PROPERTIES OF RATIONAL ARITHMETIC FUNCTIONS

VICHIAN LAOHAKOSOL AND NITTIYA PABHAPOTE

Received 13 January 2005 and in revised form 20 September 2005

Rational arithmetic functions are arithmetic functions of the form $g_1 * \cdots * g_r * h_1^{-1} * \cdots * h_s^{-1}$, where g_i, h_j are completely multiplicative functions and $*$ denotes the Dirichlet convolution. Four aspects of these functions are studied. First, some characterizations of such functions are established; second, possible Busche-Ramanujan-type identities are investigated; third, binomial-type identities are derived; and finally, properties of the Kesava Menon norm of such functions are proved.

1. Introduction

By an *arithmetic function* we mean a complex-valued function whose domain is the set of positive integers $\mathbb{N}$. We define the addition and the Dirichlet convolution of two arithmetic functions f and g, respectively, by

$$(f+g)(n) = f(n)+g(n), \qquad (f*g)(n) = \sum_{ij=n} f(i)g(j). \tag{1.1}$$

It is well known (see, e.g., [1, 5, 13, 19, 21]) that the set $(\mathcal{A},+,*)$ of all arithmetic functions is a unique factorization domain with the arithmetic function

$$I(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

being its convolution identity.

A nonzero arithmetic function $f \in \mathcal{A}$ is called *multiplicative*, denoted by $f \in \mathcal{M}$, if $f(mn) = f(m)f(n)$ whenever $(m,n) = 1$. It is called *completely multiplicative*, denoted by $f \in \mathcal{C}$, if $f(mn) = f(m)f(n)$ for all $m,n \in \mathbb{N}$.

For nonnegative integers r, s by an (r,s)-*rational arithmetic function* f, denoted by $f \in \mathcal{C}(r,s)$, we mean an arithmetic function which can be written as

$$f = g_1 * \cdots * g_r * h_1^{-1} * \cdots * h_s^{-1}, \tag{1.3}$$

International Journal of Mathematics and Mathematical Sciences 2005:24 (2005) 3997–4017
DOI: 10.1155/IJMMS.2005.3997

where each $g_i, h_j \in \mathscr{C}$. Such functions were first studied by Vaidyanathaswamy [23] in 1931, and later by several authors; see, for example, [4, 6, 7, 9, 10, 13, 16, 18, 20]. Two important classes of rational functions are $\mathscr{C}(1,1)$ whose elements are known as *totients*, and $\mathscr{C}(2,0)$ whose elements are the so-called *specially multiplicative functions*. Characterizations of these two classes can be found in [7, 10], respectively.

The present work deals with four aspects of rational arithmetic functions. In the next section, some characterizations of these functions are derived and are then used in the next sections to investigate whether two types of identities, the Busche-Ramanujan identity and the binomial identity, which are known to hold for totients and/or specially multiplicative functions, continue to hold for general rational arithmetic functions. In the last section, the Kesava Menon norm of such functions is studied.

We will find it helpful to make use of two important concepts which we now recall. For $f \in \mathscr{A}$, $f(1) \in \mathbb{R}^+$, the Rearick logarithm of f (see [11, 14, 15]), denoted by $\operatorname{Log} f \in \mathscr{A}$, is defined via

$$
\begin{gathered}
(\operatorname{Log} f)(1) = \log f(1), \\
(\operatorname{Log} f)(n) = \frac{1}{\log n} \sum_{d|n} f(d) f^{-1}\left(\frac{n}{d}\right) \log d = \frac{1}{\log n} (df * f^{-1})(n) \quad (n > 1),
\end{gathered}
\tag{1.4}
$$

where $df(n) = f(n)\log n$ denotes the log derivation of f. The Hsu's generalized Möbius function (see [2]) μ_r, $r \in \mathbb{R}$, is defined as

$$
\mu_r(n) = \prod_{p|n} \binom{r}{\nu_p(n)} (-1)^{\nu_p(n)}, \tag{1.5}
$$

where $\nu_p(n)$ is the highest power of the prime p dividing n. It is known (see [8, 12]) that for $f \in \mathscr{M}$,

$$
f \in \mathscr{C} \Longrightarrow f^r = \mu_{-r} f, \tag{1.6}
$$

and the converse holds under additional hypotheses.

2. Characterizations

In this section, r and s will generally denote nonnegative integers. Should either of them be zero, the sum and/or any other expressions connected with them are taken to be zero.

Theorem 2.1. *Let r, s be nonnegative integers and $f \in \mathscr{M}$. Then, $f \in \mathscr{C}(r,s) \Leftrightarrow$ for each prime p and each $\alpha \in \mathbb{N}$, there exist complex numbers $a_1(p), \ldots, a_r(p)$, $b_1(p), \ldots, b_s(p)$ such that*

$$
(\operatorname{Log} f)(n) = \begin{cases} \dfrac{1}{\alpha}\left[a_1(p)^\alpha + \cdots + a_r(p)^\alpha - b_1(p)^\alpha - \cdots - b_s(p)^\alpha\right] & \text{if } n = p^\alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}
$$

Proof.

$$\begin{aligned} f \in \mathcal{C}(r,s) &\Longleftrightarrow f = g_1 * \cdots * g_r * h_1^{-1} * \cdots * h_s^{-1} \quad (g_i, h_j \in \mathcal{C}) \\ &\Longleftrightarrow \operatorname{Log} f = \operatorname{Log} g_1 + \cdots + \operatorname{Log} g_r - \operatorname{Log} h_1 - \cdots - \operatorname{Log} h_s. \end{aligned} \tag{2.2}$$

The result now follows immediately from Carroll's theorem [3] which states that for $F \in \mathcal{M}$,

$$F \in \mathcal{C} \Longleftrightarrow (\operatorname{Log} F)(n) = \begin{cases} \dfrac{1}{\alpha} F(p)^{\alpha} & \text{if } n = p^{\alpha}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

□

Taking $a_1(p)^{\alpha} = f(p^{\alpha+1})/f(p)$, $b_1(p) = b(p)$ in Theorem 2.1, we get the following corollary.

Corollary 2.2. *Let $f \in \mathcal{M}$, with $f(p) \neq 0$ for each prime p. Then $f \in \mathcal{C}(1,1) \Longleftrightarrow$ for each prime p and each $\alpha \in \mathbb{N}$, there is a complex number $b(p)$ such that*

$$(\operatorname{Log} f)(n) = \begin{cases} \dfrac{1}{\alpha} \left(\dfrac{f(p^{\alpha+1})}{f(p)} - b(p)^{\alpha} \right) & \text{if } n = p^{\alpha}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Theorem 2.3. *Let r, s be nonnegative integers and $f \in \mathcal{M}$. Then $f \in \mathcal{C}(r,s) \Leftrightarrow$ for each prime p and each $\alpha \in \mathbb{N}$, there exist complex numbers $a_1(p), \dots, a_r(p)$, $b_1(p), \dots, b_s(p)$ such that for all $\alpha \geq s$,*

$$f(p^{\alpha}) = \sum_{k=0}^{s} G_{\alpha-k} H_k, \tag{2.5}$$

where

$$\begin{aligned} G_{\alpha-k} &= \sum_{j_1+\cdots+j_r=\alpha-k} a_1(p)^{j_1} \cdots a_r(p)^{j_r}, \quad G_0 = 1, \\ H_k &= (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq s} b_{i_1}(p) \cdots b_{i_k}(p), \quad H_0 = 1. \end{aligned} \tag{2.6}$$

Proof.

$$\begin{aligned} f \in \mathcal{C}(r,s) &\Longleftrightarrow f = g_1 * \cdots * g_r * h_1^{-1} * \cdots * h_s^{-1} \quad (g_i, h_j \in \mathcal{C}) \\ &\Longleftrightarrow f(p^{\alpha}) = \sum_{j_1+\cdots+j_r+k_1+\cdots+k_s=\alpha} g_1(p)^{j_1} \cdots g_r(p)^{j_r} h_1^{-1}(p^{k_1}) \cdots h_s^{-1}(p^{k_s}). \end{aligned} \tag{2.7}$$

The result now follows by grouping terms on the right-hand side and using $h^{-1}(p^k) = 0$ for $k \geq 2$. □

A few known characterizations of two particular classes of functions, namely, those in $\mathscr{C}(1,1)$, that is, totients (see [7]), and those in $\mathscr{C}(2,0)$, that is, specially multiplicative functions (see [13, Theorem 1.12]), are immediate consequences of Theorem 2.3, which we record in the following corollary together with a characterizing property of $\mathscr{C}(1,s)$ to be used later.

Corollary 2.4. *Let $f \in \mathcal{M}$. Then the following hold.*

(i) $f \in \mathscr{C}(1,1) \Leftrightarrow$ *for each prime p and each $\alpha \in \mathbb{N}$, there exists a complex number $a(p)$ such that*

$$f(p^\alpha) = a(p)^{\alpha-1} f(p). \tag{2.8}$$

(ii) $f \in \mathscr{C}(2,0) \Leftrightarrow$ *for each prime p and each $\alpha(\geq 2) \in \mathbb{N}$,*

$$f(p^{\alpha+1}) = f(p)f(p^\alpha) + f(p^{\alpha-1})[f(p^2) - f(p)^2]. \tag{2.9}$$

(iii) $f \in \mathscr{C}(1,s) \Leftrightarrow$ *for each prime p and each $\alpha \in \mathbb{N}$, there exist complex numbers $a(p)$, $b_1(p),\ldots,b_s(p)$ such that for all $\alpha \geq s$,*

$$f(p^\alpha) = \sum_{k=0}^{s} g(p)^{\alpha-k} H_k, \tag{2.10}$$

where

$$H_k = (-1)^k \sum_{1\leq i_1<i_2<\cdots<i_k\leq s} b_{i_1}(p)\cdots b_{i_k}(p), \qquad H_0 = 1. \tag{2.11}$$

Simplified characterizations for rational arithmetic functions belonging to the classes where r is 0 can similarly be obtained as in the next corollaries.

Corollary 2.5. *Let s be a nonnegative integer and $f \in \mathcal{M}$. Then $f \in \mathscr{C}(0,s) \Leftrightarrow$ for each prime p, $f(p^\alpha) = 0$ for all $\alpha > s$.*

Proof.

$$\begin{aligned} f \in \mathscr{C}(0,s) &\Longleftrightarrow f = h_1^{-1} * \cdots * h_s^{-1} \quad (h_i \in \mathscr{C}) \\ &\Longleftrightarrow f(p^\alpha) = \sum_{i_1+\cdots+i_s=\alpha} h_1^{-1}(p^{i_1})\cdots h_s^{-1}(p^{i_s}). \end{aligned} \tag{2.12}$$

The result now follows by noting that for $h \in \mathscr{C}$, we have $h^{-1}(p) = -h(p)$, $h^{-1}(p^i) = 0$ for $i \geq 2$, and that the s complex numbers $h_1(p),\ldots,h_s(p)$ are uniquely determined by the s values $f(p),\ldots,f(p^s)$, which are generally arbitrary. In fact, by elementary symmetric functions, we note that $h_1(p),\ldots,h_s(p)$ are just all the s roots of

$$X^s + f(p)X^{s-1} + \cdots + f(p^{s-1})X + f(p^s) = 0. \tag{2.13}$$

This indeed renders their existence, which was stated in the result of Theorem 2.3, to be redundant. □

Invoking upon the fact that $f \in \mathscr{C}(r,0) \Leftrightarrow f^{-1} \in \mathscr{C}(0,r)$, we easily deduce our next result which appears as [13, Problem 1.16, page 48].

Corollary 2.6. *Let r be a nonnegative integer and $f \in \mathcal{M}$. Then*

$$f \in \mathscr{C}(r,0) \Longleftrightarrow \text{for each prime } p, \qquad f^{-1}(p^\alpha) = 0 \quad \forall \alpha > r. \tag{2.14}$$

Corollary 2.7. *Let r be a nonnegative integer and $f \in \mathcal{M}$. Then $f \in \mathscr{C}(r,0) \Leftrightarrow$ for each prime p, and for all $\alpha \geq r$,*

$$f(p^{\alpha+1}) = -\left[f(p^\alpha)f^{-1}(p) + f(p^{\alpha-1})f^{-1}(p^2) + \cdots + f(p^{\alpha-r+1})f^{-1}(p^r)\right]. \tag{2.15}$$

Proof. This follows by expanding $f * f^{-1} = I$ at the prime powers p^α and applying the result of Corollary 2.6. □

Recall that totients are elements of $\mathscr{C}(1,1)$. It seems natural to further characterize a particular class of $\mathscr{C}(r,s)$, called here (r,s)-*totients*, defined by

$$f = g^r * h^{-s}, \quad g,h \in \mathscr{C}. \tag{2.16}$$

Theorem 2.8. *Let r, s be nonnegative integers, $f \in \mathcal{M}$. Then f is an (r,s)-totient $\Leftrightarrow$ for each prime p and each $\alpha(>2) \in \mathbb{N}$, there are complex numbers $a(p)$, $b(p)$ such that*

$$f(p^\alpha) = (-1)^\alpha \sum_{i=0}^{\alpha} \binom{-r}{\alpha - i}\binom{s}{i} a(p)^{\alpha-i} b(p)^i. \tag{2.17}$$

Proof. Using the definition and properties of Hsu's generalized Möbius function mentioned in Section 1, we have

$$\begin{aligned} f \text{ is an } (r,s)\text{-totient } &\Longleftrightarrow f = g^r * h^{-s} = \mu_{-r}g * \mu_s h \\ &\Longleftrightarrow f(p^\alpha) = \sum_{i=0}^{\alpha} \mu_{-r}g(p^{\alpha-i})\mu_s h(p^i). \end{aligned} \tag{2.18}$$

Taking $a(p) = g(p)$, $b(p) = h(p)$, the result follows. □

Another important characterization of $\mathscr{C}(r,s)$ involving recurrence is due to Rutkowski [18] which states that $f = g_1 * \cdots * g_r * h_1^{-1} * \cdots * h_s^{-1} \in \mathscr{C}(r,s) \Leftrightarrow$ for each prime p and each $\alpha \in \mathbb{N}$, there exist complex numbers $c_1(p),\ldots,c_r(p)$ such that

$$f(p^\alpha) = c_1(p)f(p^{\alpha-1}) + \cdots + c_r(p)f(p^{\alpha-r}) \quad (\alpha > s), \tag{2.19}$$

where

$$c_1(p) = \sum_{i=1}^{r} g_i(p),\ c_2(p) = -\sum_{1 \leq i_1 < i_2 \leq r} g_{i_1}(p)g_{i_2}(p), \ldots,\ c_r(p) = (-1)^{r+1}g_1(p)\cdots g_r(p). \tag{2.20}$$

We will have occasion to use Rutkowski's result later.

3. Busche-Ramanujan-type identities

It is well known (see, e.g., [21, page 62], [7, 10], or [13]) that

$$\begin{aligned} f \in \mathscr{C}(2,0) \Longleftrightarrow &\text{ there exists } B \in \mathscr{C} \text{ such that for all } m,n \in \mathbb{N}, \text{ we have} \\ & f(m)f(n) = \sum_{d\mid(m,n)} f\left(\frac{mn}{d^2}\right)B(d) \qquad (3.1) \\ \Longleftrightarrow &\text{ there exists } F \in \mathscr{M} \text{ such that for all } m,n \in \mathbb{N}, \text{ we have} \\ & f(mn) = \sum_{d\mid(m,n)} f\left(\frac{m}{d}\right)f\left(\frac{n}{d}\right)F(d), \qquad (3.2) \end{aligned}$$

and that

$$\begin{aligned} f \in \mathscr{C}(1,1) \Longleftrightarrow &\text{ there exists } h \in \mathscr{C} \text{ such that for all } m,n \in \mathbb{N}, \text{ we have} \\ & f(m)f(n) = \sum_{d\mid(m,n)} f\left(\frac{mn}{d}\right)h(d)\mu(d) \qquad (3.3) \\ \Longleftrightarrow &\text{ there exists } F \in \mathscr{M} \text{ such that for all } m,n \in \mathbb{N}, \text{ we have} \\ & f(mn) = \sum_{d\mid(m,n)} f\left(\frac{m}{d}\right)f\left(\frac{n}{d}\right)F(d), \qquad (3.4) \end{aligned}$$

whenever the greatest common unitary divisor $(m,n)_u = 1$, $f(p^2) \neq f(p)^2 + F(p)$, and $f(p) \neq 0$ for all primes p. For the notion of unitary divisor, see [21, page 9].

Identities (3.1) and (3.2) are known as Busche-Ramanujan identities, while (3.4) is called the restricted Busche-Ramanujan identity because of the restrictions on m, n. In this section, we ask whether similar identities hold for functions in general $\mathscr{C}(r,s)$. An earlier affirmative answer to a particular case of this problem appears in [9, Theorem 4.2] which in our terminology states that for $f = g_1 * g_2 * h^{-1} \in \mathscr{C}(2,1)$, we have

$$f(mn) = \sum_{d\mid(m,n)} (g_1 * g_2)\left(\frac{m}{d}\right)f\left(\frac{n}{d}\right)\mu(d)g_1(d)g_2(d), \qquad (3.5)$$

whenever $\gamma(m) \mid \gamma(n)$, where $\gamma(m)$ denotes *the product of all distinct primes factors of m*. We will show that there are similar Busche-Ramanujan-type identities for functions in the classes $\mathscr{C}(r,s)$ with $r = 1,2$, but are possible for $r \geq 3$ with rather artificial flavor. As to be expected, the identities are of restricted form, that is, hold with conditions on m, n.

Definition 3.1. Let s be a nonnegative integer. A pair $(m,n) \in \mathbb{N} \times \mathbb{N}$ is said to be s-excessive if for each prime p dividing (m,n), either $\nu_p(m) \geq \nu_p(n) + s$ or $\nu_p(n) \geq \nu_p(m) + s$, where $\nu_p(m)$ denotes the highest power of p appearing in m.

Note that the 0-excessive pairs are trivially all pairs of natural numbers, while the 1-excessive pairs (m,n) correspond exactly to those with the greatest common unitary divisor $(m,n)_u = 1$.

Theorem 3.2. *Let s be a nonnegative integer and $f = g * h_1^{-1} * \cdots * h_s^{-1} \in \mathscr{C}(1,s)$. For each prime p, if $g(p) \neq 0$, and $\sum_{k=0}^{s} g(p)^{s-k} H_k \neq 0$, where $H_k = (-1)^k \sum_{1\le i_1<i_2<\cdots<i_k\le s} h_{i_1}(p) \cdots h_{i_k}(p)$, $H_0 = 1$, then there exists $F \in \mathscr{M}$ such that*

$$f(mn) = \sum_{d|(m,n)} f\left(\frac{m}{d}\right) f\left(\frac{n}{d}\right) F(d), \tag{3.6}$$

for each s-excessive pair (m,n).

Proof. Since $f \in \mathscr{M}$, the identity holds for all m, n with $(m,n) = 1$. It thus remains to prove this identity when $(m,n) > 1$. For such s-excessive pair (m,n), let their prime factorizations be

$$m = p_1^{a_1} \cdots p_u^{a_u} q_{11}^{c_1} \cdots q_{1v}^{c_v}, \qquad n = p_1^{b_1} \cdots p_u^{b_u} q_{21}^{d_1} \cdots q_{2w}^{d_w}, \tag{3.7}$$

where p_i, q_{1j}, q_{2k} are distinct primes; a_i, b_j, c_k, d_l are positive integers. By multiplicativity, we can write

$$f(mn) = Q \prod_{i=1}^{u} f\left(p_i^{a_i+b_i}\right), \tag{3.8}$$

where

$$Q = f\left(q_{11}^{c_1}\right) \cdots f\left(q_{1v}^{c_v}\right) f\left(q_{21}^{d_1}\right) \cdots f\left(q_{2w}^{d_w}\right). \tag{3.9}$$

The right-hand side of the identity becomes

$$\sum_{d|(m,n)} f\left(\frac{m}{d}\right) f\left(\frac{n}{d}\right) F(d) = Q \prod_{i=1}^{u} \sum_{j=0}^{\min(a_i,b_i)} f\left(p_i^{a_i-j}\right) f\left(p_i^{b_i-j}\right) F\left(p_i^{j}\right). \tag{3.10}$$

Assuming without loss of generality that $\nu_p(m) \ge \nu_p(n) + s$, that is, $a \ge b + s$, the identity will be established if we can find $F \in \mathscr{M}$ satisfying

$$f\left(p^{a+b}\right) = \sum_{j=0}^{b} f\left(p^{a-j}\right) f\left(p^{b-j}\right) F\left(p^{j}\right), \tag{3.11}$$

for each prime p. It suffices to exhibit $F(p^j)$, the values of F at prime powers, independent of a and b, such that

$$f_{a+b} = \sum_{j=0}^{b} f_{a-j} f_{b-j} F_j, \tag{3.12}$$

where, for short, we put $f(p^i) = f_i$, $F(p^j) = F_j$. Substituting $b = 1$ into (3.12), we have

$$f_{a+1} = f_a f_1 + f_{a-1} F_1 \quad (a \ge s+1). \tag{3.13}$$

Replacing f_{a+1}, f_a, f_{a-1}, f_1 using Corollary 2.4(iii), we have

$$\sum_{k=0}^{s} g(p)^{a+1-k}H_k = \left(g(p) - \sum_{i=1}^{s} h_i(p)\right)\sum_{k=0}^{s} g(p)^{a-k}H_k + F_1\sum_{k=0}^{s} g(p)^{a-1-k}H_k, \tag{3.14}$$

yielding $F_1 = g(p)\sum_{i=1}^{s} h_i(p)$, which is independent of a, provided that $g(p)$ and $\sum_{k=0}^{s} g(p)^{a-1-k}H_k$ are nonzero. Substituting $b = 2$ into (3.12), we get

$$f_{a+2} = f_a f_2 + f_{a-1} f_1 F_1 + f_{a-2}F_2 \quad (a \geq s+2). \tag{3.15}$$

Replacing f_{a+2}, f_a, f_{a-1}, f_2, f_1, using Corollary 2.4(iii) and the value of F_1, we find that

$$F_2 = g(p)^2\left[\left(\sum_{i=1}^{s} h_i(p)\right)^2 - \sum_{1\leq i_1<i_2\leq s} h_{i_1}(p)h_{i_2}(p)\right], \tag{3.16}$$

independent of a. In general, for fixed j, from Corollary 2.4(iii), with $a - j \geq s$, we have

$$f_{a+j} = g^{2j} f_{a-j}, \qquad f_{a+j-1} = g^{2j-1} f_{a-j}, \ldots, \qquad f_{a-j+1} = g f_{a-j}. \tag{3.17}$$

Substituting these and the previous values of F_i $(i < j)$ into (3.12), and dividing by f_{a-j}, we uniquely determine F_j independent of a. Note that the division by f_{a-j} is legitimate because from $g(p)$, $f_s = \sum_{k=0}^{s} g^{s-k}H_k$ being nonzero, we immediately infer that $f_a \neq 0$ for all $a \geq s$. □

Theorem 3.3. *Let* $s \in \mathbb{N}$. *If* $f = g_1 * g_2 * h_1^{-1} * \cdots * h_s^{-1} \in \mathscr{C}(2,s)$, *then*

$$f(mn) = \sum_{d\mid(m,n)} (g_1 * g_2)\left(\frac{m}{d}\right) f\left(\frac{n}{d}\right)\mu(d)(g_1g_2)(d), \tag{3.18}$$

for each $(s-1)$*-excessive pair* (m,n) *with* $\gamma(m) \mid \gamma(n)$.

Proof. Clearly, the identity holds for all m, n with $\gamma(m) \mid \gamma(n)$ and $(m,n) = 1$. It thus remains to prove this identity when $(m,n) > 1$. For each $(s-1)$-excessive pair (m,n) with $\gamma(m) \mid \gamma(n)$, let their prime factorizations be

$$m = p_1^{a_1} \cdots p_u^{a_u}, \qquad n = p_1^{b_1} \cdots p_u^{b_u}, \tag{3.19}$$

where p_i are distinct primes, a_i *nonnegative* integers, and b_i *positive* integers, $a_i \leq b_i$ $(i = 1,2,3,\ldots,u)$. By multiplicativity, we can write

$$f(mn) = \prod_{i=1}^{u} f\left(p_i^{a_i+b_i}\right). \tag{3.20}$$

The right-hand side of the identity becomes

$$\begin{aligned}\sum_{d\mid(m,n)} &(g_1 * g_2)\left(\frac{m}{d}\right) f\left(\frac{n}{d}\right)\mu(d)(g_1g_2)(d) \\ &= \prod_{i=1}^{u}\sum_{j=0}^{a_i}(g_1 * g_2)(p_i^{a_i-j}) f(p_i^{b_i-j})\mu(p^j)(g_1g_2)(p^j).\end{aligned} \tag{3.21}$$

The identity will be established if we can show that

$$f(p^{a+b}) = \sum_{j=0}^{a}(g_1 * g_2)(p^{a-j}) f(p^{b-j})\mu(p^j)(g_1g_2)(p^j), \tag{3.22}$$

for each prime p and $a \in \mathbb{N} \cup \{0\}$, $b \in \mathbb{N}$ with $b \geq a+s-1$. To this end, it suffices to show that

$$f_{a+b} = g_a^* f_b - g_{a-1}^* f_{b-1} g_1', \tag{3.23}$$

where $f(p^i) = f_i$, $(g_1 * g_2)(p^i) = g_i^*$, $(g_1g_2)(p^j) = g_j'$.

For $a = 0$, (3.23) trivially holds. When $a = 1$, $b \geq s$, from Rutkowski's recurrence, we get

$$f_{b+1} = c_1 f_b + c_2 f_{b-1}. \tag{3.24}$$

Noting that $c_1 = g_1^*$, $c_2 = -g_1'$, (3.23) follows in this case. Now proceed by induction on a. Assume that (3.23) holds up to $a-1$. Again by Rutkowski's recurrence, when $b+a \geq s-1$, noting also that f and $g_1 * g_2$ satisfy the same recurrence, we have

$$\begin{aligned} f_{a+b} &= c_1 f_{b+a-1} + c_2 f_{b+a-2} \\ &= c_1\left(g_{a-1}^* f_b - g_{a-2}^* f_{b-1} g_1'\right) + c_2\left(g_{a-1}^* f_{b-1} - g_{a-2}^* f_{b-2} g_1'\right) \\ &= c_1 g_{a-1}^* f_b + g_{a-2}^* f_b c_2 + c_2 g_{a-1}^* f_{b-1} \\ &= g_a^* f_b - g_{a-1}^* f_{b-1} g_1', \end{aligned} \tag{3.25}$$

as required. □

Theorem 3.3 as stated does not include the case $\mathscr{C}(2,0)$ because (-1)-excessive pair is not defined. However, going through the above proof, we see that in this case, we simply get the result of Haukkanen referred to in (3.5) above. Since functions in $\mathscr{C}(2,0)$ satisfy the Busche-Ramanujan identity, a natural question to ask is whether a $\mathscr{C}(3,0)$-function enjoys such property. A trivial example of the identity function $I = I * I * I = I * I$, which belongs to both $\mathscr{C}(2,0)$ and $\mathscr{C}(3,0)$, shows that the answer is affirmative in certain cases, while $u * u * u = \mu_{-3} \in \mathscr{C}(3,0)$ does not satisfy the Busche-Ramanujan identity. Some necessary conditions for $\mathscr{C}(3,0)$-functions to satisfy the Busche-Ramanujan identity are given in the next proposition.

Proposition 3.4. *Let $f \in \mathscr{C}(3,0)$. If f satisfies the Busche-Ramanujan identity*

$$f(mn) = \sum_{d|(m,n)} f\left(\frac{m}{d}\right) f\left(\frac{n}{d}\right) F(d) \quad (m,n \in \mathbb{N}), \tag{3.26}$$

where $F \in \mathcal{M}$, then for each prime p, there are five possibilities:

(1) $f(p^n) = 0$ *for all* $n \geq 1$, *or*
(2) $f(p^n) = (f(p))^n$ *for all* $n \geq 1$, *or*
(3) $f(p^{2n}) = (f(p^2))^n$, $f(p^{2n-1}) = 0$ *for all* $n \geq 1$, *or*
(4) $f(p^n) = (1+n)(f(p)/2)^n$ *for all* $n \geq 1$, *or*
(5) $f(p^n) = (1/2)(1+f(p)/D)((f_1+D)/2)^n + (1/2)(1-f_1/D)((f_1-D)/2)^n$ *for all* $n \geq 1$, *where* $D = \sqrt{4f(p^2) - 3(f(p))^2} \neq 0$.

Proof. Proceeding as in the proof of Theorem 3.2, we are looking for necessary conditions for f to satisfy the Busche-Ramanujan identity and this amounts to finding $F \in \mathcal{M}$ such that

$$f_{a+b} = \sum_{j=0}^{b} f_{a-j} f_{b-j} F_j, \tag{3.27}$$

for each prime p and $a \geq b$, that is, assuming without loss of generality that $\nu_p(m) \geq \nu_p(n)$. Substituting $b = 1$ into (3.27), we obtain the main recurrence relation

$$f_{a+1} = f_a f_1 + f_{a-1} F_1 \quad (a \geq 1). \tag{3.28}$$

Putting $a = 1$, we get $F_1 = f_2 - f_1^2$. From Corollary 2.7,

$$f_a f_1 + f_{a-1} F_1 = f_a f_1 + f_{a-1}(f_2 - f_1^2) + f_{a-2}(f_3 - 2f_1 f_2 + f_1^3), \tag{3.29}$$

which entails

$$f_{a-1} F_1 = c_2 f_{a-1} + c_3 f_{a-2} \quad (a \geq 3), \tag{3.30}$$

where $c_2 = f_2 - f_1^2$, $c_3 = f_3 - 2f_1 f_2 + f_1^3$. Using $F_1 = f_2 - f_1^2 = c_2$, this last relation simplifies to $c_3 f_{a-2} = 0$ $(a \geq 3)$, and so either

(i) $f_n = 0$ for all $n \geq 1$, or
(ii) $0 = c_3 = f_3 - 2f_1 f_2 + f_1^3$.

In the latter situation, we divide into two cases according to $c_2 = 0$ or $c_2 \neq 0$.

Case 1 ($c_2 = 0$). In this case, it easily follows from the main recurrence relation that $f_a = f_1^a$ for all $a \geq 1$.

Case 2 ($c_2 \neq 0$). In this case, we further subdivide into two subcases according to $f_1 = 0$ or not.

Subcase 2.1 ($f_1 = 0$, and so $f_2 = c_2 \neq 0$). Using the main recurrence relation, it is easily checked that $f(p^{2n}) = (f(p^2))^n$, and $f(p^{2n-1}) = 0$ for all $n \geq 1$.

Subcase 2.2 ($f_1 \neq 0$). In this case, the main recurrence relation is a second-order recurrence with constant coefficients whose characteristic equation is $x^2 - f_1 x - c_2 = 0$, with roots $(1/2)(f_1 \pm D)$, where $D = \sqrt{f_1^2 + 4c_2}$. The solutions corresponding to $D = 0$ or $D \neq 0$ are listed as (4) and (5), respectively, in the statement of the proposition. □

In the proof of Proposition 3.4, Case 1 contains, as a special case, the identity function, while other cases contain some nontrivial $\mathscr{C}(2,0)$-functions, and some nontrivial $\mathscr{C}$-functions. Proposition 3.4 indicates somewhat that $\mathscr{C}(3,0)$-functions which satisfy reasonable Busche-Ramanujan-type identity can be artificially constructed from those satisfying conditions in any of the five cases. We now give an example to substantiate this claim. Recall from Corollary 2.7 that $f \in \mathscr{C}(3,0) \Leftrightarrow$ for each prime p and integers $e \geq 3$, we have

$$f_{e+1} = f_e f_1 + f_{e-1} A + f_{e-2} B, \tag{2.31}$$

where $f_e = f(p^e)$, $A = A(p) = f_2 - f_1^2$, $B = B(p) = f_3 - 2f_2 f_1 + f_1^3$. Should there be a Busche-Ramanujan-type identity, subject to certain conditions on m, n, proceeding as in the proof of Theorem 3.2, we deduce that there must exist $F \in \mathscr{M}$ satisfying

$$f_{a+b} = \sum_{i=0}^{b} f_{a-i} f_{b-i} F_i, \tag{2.32}$$

where $a \geq b$, $F_i = F(p^i)$. Consider the $\mathscr{C}(3,0)$-function defined by

$$f(1) = 1, \qquad f(2^a) \quad (a \geq 1) \tag{2.33}$$

satisfying (2.31) with

$$\begin{gathered} B(2) = f(2^3) - 2f(2^2)f(2^1) + f(2^3) = 0, \\ f(p^a) = 0 \quad (a \geq 1) \end{gathered} \tag{2.34}$$

for all other primes $p \geq 3$. This particular function $f \in \mathscr{C}(3,0)$ because it satisfies (2.31) with $A(2)$, $A(p)$, $B(p)$ (p prime ≥ 3) arbitrary but $B(2) = 0$. It satisfies the Busche-Ramanujan identity (2.32) with $F(2) = A(2)$, $F(2^i) = F(p^j) = 0$ ($i \geq 2$, $j \geq 1$). The situations for general $\mathscr{C}(3,s)$ and $\mathscr{C}(r,s)$ with $r \geq 3$ are analogous. The details are omitted.

Another class of identities for functions in $\mathscr{C}(2,0)$, called extended Busche-Ramanujan identity, is due to Redmond and Sivaramakrishnan [16] which states that for $f \in \mathscr{A}$, define

$$t_0(n) = t(n), \qquad t_k(n) = \begin{cases} t(n) & \text{if } n \mid k, \\ 0 & \text{otherwise.} \end{cases} \tag{2.35}$$

Let $T_0 = T$, $T_k = \mu * t_k$. If $f = g_1 * g_2 \in \mathscr{C}(2,0)$, then

$$\sum_{d|(m,n)} f\left(\frac{m}{d}\right) f\left(\frac{n}{d}\right)(g_1g_2)(d)T_k(d) = \sum_{d|(m,n,k)} t(d)(g_1g_2)(d)f\left(\frac{mn}{d^2}\right). \tag{2.36}$$

Using exactly the same proof as in [16, Theorem 13], together with the result of Theorem 3.3, we have the following theorem.

Theorem 3.5. *Let $s \in \mathbb{N}$. If $f = g_1 * g_2 * h_1^{-1} * \cdots * h_s^{-1} \in \mathscr{C}(2, s)$, then*

$$\sum_{d|(m,n)} (g_1 * g_2)\left(\frac{m}{d}\right) f\left(\frac{n}{d}\right)(g_1g_2)(d)T_k(d) = \sum_{d|(m,n,k)} t(d)(g_1g_2)(d)f\left(\frac{mn}{d^2}\right), \tag{3.37}$$

for each $(s-1)$-excessive pair (m,n) with $\gamma(m) \mid \gamma(n)$.

4. Binomial-type identities

It is known, see, for example, [16] or [21, Chapter 13], that if $f = g_1 * g_2 \in \mathscr{C}(2,0)$, then f satisfies the so-called *binomial identity*

$$f(p^k) = \sum_{j=0}^{[k/2]} (-1)^j \binom{k-j}{j} f(p)^{k-2j}(g_1(p)g_2(p))^j, \tag{4.1}$$

where p is a prime, $k \in \mathbb{N}$. In [6], another form of binomial identity is found, namely,

$$2^k f(p^k) = \sum_{i=0}^{[k/2]} \binom{k+1}{2i+1} f(p)^{k-2i}\left[f(p)^2 - 4g_1(p)g_2(p)\right]^i. \tag{4.2}$$

The derivation of (4.1) in [16] is by induction, while that of (4.2) in [6] is based on solving second-order recurrence relation. Making use of certain Chebyshev-type identities, Haukkanen also derived the following inverse forms of (4.1) and (4.2):

$$f(p)^k = \sum_{i=0}^{[k/2]} \left\{\binom{k}{i} - \binom{k}{i-1}\right\} f(p^{k-2i})(g_1(p)g_2(p))^i, \tag{4.3}$$

$$(k+1)f(p)^k = \sum_{i=0}^{[k/2]} \binom{k+1}{2i} d_{2i}2^{k-2i} f(p^{k-2i})(f(p)^2 - 4g_1(p)g_2(p))^i, \tag{4.4}$$

where d_{2i} is defined as in [17, Section 3.4], namely, via the generating series relation

$$\frac{2x}{e^x - e^{-x}} = \sum_{i=0}^{\infty} d_{2i}\frac{x^{2i}}{(2i)!}. \tag{4.5}$$

Our objective in this section is to use Rutkowski's recurrence to derive binomial-type identities and their inverse forms similar to (4.1)–(4.4) for elements in $\mathscr{C}(2,s)$. Our starting point comes from the observation that (4.1) and (4.2) are indeed equivalent through a combinatorial identity, which we now elaborate.

Starting from (4.2), we have

$$\begin{aligned} f(p^k) &= \sum_{i=0}^{[k/2]} \binom{k+1}{2i+1} \left(\frac{f(p)}{2}\right)^{k-2i} \left[\left(\frac{f(p)}{2}\right)^2 - g_1(p)g_2(p)\right]^i \\ &= \sum_{i=0}^{[k/2]} \binom{k+1}{2i+1} \left(\frac{f(p)}{2}\right)^{k-2i} \sum_{j=0}^{i} \binom{i}{j} \left(\frac{f(p)}{2}\right)^{2i-2j} \left[-g_1(p)g_2(p)\right]^j \\ &= \sum_{j=0}^{[k/2]} \sum_{i=j}^{[k/2]} \binom{k+1}{2i+1} \binom{i}{j} \left(\frac{f(p)}{2}\right)^{k-2j} \left[-g_1(p)g_2(p)\right]^j \\ &= \sum_{j=0}^{[k/2]} (-1)^j f(p)^{k-2j} \left[g_1(p)g_2(p)\right]^j \left(\frac{1}{2^{k-2j}}\right) \sum_{i=j}^{[k/2]} \binom{k+1}{2i+1} \binom{i}{j} \\ &= \sum_{j=0}^{[k/2]} (-1)^j \binom{k-j}{j} f(p)^{k-2j} \left[g_1(p)g_2(p)\right]^j, \end{aligned} \tag{4.6}$$

which is (4.1). The last equality follows from the combinatorial identity

$$\sum_{i=j}^{[k/2]} \binom{k+1}{2i+1} \binom{i}{j} = 2^{k-2j} \binom{k-j}{j} \tag{4.7}$$

which appears in Riordan [17, problem 18(c)].

Theorem 4.1. *Let $s \in \mathbb{N}$ and $f = g * G * h_1^{-1} * \cdots * h_s^{-1} \in \mathscr{C}(2,s)$. For each prime p and each $k > 0$,*

$$2^{k+s} f(p^{k+s}) = (A+B)S_{k+s}(p) - 2\big(Bg(p) + AG(p)\big)S_{k+s-1}(p), \tag{4.8}$$

$$\begin{aligned} f(p^{k+s}) = {} & (A+B) \sum_{j=0}^{[(k+s)/2]} (-1)^j \binom{k+s-j}{j} \left[f(p) + H(p)\right]^{k+s-2j} \left[g(p)G(p)\right]^j \\ & - \big(Bg(p) + AG(p)\big) \frac{S_{k+s-1}(p)}{2^{k+s-1}}, \end{aligned} \tag{4.9}$$

where

$$H(p)=\sum_{i=1}^{s}h_i(p),\qquad A=\frac{f(p^{1+s})G(p)-f(p^{2+s})}{g(p^{1+s})(G(p)-g(p))},\qquad B=\frac{f(p^{2+s})-g(p)f(p^{1+s})}{G(p^{1+s})(G(p)-g(p))},\tag{4.10}$$

$$\begin{aligned}S_{k+s}(p)&=\sum_{i=0}^{[(k+s)/2]}\binom{k+s+1}{2i+1}[f(p)+H(p)]^{k+s-2i}[g(p)-G(p)]^{2i}\\&=\sum_{i=0}^{[(k+s)/2]}\binom{k+s+1}{2i+1}[f(p)+H(p)]^{k+s-2i}\Big[(f(p)+H(p))^2-4g(p)G(p)\Big]^{i}.\end{aligned}\tag{4.11}$$

Proof. Since $f=g*G*H^{-1}$, with $H^{-1}=h_1^{-1}*\cdots*h_s^{-1}$, we have

$$f(p)=g(p)+G(p)-H(p).\tag{4.12}$$

For brevity, we put

$$f_k=f(p^k),\qquad g_k=g(p^k),\qquad G_k=G(p^k),\qquad H=H(p).\tag{4.13}$$

By Rutkowski's theorem,

$$f_k=Cf_{k-1}+Df_{k-2}\quad(k>s),\tag{4.14}$$

where

$$C=g_1+G_1=f_1+H,\qquad D=-g_1G_1.\tag{4.15}$$

The characteristic polynomial of this recurrence is r^2-Cr-D, whose two roots are g_1 and G_1. Let $\Delta=g_1-G_1$.

If $g_1\neq G_1$, then $\Delta\neq 0$, and the general solution of this recurrence is of the form

$$f_{k+s}=Ag_1^{k+s}+BG_1^{k+s}\quad(k>0).\tag{4.16}$$

Using the two initial values

$$f_{1+s}=Ag_1^{1+s}+BG_1^{1+s},\qquad f_{2+s}=Ag_1^{2+s}+BG_1^{2+s},\tag{4.17}$$

we get

$$A=\frac{-f_{1+s}G_1+f_{2+s}}{g_1^{1+s}\Delta},\qquad B=\frac{-f_{2+s}+g_1f_{1+s}}{G_1^{1+s}\Delta}.\tag{4.18}$$

Thus,

$$f_{k+s}=A\left(\frac{C+\Delta}{2}\right)^{k+s}+B\left(\frac{C-\Delta}{2}\right)^{k+s},\tag{4.19}$$

and so

$$\begin{aligned}
2^{k+s} f_{k+s} &= A \sum_{i=0}^{k+s} \binom{k+s}{i} C^{k+s-i} \Delta^i + B \sum_{i=0}^{k+s} \binom{k+s}{i} C^{k+s-i} (-\Delta)^i \\
&= (A+B) \sum_{i=0}^{[(k+s)/2]} \binom{k+s}{2i} C^{k+s-2i} \Delta^{2i} \\
&\quad + (A-B) \sum_{i=0}^{[(k+s-1)/2]} \binom{k+s}{2i+1} C^{k+s-2i-1} \Delta^{2i+1} \\
&= (A+B) \sum_{i=0}^{[(k+s)/2]} \binom{k+s}{2i} C^{k+s-2i} \Delta^{2i} \\
&\quad + \left(A+B-2\left(\frac{Bg_1+AG_1}{C}\right)\right) \sum_{i=0}^{[(k+s-1)/2]} \binom{k+s}{2i+1} C^{k+s-2i} \Delta^{2i} \\
&= (A+B) \sum_{i=0}^{[(k+s)/2]} \binom{k+s+1}{2i+1} C^{k+s-2i} \Delta^{2i} \\
&\quad - 2(Bg_1+AG_1) \sum_{i=0}^{[(k+s-1)/2]} \binom{k+s}{2i+1} C^{k+s-2i-1} \Delta^{2i}.
\end{aligned} \tag{4.20}$$

If $g_1 = G_1$, then $\Delta = 0$. Without loss of generality, we may assume that $g_1 = G_1 := r \neq 0$, for otherwise the desired result is trivial. The general solution of our recurrence now takes the shape

$$f_{k+s} = A' r^{k+s} + (k+s) B' r^{k+s} \quad (k > 0). \tag{4.21}$$

Using the initial conditions

$$f_{1+s} = A' r^{1+s} + (1+s) B' r^{1+s}, \qquad f_{2+s} = A' r^{2+s} + (2+s) B' r^{2+s}, \tag{4.22}$$

we get

$$A' = \frac{(2+s) r f_{1+s} - (1+s) f_{2+s}}{r^{2+s}}, \quad B' = \frac{f_{2+s} - r f_{1+s}}{r^{2+s}}, \quad r = \frac{C}{2}. \tag{4.23}$$

Therefore,

$$2^{k+s} f_{k+s} = \left(A' + (k+s) B'\right) C^{k+s}, \tag{4.24}$$

which agrees with (4.20) under the limit $\Delta \to 0$, and the first identity is established. To establish the second identity, we proceed to use the combinatorial identity alluded to above. Since

$$\begin{aligned}
2^{k+s} f\left(p^{k+s}\right) = (A+B) &\sum_{i=0}^{[(k+s)/2]} \binom{k+s+1}{2i+1} C^{k+s-2i} \Delta^{2i} \\
&- 2(Bg_1+AG_1) S_{k+s-1}(p),
\end{aligned} \tag{4.25}$$

we have

$$\begin{aligned}
f(p^{k+s}) &= (A+B)\sum_{i=0}^{[(k+s)/2]}\binom{k+s+1}{2i+1}\left(\frac{C}{2}\right)^{k+s-2i}\left[\left(\frac{C}{2}\right)^2+D\right]^i \\
&\quad-(Bg_1+AG_1)\frac{S_{k+s-1}(p)}{2^{k+s-1}} \\
&= (A+B)\sum_{i=0}^{[(k+s)/2]}\binom{k+s+1}{2i+1}\left(\frac{C}{2}\right)^{k+s-2i}\sum_{j=0}^{i}\binom{i}{j}\left(\frac{C}{2}\right)^{2i-2j}D^j \\
&\quad-(Bg_1+AG_1)\frac{S_{k+s-1}(p)}{2^{k+s-1}} \qquad (4.26)\\
&= (A+B)\sum_{j=0}^{[(k+s)/2]}C^{k+s-2j}D^j\left(\frac{1}{2^{k+s-2j}}\right)\sum_{i=j}^{[(k+s)/2]}\binom{k+s+1}{2i+1}\binom{i}{j} \\
&\quad-(Bg_1+AG_1)\frac{S_{k+s-1}(p)}{2^{k+s-1}} \\
&= (A+B)\sum_{j=0}^{[(k+s)/2]}\binom{k+s-j}{j}C^{k+s-2j}D^j-(Bg_1+AG_1)\frac{S_{k+s-1}(p)}{2^{k+s-1}},
\end{aligned}$$

where the last equality follows from the identity of Riordan [17, Problem 18(c), page 87]. □

The results of Theorem 4.1 reduce to the identities (4.1) and (4.2) when $s=0$, because then $A+B=1$ and $Bg(p)+AG(p)=H(p)=0$. It remains to establish inverse forms of the two identities of Theorem 4.1.

Theorem 4.2. *Let $s\in\mathbb{N}$ and $f=g*G*h_1^{-1}*\cdots*h_s^{-1}\in\mathscr{C}(2,s)$. For each prime p and each integer $k>0$,*

$$\begin{aligned}
S_{k+s}(p) &= \frac{2^{k+s}}{A+B}\sum_{i=0}^{k-1}\left(\frac{Bg(p)+AG(p)}{A+B}\right)^i f(p^{k+s-i})+\left(\frac{2(Bg(p)+AG(p))}{A+B}\right)^k S_s, \\
(k+s+1)(f(p)+H(p))^{k+s} &= \sum_{i=0}^{[(k+s)/2]}\binom{k+s+1}{2i}d_{2i}S_{k+s-2i}(p) \\
&\quad\times\left[(f(p)+H(p))^2-4g(p)G(p)\right]^i, \\
(f(p)+H(p))^{k+s} & \\
&= \frac{1}{A+B}\sum_{i=0}^{[(k+s)/2]}\left[\binom{k+s}{i}-\binom{k+s}{i-1}\right] \\
&\quad\times\left[f(p^{k+s-2i})+(Bg(p)+AG(p))\frac{S_{k+s-2i-1}(p)}{2^{k+s-2i-1}}\right](g(p)G(p))^i,
\end{aligned} \qquad (4.27)$$

where $S_{k+s}(p)$, $H(p)$, A, B are as defined in Theorem 4.1 and d_{2i} is as defined in (4.4).

Proof. The first identity, for $S_{k+s}(p)$, comes immediately from solving the first-order nonhomogeneous recurrence (4.8) in Theorem 4.1, where we define $g_s := f_s := S_s(p)/2^s$. The second identity, for $(k+s+1)(f(p)+H(p))^{k+s}$, follows from the inverse relation, see, for example, [6, page 160],

$$a_k = \sum_{i=0}^{[k/2]} \binom{k+1}{2i+1} b_{k-2i}c^i \Longleftrightarrow (k+1)b_k = \sum_{i=0}^{[k/2]} \binom{k+1}{2i} d_{2i}a_{k-2i}c^i, \tag{4.28}$$

applied to (4.11) of Theorem 4.1. The third identity, for $(f(p)+H(p))^{k+s}$, follows from the inverse relation, see, for example, [6, page 159],

$$a_k = \sum_{i=0}^{[k/2]} (-1)^i \binom{k-i}{i} b_{k-2i}c^i \Longleftrightarrow b_k = \sum_{i=0}^{[k/2]} \left[\binom{k}{i} - \binom{k}{i-1} \right] a_{k-2i}c^i, \tag{4.29}$$

applied to (4.9) of Theorem 4.1. □

We end this section by remarking that it seems unlikely for functions in $\mathscr{C}(r,s)$ with $r>2$ to have similar binomial-type identities.

5. Kesava Menon norm

For $f \in \mathcal{M}$, its Kesava Menon norm Nf is an arithmetic function defined by (see [16, 20])

$$Nf(n) := (f * \lambda f)(n^2), \tag{5.1}$$

where λ is the well-known Liouville's function, $\lambda(n) = (-1)^{\Omega(n)}$, $\Omega(n)$ being the total number of prime factors of n counted with multiplicity. Observe that $Nf \in \mathcal{M}$ and $\lambda \in \mathscr{C}$ which implies (see [23]) that $\lambda(f*g) = \lambda f * \lambda g$ when $f,g \in \mathcal{M}$. For nonnegative integer m, the mth power (Kesava Menon) norm of $f \in \mathcal{M}$ is inductively defined by

$$N^0 f = f, \qquad N^1 f = Nf, \qquad N^m f = N(N^{m-1} f). \tag{5.2}$$

It is shown in [20, Theorem 3.3, page 160] that

$$f \in \mathscr{C}(2,0) \Longrightarrow Nf \in \mathscr{C}(2,0), \tag{5.3}$$

and in [16, Theorem 3, page 214] that for nonnegative integer m,

$$f_1, f_2 \in \mathscr{C}(2,0) \Longrightarrow N^m(f_1 * f_2) = N^m f_1 * N^m f_2. \tag{5.4}$$

In this section, we prove that both of these properties hold for elements in general $\mathscr{C}(r,s)$.

Theorem 5.1. *Let r, s be positive integers and $f \in \mathscr{C}(r,s)$. Then $Nf \in \mathscr{C}(r,s)$.*

Proof. Let $f = g_1 * \cdots * g_r * h_1^{-1} * \cdots * h_s^{-1}$. By the distributivity of completely multiplicative functions, we get

$$\begin{aligned} f * \lambda f &= (g_1 * \cdots * g_r * h_1^{-1} * \cdots * h_s^{-1}) * (\lambda(g_1 * \cdots * g_r * h_1^{-1} * \cdots * h_s^{-1})) \\ &= (g_1 * \lambda g_1) * \cdots * (g_r * \lambda g_r) * (h_1^{-1} * \lambda h_1^{-1}) * \cdots * (h_s^{-1} * \lambda h_s^{-1}) \\ &= (g_1(u * \lambda)) * \cdots * (g_r(u * \lambda)) * (h_1^{-1} * \lambda h_1^{-1}) * \cdots * (h_s^{-1} * \lambda h_s^{-1}), \end{aligned} \tag{5.5}$$

where $u \in \mathscr{C}$ is the unit function, $u(n) = 1$ $(n \in \mathbb{N})$. Using

$$(u * \lambda)(n) = \begin{cases} 1 & \text{if } n \text{ is a perfect square,} \\ 0 & \text{otherwise,} \end{cases} \tag{5.6}$$

and for $h \in \mathscr{C}$, p prime,

$$(h^{-1} * \lambda h^{-1})(p^k) = \begin{cases} 1 & \text{if } k = 0, \\ -h(p)^2 & \text{if } k = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{5.7}$$

we have for each prime p and $k \in \mathbb{N} \cup \{0\}$ that

$$\begin{aligned} Nf(p^k) &= (f * \lambda f)(p^{2k}) \\ &= \sum_{(2k)} g_1(u * \lambda)(p^{i_1}) \cdots g_r(u * \lambda)(p^{i_r})(h_1^{-1} * \lambda h_1^{-1})(p^{j_1}) \cdots (h_1^{-s} * \lambda h_s^{-1})(p^{j_s}) \\ &= \sum_{(k)} g_1 g_1(p^{i_1}) \cdots g_r g_r(p^{i_r})(h_1 h_1)^{-1}(p^{j_1}) \cdots (h_s h_s)^{-1}(p^{j_s}), \end{aligned} \tag{5.8}$$

where $\sum_{(l)}$ denotes the sum taken over all $(r+s)$-tuples of nonnegative integers $(i_1,\dots,i_r, j_1,\dots,j_s)$ such that $i_1 + \cdots + i_s + j_1 + \cdots + j_s = l$. Since $Nf \in \mathscr{M}$ and $g_i g_i, h_j h_j \in \mathscr{C}$, it follows that $Nf \in \mathscr{C}(r,s)$. $\square$

The gist of Theorem 5.1 is that

$$\begin{aligned} & f = g_1 * \cdots * g_r * h_1^{-1} * \cdots * h_s^{-1} \\ & \Longrightarrow Nf = (g_1 g_1) * \cdots * (g_r g_r) * (h_1 h_1)^{-1} * \cdots * (h_s h_s)^{-1}. \end{aligned} \tag{5.9}$$

Theorem 5.1 remains valid when r and/or s is 0 for we can always, if needed, convolute by I or I^{-1}.

Immediate from these remarks is the following corollary.

Corollary 5.2. *Let $m, r, s \in \mathbb{N} \cup \{0\}$. Then the following hold.*

(i) $f = g_1 * \cdots * g_r * h_1^{-1} * \cdots * h_s^{-1} \in \mathscr{C}(r,s) \Rightarrow N^m f = (g_1)^{2^m} * \cdots * (g_r)^{2^m} * ((h_1)^{2^m})^{-1} * \cdots * ((h_s)^{2^m})^{-1}$, *where* $(g_i)^m = g_i \cdots g_i$ *(m times).*

(ii) $f_1, f_2 \in \mathscr{C}(r,s) \Rightarrow N^m(f_1 * f_2) = N^m f_1 * N^m f_2$.

The Kesava Menon norm of $f \in \mathcal{M}$ is closely related to its (ordinary) square $(f)^2$ as seen from the following two identities of Sivaramakrishnan [20]. If $f = g_1 * g_2 \in \mathscr{C}(2,0)$, then

$$f(n)^2 = \sum_{d|n} Nf\left(\frac{n}{d}\right)\theta(d)(g_1g_2)(d), \tag{5.10}$$

$$\sum_{d|n}\lambda(d)f(d)^2 f\left(\frac{n}{d}\right)^2 = \sum_{d|n}\lambda(d)Nf(d)Nf\left(\frac{n}{d}\right), \tag{5.11}$$

where $\theta(n) = 2^{\omega(n)}$, $\omega(n)$ being the number of distinct prime factors of n. We next show that similar identities hold for functions in $\mathscr{C}(2,1)$.

Theorem 5.3. *Let $f = g_1 * g_2 * h^{-1} \in \mathscr{C}(2,1)$ and let Nf be its Kesava Menon norm. Then there exists $G \in \mathcal{M}$ such that*

$$\begin{gathered}(g_1 * g_2)f = Nf * G * (g_1g_2),\\ (g_1 * g_2)f * \lambda(g_1 * g_2)f = (Nf * \lambda Nf) * (G * \lambda G) * g_1g_2(u * \lambda).\end{gathered} \tag{5.12}$$

Further, G is defined on prime powers by $G(p^e) = f^{-1}(p^{2e})$ $(e \in \mathbb{N})$, and $u(n) = 1$ $(n \in \mathbb{N})$.

Proof. Define $\bar{f} = f * \lambda f \in \mathcal{M}$. Observe that

$$\bar{f}(n) = \begin{cases} Nf(\sqrt{n}) & \text{if } n \text{ is a perfect square,}\\ 0 & \text{otherwise,}\end{cases} \tag{5.13}$$

and that

$$f(n^2) = (\bar{f} * \lambda f^{-1})(n^2) = \sum_{i|n^2}\bar{f}(i)\lambda f^{-1}\left(\frac{n^2}{i}\right) = \sum_{j|n} Nf(j)f^{-1}\left(\frac{n^2}{j^2}\right). \tag{5.14}$$

Define $G \in \mathcal{M}$ by $G(1) = 1$, $G(p^e) = f^{-1}(p^{2e})$ (p prime, $e \in \mathbb{N}$), and extend it by multiplicativity to all positive integers. Thus, $f(n^2) = (Nf * G)(n)$. On the other hand, by Theorem 3.2, when $s = 1$, we have

$$f(n^2) = \sum_{d|n}((g_1 * g_2)f)\left(\frac{n}{d}\right)(\mu g_1g_2)(d). \tag{5.15}$$

Thus $(g_1 * g_2)f * (\mu g_1g_2) = Nf * G$, and the first identity follows by noting that as $g_1g_2 \in \mathscr{C}$, then $(\mu g_1g_2)^{-1} = g_1g_2$.

To prove the second identity, using the first identity and the distributivity of $\lambda \in \mathscr{C}$, we have

$$\lambda(g_1 * g_2)f = \lambda(Nf * G * g_1g_2) = \lambda Nf * \lambda G * \lambda g_1g_2. \tag{5.16}$$

Thus,

$$(g_1 * g_2)f * \lambda(g_1 * g_2)f = (Nf * \lambda Nf) * (G * \lambda G) * (g_1g_2 * \lambda g_1g_2), \tag{5.17}$$

and the desired result follows from the distributivity of $g_1g_2 \in \mathscr{C}$. $\square$

Theorem 5.3 yields the following immediate consequences.

Corollary 5.4. *If* $f = g_1 * g_2 * h^{-1} \in \mathscr{C}(2,1)$, Nf *is its Kesava Menon norm, and* $G \in \mathcal{M}$ *as defined in Theorem 5.3, then*

$$f(n)(g_1 * g_2)(n) = \sum_{ijk=n} Nf(i)G(j)(g_1g_2)(k),$$

$$\sum_{d|n}((g_1 * g_2)f)\left(\frac{n}{d}\right)(\lambda(g_1 * g_2)f)(d) = \sum_{ijk=n}(Nf * \lambda Nf)(i)(G * \lambda G)(j)(g_1g_2(u * \lambda))(k). \tag{5.18}$$

Note that for $f = g_1 * g_2 \in \mathscr{C}(2,0)$, if we interpret $G \in \mathcal{M}$ by

$$G(1) = 1, \qquad G(p) = (g_1g_2)(p), \quad G(p^e) = 0 \quad \text{for prime } p \text{ and integer } e \geq 2, \tag{5.19}$$

then $G * (g_1g_2) = \theta g_1g_2$ and $(G * \lambda G) * (g_1g_2)(u * \lambda) = I$, where $\theta(n) = 2^{\omega(n)}$, I is the convolution identity, and so the identities in Corollary 5.4 reduce to (5.10) and (5.11), respectively.

Added note. Regarding Theorem 5.1, it has been pointed out by one of the referees that Nf for rational arithmetic functions f of order (r,s) has already been given in P. Haukkanen's review on [22].

Acknowledgment

This work was partially supported by the University of the Thai Chamber of Commerce.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer, New York, 1976.

[2] T. C. Brown, L. C. Hsu, J. Wang, and P. J.-S. Shiue, *On a certain kind of generalized number-theoretical Möbius function*, Math. Sci. **25** (2000), no. 2, 72–77.

[3] T. B. Carroll, *A characterization of completely multiplicative arithmetic functions*, Amer. Math. Monthly **81** (1974), no. 9, 993–995.

[4] T. B. Carroll and A. A. Gioia, *On a subgroup of the group of multiplicative arithmetic functions*, J. Austral. Math. Soc. Ser. A **20** (1975), no. 3, 348–358.

[5] E. D. Cashwell and C. J. Everett, *The ring of number-theoretic functions*, Pacific J. Math. **9** (1959), no. 4, 975–985.

[6] P. Haukkanen, *Binomial formulas for specially multiplicative functions*, Math. Student **64** (1995), no. 1–4, 155–161.

[7] ———, *Some characterizations of totients*, Int. J. Math. Math. Sci. **19** (1996), no. 2, 209–217.

[8] ———, *On the real powers of completely multiplicative arithmetical functions*, Nieuw Arch. Wiskd. (4) **15** (1997), no. 1-2, 73–77.

[9] ———, *Rational arithmetical functions of order* (2,1) *with respect to regular convolutions*, Port. Math. **56** (1999), no. 3, 329–343.

[10] ———, *Some characterizations of specially multiplicative functions*, Int. J. Math. Math. Sci. **2003** (2003), no. 37, 2335–2344.

[11] V. Laohakosol, N. Pabhapote, and N. Wechwiriyakul, *Logarithmic operators and characterizations of completely multiplicative functions*, Southeast Asian Bull. Math. **25** (2001), no. 2, 273–281.
[12] ______, *Characterizing completely multiplicative functions by generalized Möbius functions*, Int. J. Math. Math. Sci. **29** (2002), no. 11, 633–639.
[13] P. J. McCarthy, *Introduction to Arithmetical Functions*, Universitext, Springer, New York, 1986.
[14] D. Rearick, *Operators on algebras of arithmetic functions*, Duke Math. J. **35** (1968), no. 4, 761–766.
[15] ______, *The trigonometry of numbers*, Duke Math. J. **35** (1968), no. 4, 767–776.
[16] D. Redmond and R. Sivaramakrishnan, *Some properties of specially multiplicative functions*, J. Number Theory **13** (1981), no. 2, 210–227.
[17] J. Riordan, *Combinatorial Identities*, John Wiley & Sons, New York, 1968.
[18] J. Rutkowski, *On recurrence characterization of rational arithmetic functions*, Funct. Approx. Comment. Math. **9** (1980), 45–47.
[19] H. N. Shapiro, *Introduction to the Theory of Numbers*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1983.
[20] R. Sivaramakrishnan, *On a class of multiplicative arithmetic functions*, J. reine angew. Math. **280** (1976), 157–162.
[21] ______, *Classical Theory of Arithmetic Functions*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 126, Marcel Dekker, New York, 1989.
[22] S. Thajoddin and S. Vangipuram, *A note on Jordan's totient function*, Indian J. Pure Appl. Math. **19** (1988), no. 12, 1156–1161.
[23] R. Vaidyanathaswamy, *The theory of multiplicative arithmetic functions*, Trans. Amer. Math. Soc. **33** (1931), no. 2, 579–662.

Vichian Laohakosol: Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand
E-mail address: fscivil@ku.ac.th

Nittiya Pabhapote: Department of Mathematics, School of Science, University of the Thai Chamber of Commerce, Bangkok 10400, Thailand
E-mail address: nittiya_pab@utcc.ac.th