6\beta e_1\beta - \mu e_1\beta = 0. \tag{5.40}$$

Combining (5.22), (5.37), and (5.40) gives

$$27\alpha^2 + 12\beta^2 - 2\beta\mu = 0. \tag{5.41}$$

However, (5.39) and (5.41) imply that α and β must be 0. It is a contradiction. Thus, the interior of W_2 is empty. Therefore $\gamma = 0$ on W_1. But we have already seen that $\gamma = 0$ on $M^2 - W_1$. Thus, $\gamma = 0$ on M^2. By (5.21), we have $\beta = 1$ or $\mu/2$. Anyway, α^2 is constant from (5.13) and (5.32). Assume that $\beta = (\mu/2)(\neq 1)$. Then by (5.34), we get

$$4\alpha^2 = -\mu^2 + \mu + \frac{1}{2}(f^*S - 2\kappa). \tag{5.42}$$

Since $S = 2(\kappa - \mu)$ on $M^3(\kappa,\mu)$ with $\kappa \neq 1$ (see (2.11)), by (5.42), we have $\alpha^2 = \mu^2 = 0$. It is a contradiction. Therefore $\beta \neq \mu/2$. Thus, $\beta = 1$.

From (5.15) we obtain $\kappa = 1$, and hence, $M^3(\kappa,\mu)$ is a Sasakian manifold. Furthermore from (5.34), we find that α^2 is equal to

$$\frac{f^*S - 6}{8}. \tag{5.43}$$

Conversely, if $M^3(\kappa,\mu)$ is a Sasakian manifold and $\alpha^2 = (1/8)(f^*S - 6) = \text{constant}$, we can easily see that M^2 satisfy (5.32)–(5.34). Actually, since α is a nonzero constant, it follows from (5.13) and (5.22) that $\gamma = 0$ on M^2. Therefore, (5.14) and (5.22) imply that β is also a constant, so that (5.32) and (5.33) are trivially satisfied. Moreover, from (5.15) and (5.21), with $\gamma = 0$ and $\kappa = 1$, we conclude that $\beta = 1$, and (5.34) is also satisfied. This completes the proof. □

Corollary 5.7. *There exist no proper biharmonic anti-invariant surfaces in Sasakian 3-manifolds with $S \leq 6$.*

Corollary 5.8. *Let $f : M^2 \to N^3(c)$ be a nonminimal anti-invariant surface of Sasakian space forms of constant ϕ-sectional curvature c. Then M^2 is biharmonic if and only if $c > 1$ and $|H|^2 = (c-1)/4$.*

Proof. From (4.7), we see that the scalar curvature of 3-dimensional Sasakian space form $M^3(c)$ is equal to $4 + 2c$. Hence by applying Theorem 5.6, we get Corollary 5.8. □

Remark 5.9. Corollaries 4.5 and 5.8 imply that there are no proper biharmonic Legendre curves and anti-invariant surfaces in the unit 3-sphere (cf. [6]). However, there are proper biharmonic Legendre surfaces and anti-invariant submanifolds in the unit 5-sphere (see [1, 11]). It is an interesting phenomenon.

Acknowledgment

This paper is prepared during the fourth author's visit to the Uludag University, Bursa, Turkey in March 2004.

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