COASSOCIATIVE GRAMMAR, PERIODIC ORBITS, AND QUANTUM RANDOM WALK OVER ℤ

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Inspired by a work of Joni and Rota, we show that the combinatorics generated by a quantisation of the Bernoulli random walk over ℤ can be described from a coassociative coalgebra. Relationships between this coalgebra and the set of periodic orbits of the classical chaotic system $x \mapsto 2x \bmod 1$, $x \in [0,1]$, are also given.

1. Introduction and notation

Motivated by the success of classical random walks and chaotic dynamical systems, we study the quantisation of the random walk over ℤ and its relationships with a classical chaotic system $x \mapsto 2x \bmod 1$, $x \in [0,1]$. In the physics literature, quantum random walks have been studied, for instance, by Ambainis et al. [1] and Konno et al. [8, 9]. In [8], Konno shows that the quantum random walk over ℤ called the Hadamard random walk generates a particular combinatorics. In [6], Joni and Rota showed that some combinatorics can be recovered from coproducts of coassociative coalgebras. Therefore, is it possible to create a coassociative coalgebra which recovers the combinatorics generated by the Hadamard random walk? We start in Section 2 with briefly recalling a new formalism, inspired by weighted directed graph theory. In Section 3, we present a mathematical framework for studying the Hadamard random walk over ℤ. In Section 4, we construct a coassociative coalgebra based on results on graphs developed in Section 2. We show that the combinatorics generated by the Hadamard random walk over ℤ can be recovered from this coalgebra. In Section 5, we present the notion of quantum graphs developed in [15] and point out a relation between a quantum graph, the classical Bernoulli random walk, the Hadamard random walk, and the periodic orbits of the classical chaotic system $x \mapsto 2x \bmod 1$ with $x \in [0,1]$. Briefly speaking, we show how to relate periodic orbits of this classical chaotic system to polynomials describing a quantisation of the Bernoulli walk.

2. L-coalgebras

In the sequel, k denotes a field. A *directed graph G* is a quadruple [16] (G_0, G_1, s, t), where G_0 and G_1 are two countable sets, respectively, called the *vertex set* and the *arrow set*.

No multiple arrows are allowed in this work. The two mappings $s, t : G_1 \to G_0$ are, respectively, called *source* and *terminus*. A vertex $v \in G_0$ is a *source* (resp., a *sink*) if $t^{-1}(\{v\})$ (resp., $s^{-1}(\{v\})$) is empty. A graph G is said to be *locally-finite* (resp., *row-finite*) if $t^{-1}(\{v\})$ is finite (resp., $s^{-1}(\{v\})$ is finite). Let us fix a vertex $v \in G_0$. Define the set $F_v := \{a \in G_1, s(a) = v\}$. A *weight* associated with the vertex v is a mapping $w_v : F_v \to k$. A directed graph equipped with a family of weights $w := (w_v)_{v \in G_0}$ is called a *weighted directed graph*. In the sequel, directed graphs will be supposed locally-finite and row-finite. Let us introduce particular coalgebras named *L-coalgebras*, introduced in [12] and developed in [13, 14], and explain why this notion is interesting. An *L-coalgebra* $(L, \Delta, \tilde{\Delta})$ over a field k is a k-vector space composed of a right part (L, Δ), where $\Delta : L \to L^{\otimes 2}$ is called the right coproduct, and a left part $(L, \tilde{\Delta})$, where $\tilde{\Delta} : L \to L^{\otimes 2}$ is called the left coproduct, such that the coassociativity breaking equation, $(\tilde{\Delta} \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\tilde{\Delta}$, is verified. If $\Delta = \tilde{\Delta}$, the L-coalgebra is said to be *degenerate*. An L-coalgebra may have two counits, the right counit $\epsilon : L \to k$, verifying $(\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}$, and the left counit $\tilde{\epsilon} : L \to k$, verifying $(\tilde{\epsilon} \otimes \mathrm{id})\tilde{\Delta} = \mathrm{id}$. An L-coalgebra is said to be *coassociative* if its two coproducts are coassociative. In this case, the equation, $(\tilde{\Delta} \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\tilde{\Delta}$, is called the *entanglement equation* and we will say that its right part (L, Δ) is entangled to its left part $(L, \tilde{\Delta})$. Denote by τ the *transposition* mapping, that is, $L^{\otimes 2} \overset{\tau}{\to} L^{\otimes 2}$ such that $\tau(x \otimes y) = y \otimes x$, for all $x, y \in L$. The L-coalgebra L is said to be *L-cocommutative* if $\Delta = \tau \tilde{\Delta}$. Let $G = (G_0, G_1, s, t)$ be a directed graph equipped with a family of weights $(w_v)_{v \in G_0}$. Let us consider the free k-vector space kG_0. The set G_1 is then viewed as a subset of $(kG_0)^{\otimes 2}$ by identifying $a \in G_1$ with $s(a) \otimes t(a)$. The mappings source and terminus are then linear mappings still called source and terminus $s, t : (kG_0)^{\otimes 2} \to kG_0$, such that $s(u \otimes v) = u$ and $t(u \otimes v) = v$, for all $u, v \in G_0$. The family of weights is given by $(w_v : F_v \to k)_{v \in G_0}$. Let $v \in G_0$. Define the right coproduct $\Delta_M : kG_0 \to (kG_0)^{\otimes 2}$, such that $\Delta_M(v) := \sum_{i : a_i \in F_v} w_v(a_i) v \otimes t(a_i)$, and the left coproduct $\tilde{\Delta}_M : kG_0 \to (kG_0)^{\otimes 2}$, such that $\tilde{\Delta}_M(v) := \sum_{i : a_i \in P_v} w_{s(a_i)}(a_i) s(a_i) \otimes v$, where P_v is the set $\{a \in G_1, t(a) = v\}$. With these definitions the k-vector space kG_0 is an L-coalgebra called a *(finite) Markov L-coalgebra* since its coproducts Δ_M and $\tilde{\Delta}_M$ verify the coassociativity breaking equation $(\tilde{\Delta}_M \otimes \mathrm{id})\Delta_M = (\mathrm{id} \otimes \Delta_M)\tilde{\Delta}_M$. In addition, this particular coalgebra is called *finite* since for all $v \in G_0$, the sets F_v and P_v are finite and the coproducts have the form $\Delta_M(v) := v \otimes \cdots$ and $\tilde{\Delta}_M(v) := \cdots \otimes v$, see [12]. Consider the Markov L-coalgebra just described and associate with each tensor product $\lambda u \otimes v$, where $\lambda \in k$ and $u, v \in G_0$, appearing in the definition of the coproducts, a directed arrow $u \overset{\lambda}{\to} v$. The weighted directed graph so obtained, called the *geometric support* of this L-coalgebra, is up to a graph isomorphism, the directed graph we started with. Therefore, general L-coalgebras generalise the notion of weighted directed graphs. If $(L, \Delta, \tilde{\Delta})$ is an L-coalgebra spanned as a k-vector space by a set L_0, then its geometric support $\mathrm{Gr}(L)$ is a directed graph with vertex set $\mathrm{Gr}(L)_0 = L_0$ and with arrow set $\mathrm{Gr}(L)_1$, the set of tensor products $u \otimes v$, with $u, v \in L_0$, appearing in the definition of the coproducts of L. As a coassociative coalgebra is a particular L-coalgebra, we naturally construct its directed graph. We draw attention to the fact that a directed graph can be the geometric support of different L-coalgebras.

Example 2.1. The directed graph shown in Figure 2.1 is the geometric support associated with the degenerate L-coalgebra or coassociative coalgebra $\mathcal{E}$, spanned by the basis a, b, c, and d, as a k-vector space, and described by the coproduct $\Delta(a) = a \otimes a + b \otimes c$,
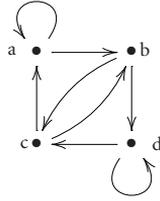
Figure 2.1. The graph Gr($\mathscr{E}$).

Figure 2.2. The $(2,1)$-De Bruijn graph.

$\Delta(b) = a \otimes b + b \otimes d$, $\Delta(c) = d \otimes c + c \otimes a$, $\Delta(d) = d \otimes d + c \otimes b$ and the geometric support of the finite Markov L-coalgebra, spanned by the basis a, b, c, and d, as a k-vector space, and described by the right coproduct $\Delta_M(a) = a \otimes (a+b)$, $\Delta_M(b) = b \otimes (c+d)$, $\Delta_M(c) = c \otimes (a+b)$, $\Delta_M(d) = d \otimes (c+d)$ and the left coproduct $\tilde{\Delta}_M(a) = (a+c) \otimes a$, $\tilde{\Delta}_M(b) = (a+c) \otimes b$, $\tilde{\Delta}_M(c) = (b+d) \otimes c$, $\tilde{\Delta}_M(d) = (b+d) \otimes d$.

Example 2.2 (the $(2,1)$-De Bruijn graph). The graph shown in Figure 2.2, also called the $(2,1)$-De Bruijn graph, is the geometric support of a Markov L-coalgebra, spanned by the basis P and Q. The (coassociative) coproducts are $\Delta(P) = P \otimes P + P \otimes Q$ and $\Delta(Q) = Q \otimes P + Q \otimes Q$, $\tilde{\Delta}(P) = Q \otimes P + P \otimes P$ and $\tilde{\Delta}(Q) = P \otimes Q + Q \otimes Q$.

Let G be a finite Markov L-coalgebra. If the family of weights used for describing right and left coproducts takes values in $\mathbb{R}_+$ and if the right counit $\epsilon : v \mapsto 1$ exists, then the geometric support associated with G is a directed graph equipped with a family of probability vectors. In addition, to enlarge the coassociative coalgebra setting, this algebraic formalism takes also into account the description of weighted paths on a given directed graph. We recall that the sequence $\Delta_1 \equiv \Delta$, $\Delta_2 = \mathrm{id} \otimes \Delta$, $\Delta_3 = \mathrm{id} \otimes \mathrm{id} \otimes \Delta, \dots$ generates all possible weighted paths starting at any vertex. Similarly, the sequence of powers of $\tilde{\Delta}$ generates all the possible weighted paths arriving at a given vertex. We end this part on graph theory by recalling the definition of the line-extension. The *line-extension* of a directed graph $G := (G_0, G_1, s, t)$ with a denumerable vertex set G_0 and a denumerable arrow set $G_1 \subseteq G_0 \times G_0$ is the directed graph with vertex set G_1 and arrow set $G_2 \subseteq G_1 \times G_1$ defined by $(v, w) \in G_1 \times G_1$ belonging to G_2 if and only if $t(v) = s(w)$. This directed graph, called the line-directed graph of G, is denoted by $E(G)$. The line-extension of the $(2,1)$-De Bruijn graph is Gr($\mathscr{E}$).

3. Quantum random walk over $\mathbb{Z}$

In the physics literature, quantum random walks were studied, for instance, by Ambainis et al. [1] and Konno et al. [8, 9]. Here, we propose a mathematical framework for the
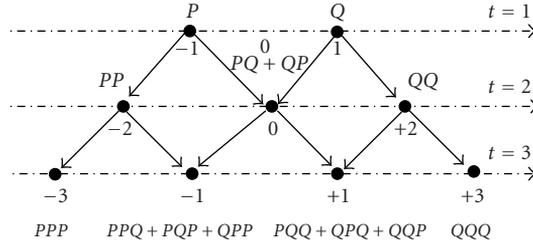
Figure 3.1. Combinatorics from the quantum random walk over $\mathbb{Z}$, up to $t = 3$.

quantum random walk over $\mathbb{Z}$ and show that the combinatorics (we keep the notation of [8]) of this walk can be recovered by using the coproduct of $\mathscr{C}$. Let $\mathscr{H}$ be a separable Hilbert space of infinite dimension with $(|n\rangle)_{n \in \mathbb{Z}}$ as an orthonormal basis. Consider the trivial tensor bundle $\mathscr{H} \otimes M_2(\mathbb{C})$. Fix a unitary matrix U and consider the operators P and Q, such that $U = P + Q$, with

$$P = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ \gamma & \delta \end{pmatrix}. \tag{3.1}$$

They verify the following algebraic relations [8]: $P^2 = \alpha P$, $Q^2 = \delta Q$, $PQP = \beta\gamma P$, $QPQ = \beta\gamma Q$.

Remark 3.1. Suppose $\alpha\delta \neq 0$. Consider $e_1 := (1/\alpha)P$ and $e_2 := (1/\delta)Q$. We get $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 e_2 e_1 = \lambda e_1$, and $e_2 e_1 e_2 = \lambda e_2$, where $\lambda := \gamma\beta/\delta\alpha$, that is, the algebra generated by e_1, e_2 is a Jones algebra [5].

Consider the algebra $\mathbb{C}\langle P, Q \rangle$, that is, the noncommutative polynomials in P and Q, and denote by $\mathscr{D}_-$, $\mathscr{D}_+$ the diffusion operators, that is, the linear maps

$$\mathscr{D}_-, \mathscr{D}_+ : \mathscr{H} \otimes \mathbb{C}\langle P, Q \rangle \longrightarrow \mathscr{H} \otimes \mathbb{C}\langle P, Q \rangle, \tag{3.2}$$

which are defined for all $k \in \mathbb{Z}$ and for all discrete time $n \in \mathbb{Z}$ such that

$$\mathscr{D}_-\big(|k+1\rangle \otimes \Xi_{[k+1;n]}\big) = |k\rangle \otimes \Xi_{[k+1;n]}P, \qquad \mathscr{D}_+\big(|k-1\rangle \otimes \Xi_{[k-1;n]}\big) = |k\rangle \otimes \Xi_{[k-1;n]}Q, \tag{3.3}$$

where $\Xi_{[0;0]} = \mathrm{id}$, $\Xi_{[-1;1]} = P$, $\Xi_{[+1;1]} = Q$, and so on. The dynamics is defined by

$$|k\rangle \otimes \Xi_{[k;n+1]} := \mathscr{D}_-\big(|k+1\rangle \otimes \Xi_{[k+1;n]}\big) + \mathscr{D}_+\big(|k-1\rangle \otimes \Xi_{[k-1;n]}\big). \tag{3.4}$$

That is shown in Figure 3.1 (graphically, the Hilbert space $\mathcal{H}$ will be represented by the usual representation of $\mathbb{Z}$, a vertex $n \in \mathbb{Z}$ meaning $|n\rangle \in \mathcal{H}$).

Example 3.2. We give here the nonzero polynomials $\Xi_{[k;n+1]}$ at time $t = 0,\dots,4$. At time $t = 0$, we get, by convention, $\Xi_{[0;0]} = \text{id}$. At time $t = 1$, $\Xi_{[-1;1]} = P$, $\Xi_{[+1;1]} = Q$. At time $t = 2$, $\Xi_{[-2;2]} = P^2$, $\Xi_{[0;2]} = PQ + QP$, and $\Xi_{[+2;2]} = Q^2$. At time $t = 3$, $\Xi_{[-3;3]} = P^3$, $\Xi_{[-1;3]} = QP^2 + PQP + P^2Q$, $\Xi_{[+1;3]} = PQ^2 + QPQ + Q^2P$, and $\Xi_{[3;3]} = Q^3$. At time $t = 4$, $\Xi_{[-4;4]} = P^4$, $\Xi_{[-2;4]} = QP^3 + PQP^2 + P^2QP + P^3Q$, $\Xi_{[0;4]} = P^2Q^2 + PQPQ + PQ^2P + Q^2P^2 + QPQP + QP^2Q$, $\Xi_{[+2;4]} = PQ^3 + QPQ^2 + Q^2PQ + Q^3P$, and $\Xi_{[+4;4]} = Q^4$.

Denote by $S(\mathbb{C}^2)$ the set of vectors ψ of $S(\mathbb{C}^2)$ such that $\psi^\dagger\psi = 1$. The quantum random walk over $\mathbb{Z}$ from a state $\psi \in S(\mathbb{C}^2)$ is defined by the initial condition $\Psi_{\text{space}=0,\text{time}=0} := |0\rangle \otimes \psi$. At time n, this state will spread and the probability amplitude at position k described by $|k\rangle$ will be $\Psi_{k,n} := |k\rangle \otimes \Xi_{[k;n+1]}\psi$ (since $P^\dagger P + Q^\dagger Q = I$, the norm of the initial state is preserved). We have an action from the bundle $\mathcal{H}_\mathbb{Z} \otimes \mathbb{C}\langle P,Q\rangle$ on $\mathbb{C}^2$ described by

$$\mathcal{R}\mathcal{W} : \mathcal{H}_\mathbb{Z} \otimes \mathbb{C}\langle P,Q\rangle \times \mathbb{C}^2 \longrightarrow \mathcal{H}_\mathbb{Z} \otimes \mathbb{C}^2, \qquad (|k\rangle \otimes \Xi_{[k;n+1]}, \psi) \longmapsto |k\rangle \otimes \Xi_{[k;n+1]}\psi. \tag{3.5}$$

The total state is $\Psi^n_{\text{total}} := \sum_k \Psi_{k,n}$.

Remark 3.3. For all $x \in \mathbb{C}\langle P, Q\rangle$, we define the right polynomial multiplication $R_x : \mathbb{C}\langle P,Q\rangle \to \mathbb{C}\langle P,Q\rangle$, $y \mapsto yx$; we have $[\mathcal{D}_+, \mathcal{D}_-] = \text{id} \otimes R_{[Q,P]}$.

4. Noncommutative polynomials and the reading of periodic orbits of $\text{Gr}(\mathcal{E})$

Remark 4.1. From now on, we forget the algebraic relation between P and Q. These monomials will be treaded simply as noncommutative symbols with no relation between them. Observe also that any monomial in $\mathbb{C}\langle P,Q\rangle$ is in one-to-one correspondence with a path of the $(2,1)$-De Bruijn graph. For instance, the monomial $PPPQP$ corresponds to the path shown in Figure 4.1 of the $(2,1)$-De Bruijn graph.

The aim of this section is to recover the polynomials $\Xi_{[k;n]}$, involved in the quantum random walk, from the periodic orbits of $\text{Gr}(\mathcal{E})$. We will show that the periodic orbits of this directed graph allow us to recover the combinatorics generated by the quantum random walk over $\mathbb{Z}$ and that this combinatorics is generated by the coassociative coproduct of $\mathcal{E}$. Recall that this coproduct is defined by $\Delta(a) = a \otimes a + b \otimes c$, $\Delta(b) = a \otimes b + b \otimes d$, $\Delta(c) = d \otimes c + c \otimes a$, $\Delta(d) = d \otimes d + c \otimes b$. However, this directed graph can be also embedded into its natural Markov L-coalgebra. Recall that by definition the right coproduct Δ_M verifies $\Delta_M(a) = a \otimes (a + b)$, $\Delta_M(b) = b \otimes (c + d)$, $\Delta_M(c) = c \otimes (a + b)$, $\Delta_M(d) = d \otimes (c + d)$.

Definition 4.2 (path space). Denote by $\mathcal{F}_n$ the free k-vector space spanned by all the monomials $x_1 \otimes \cdots \otimes x_n$, where for all $1 \le i \le n$, x_i stands for a, b, c, d and such that $x_1 \cdots x_n$ represents a path of length n of the graph $\text{Gr}(\mathcal{E})$. For instance, $\mathcal{F}_0 := k\langle a,b,c,d\rangle$. The *path space* of the graph $\text{Gr}(\mathcal{E})$ is called the k-vector space $\mathcal{F} := \bigoplus_{n\ge0} \mathcal{F}_n$.

$$PPPQP =$$

$$P \qquad\qquad\qquad Q$$

Figure 4.1. Path associated with *PPPQP*.

Remark 4.3. As described in Section 2, we associate with each tensor product $P \otimes Q$, appearing in the definition of the coproducts, a directed arrow $P \to Q$, the relationship between the $(2,1)$-De Bruijn graph, whose vertex set is $\{P, Q\}$, and its line-extension is thus given by setting $a := P \otimes P$, $b := P \otimes Q$, $c := Q \otimes P$, and $d := Q \otimes Q$.

Often, for simplifying notation and only in the path space $\mathcal{F}$, we will write, for instance, xy instead of $x \otimes y$, where x, y stand for a, b, c, d. No confusion is possible since, in the sequel, we forget the algebraic relations between letters. Define now the contraction map.

Definition 4.4 (contraction map). For all $n > 1$, the *contraction map* is the linear map

$$\mathcal{C} : \mathcal{F} \longrightarrow \mathbb{C}\langle P, Q\rangle, \qquad (y_1 \otimes y_2) \otimes (y_2 \otimes y_3) \otimes (y_3 \otimes y_4) \cdots (y_{n-1} \otimes y_n) \longmapsto y_1 y_2 \cdots y_n, \tag{4.1}$$

where for all $1 \le i \le n$, y_i stands for P and Q. For instance, the contraction of $a \otimes b \otimes c :=$ $(P \otimes P) \otimes (P \otimes Q) \otimes (Q \otimes P)$ is equal to $PPQP$.

PROPOSITION 4.5. *Fix a time $t > 1$. To any monomial Ξ constructed from P and Q in the algebra $\mathbb{C}\langle P, Q\rangle$, excepted of course P and Q, corresponds a unique monomial ω in $\mathcal{F}$ such that $\mathcal{C}(\omega) = \Xi$.*

Proof. Any monomial Ξ constructed from P and Q in the algebra $\mathbb{C}\langle P, Q\rangle$ corresponds to a unique path of the $(2,1)$-De Bruijn graph, that is, a unique path of its line-extension. $\square$

LEMMA 4.6. *If x stands for a, b, c or d, the following equalities hold:*

$$\mathcal{C}(x \otimes a)P = \mathcal{C}(x \otimes a \otimes a), \qquad \mathcal{C}(x \otimes a)Q = \mathcal{C}(x \otimes a \otimes b),$$

$$\mathcal{C}(x \otimes b)P = \mathcal{C}(x \otimes b \otimes c), \qquad \mathcal{C}(x \otimes b)Q = \mathcal{C}(x \otimes b \otimes d),$$

$$\mathcal{C}(x \otimes c)P = \mathcal{C}(x \otimes c \otimes a), \qquad \mathcal{C}(x \otimes c)Q = \mathcal{C}(x \otimes c \otimes b), \tag{4.2}$$

$$\mathcal{C}(x \otimes d)P = \mathcal{C}(x \otimes d \otimes c), \qquad \mathcal{C}(x \otimes d)Q = \mathcal{C}(x \otimes d \otimes d).$$

Let $x \in \{a, b, c, d\}$. Then $\mathcal{C}(x)(P + Q) = \mathcal{C}(\Delta_M(x))$.

Proof. The proof comes from simple computations. $\square$

Define $(\Delta_M)_0 = \mathrm{id}$, $(\Delta_M)_1 := \Delta_M$, $(\Delta_M)_2 := (\mathrm{id} \otimes \Delta_M)\Delta_M$ and, more generally, for all $n > 0$, $(\Delta_M)_n := (\underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n-1} \otimes \Delta_M)(\Delta_M)_{n-1}$, similarly for the coassociative coproduct Δ.
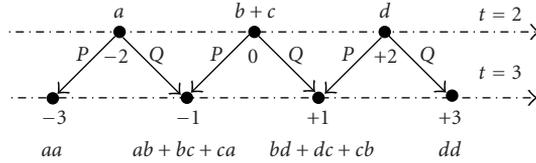
Figure 4.2. The quantum random walk, coded in terms of the path space $\mathscr{F}$.

PROPOSITION 4.7. *The following equalities* $\Delta_M(a + b) = \Delta(a + b)$ *and* $\Delta_M(c + d) = \Delta(c + d)$ *hold. Moreover,* $(\mathrm{id} \otimes \Delta)\Delta_M = (\mathrm{id} \otimes \Delta_M)\Delta_M$. *The equalities* $(\mathrm{id} \otimes \Delta)\Delta(a + b) = (\mathrm{id} \otimes \Delta_M)\Delta_M(a + b)$ *and* $(\mathrm{id} \otimes \Delta)\Delta(c + d) = (\mathrm{id} \otimes \Delta_M)\Delta_M(c + d)$ *imply that* $(\Delta_M)_n(a + b + c + d) = (\Delta)_n(a + b + c + d)$ *for all* $n \geq 0$.

Proof. It is straightforward. □

COROLLARY 4.8. *Fix* $n > 1$. *Denote* $\Sigma_n := \sum_{-n \leq k \leq n} \Xi_{[k,n]}$. *Then* $\Sigma_n = \mathscr{C}((\Delta_M)_{n-2}(a + b + c + d)) = \mathscr{C}((\Delta)_{n-2}(a + b + c + d))$.

Remark 4.9. Set $n > 1$. The polynomial Σ_n is the sum of all the monomials $\Xi_{[k,n]}$, appearing exactly one time because of Lemma 4.6, generated by the combinatorics of the quantum random walk over $\mathbb{Z}$. This sum can be computed by contraction of all the monomials from $\mathscr{F}$ present at time n and obtained either by applying the operator $(\Delta_M)_{n-2} \cdots (\Delta_M)_2 (\Delta_M)_1$ to $a + b + c + d$ or by applying the operator $\Delta_{n-2} \cdots \Delta_2 \Delta_1$ to $a + b + c + d$.

Here is the beginning of the combinatorics generated by the quantum random walk over $\mathbb{Z}$, viewed from the path space $\mathscr{F}$, as shown in Figure 4.2.

For the moment, we get all the sums Σ_n of monomials created by the walk. If a monomial is picked up from $\mathscr{F}$ or from a sum Σ_n, how can we say that it has to belong to such or such vertex? We have to enlarge the definition of $\mathscr{F}$ by defining an index map and an index path space. From now on, we denote by convention $x_{-1,-1} := a$, $x_{-1,+1} := b$, $x_{+1,-1} := c$, $x_{+1,+1} := d$ and observe that a monomial from $\mathscr{F}$ can be always written like $\omega := x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n}$. The *index path space* $\widehat{\mathscr{F}}$ is by definition the space $\mathscr{H} \otimes \mathscr{F}$.

Definition 4.10 (index map). Let $\omega \in \mathscr{F}$, say $\omega := x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n}$. The (linear) *index map* is defined as $\widehat{\mathrm{ind}} : \mathscr{F} \to \widehat{\mathscr{F}}$, $\omega \mapsto (|\mathrm{ind}(\omega)\rangle \otimes \omega)$, with $\mathrm{ind}(\omega) = \mathrm{ind}(x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n}) := \sum_{k=1}^n i_k$.

PROPOSITION 4.11. *Let* $\omega := x_{i_1 i_2} \cdots x_{i_{n-1} i_n} \in \mathscr{F}$. *The index* $\mathrm{ind}(\omega)$ *is equal to the number of Q minus the number of P obtained in the contraction of the monomial* ω. *Therefore, the index map fixes the vertex attributed by the quantum random walk over* $\mathbb{Z}$.

Proof. We will proceed by induction. It is true for $n = 2$, that is, for a, b, c, d. Let ω be a monomial present at vertex k and at time $t = n > 2$. We suppose $\omega = x_{i_1 i_2} \cdots x_{i_{n-1} i_n}$ and the index $\mathrm{ind}(\omega) = k$ does indicate the number of Q minus the number of P obtained in the contraction of this monomial. At time $t = n + 1$, $\omega \mapsto \omega \otimes x_{i_n, i_{n+1}}$. By definition of the quantum random walk, this monomial will be at vertex $k + 1$ if $x_{i_n, i_{n+1}}$ is equal to Q or $k - 1$ if $x_{i_n, i_{n+1}}$ is equal to P. Now, $\mathrm{ind}(\omega \otimes x_{i_n, i_{n+1}}) = \mathrm{ind}(\omega) + i_{n+1}$. By definition, $i_{n+1} = +1$ for b
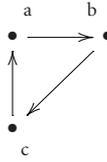
Figure 4.3. Periodic orbit, $\langle a \otimes b \otimes c \rangle$, with pattern of length 3.

and d which are monomials ending by Q and $i_{n+1} = -1$ for a and c which are monomials ending by P.   □

Example 4.12. For instance, $\mathrm{ind}(a) = -2$ and $\mathscr{C}(a) = P^2$. Therefore, the monomial a has to be present at time $t = 2$. Moreover, its contraction yielding the monomial P^2, a is at vertex -2, as expected.

By using the projector $|k\rangle\langle k| \otimes \mathrm{id}$, we will recover all the monomials ω of the sum Σ_n with index $\mathrm{ind}(\omega) = k$. The next question is how can we produce all these monomials of $\mathscr{F}$ from the notion of periodic orbits of the $(2,1)$-De Bruijn graph.

Definition 4.13 (periodic orbits, pattern). The equivalence relation $\sim$ in $\mathscr{F}$ is defined by saying that $\omega_1 \sim \omega_2$ if and only if $\omega_1 = x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_1}$, for some n and there exist m, $\tau^m(\omega_1) = \omega_2$, where $\tau^m : \mathscr{F} \to \mathscr{F}$ is such that $y_1 \cdots f_m \mapsto f_{p-m+1} \cdots f_p f_1 \cdots f_{p-m}$. The set $\mathbb{PO} = \mathscr{F}/\sim$ is the set of *periodic orbits* of the directed graph $\mathrm{Gr}(\mathscr{E})$. Denote by $\langle \omega \rangle$ the *pattern* of an equivalence class associated with ω and its permutations, that is, $\langle \omega \rangle :=$ $\langle x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_1} \rangle$. A *periodic orbit* is just the graphical representation of the pattern. Often, the two words will be confounded.

Remark 4.14. Fix a time $t > 1$. It is straightforward that the length of the pattern of a periodic orbit $\langle \omega \rangle$ present at t denoted by $l(\langle \omega \rangle)$ is equal to t.

Example 4.15. We have $a \otimes b \otimes c \sim c \otimes a \otimes b \sim b \otimes c \otimes a$. The equivalent class is designed by the pattern $\langle a \otimes b \otimes c \rangle$ and the associated periodic orbit is . . . *abcabcabcabc*. . . as shown in Figure 4.3.

This periodic orbit can be also represented by the pattern $\langle a \otimes b \otimes c \otimes a \otimes b \otimes c \rangle$, that is, we cover two times the triangle. Similarly, we have to enlarge the k-vector space of the periodic orbits $\mathbb{PO}$ to keep the notion of vertex attributed by the quantum walk to each periodic orbit. Denote by $\widehat{\mathbb{PO}} := \mathscr{H} \otimes \mathbb{PO}$ such a set.

Definition 4.16 (index map in $\mathbb{PO}$). Define the (linear) *index map* $\widehat{\mathrm{Ind}} : \mathbb{PO} \to \widehat{\mathbb{PO}}$, $\langle \omega \rangle \mapsto$ $|\mathrm{Ind}(\langle \omega \rangle)\rangle \otimes \langle \omega \rangle$ with

$$\mathrm{Ind}\left(\langle \omega \rangle\right) := \mathrm{Ind}\left(\langle x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_1} \rangle\right)$$

$$:= \frac{1}{2}\left((i_1 + i_2) + (i_2 + i_3) + \cdots + (i_{n-1} + i_1)\right) = \sum_{k=1}^{n-1} i_k. \tag{4.3}$$
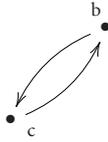
Figure 4.4. Periodic orbit of period 4, $\langle bcbc \rangle$, with pattern of length 4.
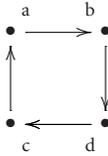
Figure 4.5. Periodic orbit, $\langle abdc \rangle$, with pattern of length period 4.

This definition does not depend on the choice of the representative of the equivalent classe. Once we have the definition of periodic orbits, we have to read them to obtain information.

Definition 4.17 (reading map). Let $\langle \omega \rangle := \langle x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_n i_1} \rangle$ be a periodic orbit. The *reading map* is denoted by $R : \mathbb{PO} \to F$ with $x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_n i_1} \mapsto \sum_{k=1}^{N} x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \cdots x_{i_{n+k-2} i_{k+n-1}}$, the labels being understood modulo n. Define $X_k := x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \cdots x_{i_{n+k-2} i_{k+n-1}}$ for all $1 \leq k \leq n$, the integer N is equal to $\min\{k; X_i \neq X_j, 1 \leq i, j \leq k\}$. The reading map does not depend on the choice of the representative of the equivalent classe.

PROPOSITION 4.18. *Under notation in the definition of the reading map,* $X_{N+k \bmod n} = X_k$.

Proof. Let us show that $X_{N+1} = X_1$. Indeed, if $k \leq N$ is such that $X_{N+1} = X_k$, then it is straightforward to show that $X_{N+2-k} = X_1$ and $N + 1 - k = \min\{l; X_i \neq X_j, 1 \leq i, j \leq l\}$. Therefore, we get $N + 2 - k = N + 1$, only possible for $k = 1$. □

Example 4.19. Consider the periodic orbit $\langle abc \rangle$. Its index is -1 and its reading yields $ab + bc + ca$. By contraction, we obtain $PPQ + PQP + QPP$, which is exactly the polynomial expected at time $t = 3$ and at vertex -1.

Example 4.20. The reading of the periodic orbit $\langle b \otimes c \otimes b \otimes c \rangle$, with a pattern of length 4, as shown in Figure 4.4, yields the monomials $X_1 = b \otimes c \otimes b$, $X_2 = c \otimes b \otimes c$, $X_3 = b \otimes c \otimes b$, and $X_4 = c \otimes b \otimes c$. Therefore, $N = 2$ and $R \langle b \otimes c \otimes b \otimes c \rangle := b \otimes c \otimes b + c \otimes b \otimes c$. By contraction, we obtain $PQPQ + QPQP$. Moreover, its index is 0.

The reading of the periodic orbit $\langle a \otimes b \otimes d \otimes c \rangle$, with pattern of length 4, as shown in Figure 4.5, yields $a \otimes b \otimes d + b \otimes d \otimes c + d \otimes c \otimes a + c \otimes a \otimes b$. By contraction, we obtain $PPQQ + PQQP + QQPP + QPPQ$. Its index is 0.

That is, the sum of these two periodic orbits gives the polynomials expected at time $t = 4$ and at vertex 0.
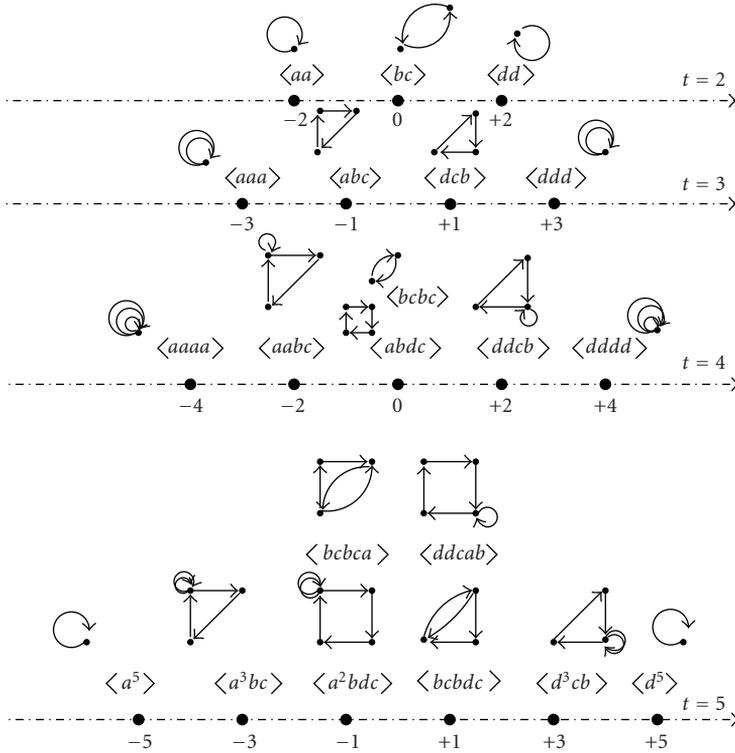
Figure 4.6. Periodic orbits and their associated pictures.

So as to give a minorant of the number of periodic orbits at a given time n at vertex k, we have to link the number k to the number x_P of P and the number x_Q of Q. Suppose k positive (as the walk is symetric, we have as many polynomials at vertex k as at vertex $-k$). This means that $x_Q = x_P + k$. As $x_P + x_Q = n$, we get $x_P = (n-k)/2$. As this solution has to be an integer, this will fix the possibilities of k, that is, the possible vertices reached by the quantum random walk at time n. Set $\kappa := (n-k)/2$.

PROPOSITION 4.21. *Set $\varpi := (1/n)(n!/\kappa!(n-\kappa)!)$. The number of periodic orbits at the vertex k and at time n is greater than or equal to ϖ if this number is an integer and greater than or equal to the integer part of $\varpi + 1$ if ϖ is not an integer.*

Proof. For a given vertex k and a given time n, we have to have $n!/\kappa!(n-\kappa)!$ polynomials in P and Q and by definition we know that the reading of these periodic orbits yields at most n different monomials.   □

In Figure 4.6, we indicate the periodic orbits (of the line-extension) of the $(2,1)$-De Bruijn graph involved in the quantum walk over $\mathbb{Z}$, up to $t = 5$.

PROPOSITION 4.22. *Let $\langle \omega \rangle := \langle x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_n i_1} \rangle$ be the pattern of a periodic orbit. Its reading yields $R\langle \omega \rangle := \sum_{k=1}^{N} X_k$, where X_k is the monomial $x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \cdots x_{i_{k-2+n \bmod n} i_{k-1+n \bmod n}}$.*

We have $\mathrm{Ind}(\langle \omega \rangle) := \mathrm{ind}(X_k)$, *for all* $k = 1,\ldots,n$, *that is, the reading map decomposes the periodic orbit into monomials with same index.*

Proof. Let $\langle \omega \rangle := \langle x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_n i_1} \rangle$ be a periodic orbit and $X_1,\ldots,X_N$ its decomposition under the reading map. We have $s_1 := \mathrm{ind}(X_1) := \mathrm{ind}(x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n}) = \sum_{k=1}^{n} i_k$, $s_2 := \mathrm{ind}(X_2) := \mathrm{ind}(x_{i_2 i_3} x_{i_3 i_4} \cdots x_{i_{n-1} i_n} x_{i_n i_1}) = \sum_{k=2}^{n} i_k + i_1$. Thus $s_1 = s_2$. Similarly, $s_k := \mathrm{ind}(X_k) := i_k + i_{k+1} + i_{k+2} + \cdots + i_n + i_1 + \cdots + i_{k-1} = s_1$. Besides, by definition, $\mathrm{Ind}(\langle \omega \rangle) := (1/2)(2\,\mathrm{ind}(\omega))$. □

Remark 4.23. A periodic orbit with index k will be assigned to the vertex k, by applying, for example, the projector $|k\rangle \langle k| \otimes \mathrm{id} : \widehat{\mathbb{PO}} \to \widehat{\mathbb{PO}}$. Its reading will yield monomials of index k. For instance, the reading of the periodic orbit $\langle abc \rangle$ is $R(\langle abc \rangle) := ab + bc + ca$. We have $\mathrm{Ind}(\langle abc \rangle) = -1$ and $\mathrm{ind}(ab) = \mathrm{ind}(ca) = \mathrm{ind}(bc) = -1$.

Definition 4.24 (completion). For a time $t = n$ and for a given vertex k, all the polynomials can be recovered from the contraction of monomials of $\mathscr{F}$ and having the same index k. Suppose we pick up one of the monomials present at this vertex k, say $x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n}$. The *completion* Comp maps $x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n}$ to $\langle x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n} x_{i_n i_1} \rangle$, thus it is a map from $\mathscr{F}$ to $\mathbb{PO}$. Similarly, $\widehat{\mathrm{Comp}} : \widehat{\mathscr{F}} \to \widehat{\mathbb{PO}}$ can be defined by $\widehat{\mathrm{Comp}} := \mathrm{id} \otimes \mathrm{Comp}$ thanks to Proposition 4.25.

PROPOSITION 4.25. $\mathrm{Ind}(\mathrm{Comp}(x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n})) = \mathrm{ind}(x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n})$.

Proof. It is straightforward. □

The reading of the completion of a monomial of $\mathscr{F}$ present at vertex k will yield many monomials of $\mathscr{F}$ present at this vertex. By contraction, we will recover polynomials in P and Q present at vertex k. As all these polynomials in P and Q are coded bijectively through monomials of $\mathscr{F}$ present at the vertex k, the completion of all these monomials will form all the necessary periodic orbits whose reading will yield all the monomials of $\mathscr{F}$ present at k. For the time being, we started with the polynomial algebra $\mathbb{C}\langle P, Q \rangle$ and arrived at the periodic orbits set of the $(2, 1)$-De Bruijn graph. We showed that the reading of all the patterns of periodic orbits present at a vertex k yielded all the monomials of $\mathscr{F}$ present at k and whose contraction gave the polynomials in P and Q. Let us now equip the periodic orbits with the coproduct of $\mathscr{E}$. Thanks to this coproduct, we will be able to speak of the growth of periodic orbits and to recover the quantum random walk over $\mathbb{Z}$ from their reading.

Definition 4.26 (growth of periodic orbit). The *growth operator* $\mathscr{G} : \mathbb{PO} \to \mathbb{PO}$ is defined by

$$\mathscr{G}(\langle \omega \rangle) = \sum_{k=1}^{l(\langle \omega \rangle)} \langle {}^k \delta(\omega) \rangle, \tag{4.4}$$

where ${}^k \delta = \mathrm{id} \otimes \cdots \otimes \underbrace{\Delta}_{k} \otimes \cdots \otimes \mathrm{id}$ and $l(\langle \omega \rangle)$ is the length of the pattern of the periodic orbit $\langle \omega \rangle$.

Remark 4.27. As $\Delta(x_{ij}) := \sum_{m=+1,-1} x_{im} \otimes x_{mj}$, it is easy to see that $\mathcal{G} : \mathbb{PO} \to \mathbb{PO}$. We take into account all the possible substitutions coming from the coassociative coproduct. Observe that $\sum_{k=1}^{l(\langle\omega\rangle)} \langle^k \delta(\omega)\rangle$ can be decomposed into $\sum_{k=1}^{l(\langle\omega\rangle)} Y_{+1}^k$ and $\sum_{k=1}^{l(\langle\omega\rangle)} Y_{-1}^k$, where $Y_{\pm1}^k \in \mathbb{PO}$ and where ±1 corresponds to the decomposition of the coproduct Δ.

Remark 4.28. Observe also that the coproduct transforms a letter of index k into two letters of index $k+1$ and $k-1$, that is, the coassociative coproduct leaves the index invariant. For instance, $\Delta(b) = ab + bd$ and $\text{Ind}(b) = 0 \mapsto (\text{Ind}(ab) = -1) + (\text{Ind}(bd) = +1)$.

THEOREM 4.29. *The growth operator applied to all the periodic orbits at time $t = n$ will yield all the periodic orbits, perhaps with repetitions, at time $t = n+1$.*

Proof. All the labels i in x_i will be understood modulo n, that is, $x_{i \bmod n}$. Let $\omega = x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n}$ be a monomial of $\mathcal{F}$ present at time $t = n$, its completion yields the periodic orbit $\langle x_{i_1 i_2} \cdots x_{i_{n-1} i_n} x_{i_n i_1}\rangle$ whose reading will give us the monomials $x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \cdots x_{i_{k+n-2} i_{k+n-1}}$, for $k = 1, \ldots, N$. By definition of the quantum random walk, we have to multiply them by P and Q to have the new polynomials present at time $t = n+1$. Let see how it works on ω itself. We get $\mathcal{C}(x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n})P$ and $\mathcal{C}(x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n})Q$. These two polynomials come from the contraction of two monomials present at time $t = n+1$, $x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n} x_{i_n i_m}$ and $x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n} x_{i_n i_{m'}}$ with $m \neq m'$. By completion of these two monomials, we get $\langle x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n} x_{i_n i_m} x_{i_m i_1}\rangle$ and $\langle x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n} x_{i_n i_{m'}} x_{i_{m'} i_1}\rangle$. The sum of these two periodic orbits is obviously equal to $\langle x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n} \Delta(x_{i_n i_1})\rangle$. Now, fix k. By computing the labels modulo n, the contraction of the monomial $x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \cdots x_{i_{k+n-2} i_{k+n-1}}$ multiplied by P and Q comes from the contraction of the monomial $x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \cdots x_{i_{k+n-2} i_{k+n-1}} x_{i_{k+n-1} i_m}$ and the monomial $x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \cdots x_{i_{k+n-2} i_{k+n-1}} x_{i_{k+n-1} i_{m'}}$. The completion of these two monomials comes from the periodic orbit $\langle x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \cdots x_{i_{k+n-2} i_{k+n-1}} \Delta(x_{i_{k+n-1} i_k})\rangle$. The last term $\Delta(x_{i_{k+n-1} i_k})$ is equal to $\Delta(x_{i_{k-1} i_k})$. That is why the growth operator works on periodic orbits to recover monomials at time $t = n+1$ from the periodic orbits at time $t = n$. $\square$

Remark 4.30. It is worth noticing that the Markovian coproduct is closely related to the coassociative coproduct. Indeed, a monomial $x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{n-1} i_n}$ multiplied by P and Q will give the monomials $x_{i_1 i_2} x_{i_2 i_3} \cdots \Delta_M(x_{i_{n-1} i_n})$, see Lemma 4.6, whereas its completion will use the coassociative coproduct. For instance, consider the monomial ab. The contraction of ab multiplied by P and Q comes from the contraction of abc and abd. Its completion will yield $abca$ and $abdc$, which is $\langle ab\Delta(c)\rangle$. Therefore, $ab \to abc + abd = a\Delta_M(b) \xrightarrow{\text{Comp}} \langle ab\Delta(c)\rangle$.

4.1. Reconstruction of the quantum walk from periodic orbits of $\text{Gr}(\mathcal{E})$.

With the results we have obtained, we will now start with the set of periodic orbits of $\text{Gr}(\mathcal{E})$ to recover the combinatorics of the quantum random walk. We now consider the space $\widehat{\mathbb{PO}}$, subvector space of $\mathcal{H} \otimes \mathbb{PO} := \widehat{\mathbb{PO}}$, spanned by the vectors $|\text{Ind}(\langle\omega\rangle)\rangle \otimes \langle\omega\rangle$, where $\langle\omega\rangle \in \mathbb{PO}$. The growth operator is now extended to $\widehat{\mathbb{PO}}$, that is, $\hat{\mathcal{G}} : \widehat{\mathbb{PO}} \to \widehat{\mathbb{PO}}$ and such that $\hat{\mathcal{G}}|\text{Ind}(\langle\omega\rangle)\rangle \otimes \langle\omega\rangle := \sum_{k=1}^{l(\langle\omega\rangle)}(|\text{Ind}(Y_{+1}^k)\rangle \otimes Y_{+1}^k + |\text{Ind}(Y_{-1}^k)\rangle \otimes Y_{-1}^k)$. Define, for all $n > 1$, $\mathbb{PO}_n$ the set of all patterns of index n and by convention $\mathbb{PO}_2 := \{\langle aa\rangle, \langle bc\rangle, \langle dd\rangle\}$.

Denote $\Sigma_n' := \sum_{\langle\omega\rangle\in\mathbb{PO}_{n-1}} \widehat{\mathcal{G}}(|\operatorname{Ind}(\langle\omega\rangle)\rangle \otimes \langle\omega\rangle)$, $n > 2$. This sum is the sum of all patterns of periodic orbits, obtained perharps with repetition, present at time n. To avoid redundancy of information, define the (nonlinear) operator $J : \Sigma_n' := \sum \lambda |\operatorname{Ind}(\langle\omega\rangle)\rangle \otimes \langle\omega\rangle \mapsto \sum |\operatorname{Ind}(\langle\omega\rangle)\rangle \otimes \langle\omega\rangle$, where λ are integers, in such a way that a pattern $\langle\omega\rangle$, present in Σ_n' is present only one time in $J\Sigma_n'$. Therefore, for all $n > 1$, $J\Sigma_n'$ is the sum of all patterns of periodic orbits, present at time n. Apply now the projector $|k\rangle\langle k| \otimes \operatorname{id}$ on $J\Sigma_n'$. Such a projector will yield all the orbits present at vertex k and at time n. By reading them, all the monomials in $\mathcal{F}$ present at time n at vertex k will be obtained. By contraction, we will obtain all the polynomials at time n at vertex k generated by the quantum random walk.

At time $t = 2$, we have two loops $\langle aa\rangle$, $\langle dd\rangle$, and $\langle bc\rangle$. Their readings yield a, d, and $b + c$. Their indexes are -2, 2, and 0. Their contractions yield PP, QQ, and $PQ + QP$. At time $t = 3$, the growth operator yields $\widehat{\mathcal{G}}(|(-2)\rangle \otimes \langle aa\rangle) \mapsto 2|(-3)\rangle \otimes \langle aaa\rangle + 2|(-1)\rangle \otimes \langle bca\rangle$ and $\widehat{\mathcal{G}}(|(2)\rangle \otimes \langle dd\rangle) \mapsto 2|(3)\rangle \otimes \langle ddd\rangle + 2|(1)\rangle \otimes \langle cbd\rangle$. Similarly, we obtain $\widehat{\mathcal{G}}(|(0)\rangle \otimes \langle bc\rangle)$. By applying J to Σ_3', we find $J\Sigma_3' = |(-3)\rangle \otimes \langle aaa\rangle + |(-1)\rangle \otimes \langle bca\rangle + |(3)\rangle \otimes \langle ddd\rangle + |(1)\rangle \otimes \langle cbd\rangle$.

For instance, apply the projector $|-1\rangle\langle -1| \otimes \operatorname{id}$ to $J\Sigma_3'$. We find $|(-1)\rangle \otimes \langle bca\rangle$. This pattern has to be present at time $t = 3$ and at vertex -1. Its reading yields $ab + bc + ca$ all of index -1. Its contraction yields at time $t = 3$ and at vertex -1 the polynomial $PPQ + PQP + QPP$ as expected. By applying the growth operator at time $t = 3$, we will still obtain all the orbits present at time $t = 4$, and so forth.

5. Relationships with classical systems

We would like to establish a link between the quantum random walk we studied and their classical counterparts. For convenience, we recall in the following two subsections some results.

5.1. Classical random walk over $\mathbb{Z}$ and the Bernoulli shift.
We consider the random walk over $\mathbb{Z}$, that is, we consider $\Omega = \{-1, 1\}^{\mathbb{N}}$ equipped with the product measure $\mu^{\otimes\mathbb{N}}$, where $\mu = (1/2)\delta_{-1} + (1/2)\delta_1$. We consider the sequence of i.i.d. random variables $(X_n)_{n\in\mathbb{N}}$, with

$$X_n : \Omega \longrightarrow \{-1, +1\}, \quad \text{such that } X_n(\omega) = \omega_n. \tag{5.1}$$

It is well known that this process and the symbolic dynamics generated by $x \mapsto 2x \bmod 1$ are isomorphic. Indeed consider the i.i.d. process defined by, $Y_n : \Omega \to \{0, +1\}$, such that $Y_n(\omega) = \omega_n' := (\omega_n + 1)/2$. The set $\Omega = \{-1, 1\}^{\mathbb{N}}$ becomes $\Omega' = \{0, 1\}^{\mathbb{N}}$ and μ becomes $\mu' = (1/2)\delta_0 + (1/2)\delta_1$. Consider the measurable function

$$\Phi : \Omega' \longrightarrow [0, 1[, \quad \omega' \longmapsto \sum_{n=0}^{\infty} \frac{\omega_n'}{2^{n+1}}. \tag{5.2}$$

The cylinder $C = [Y_0 = \omega_0'; \dots ; Y_l = \omega_l']$, being the set of sequences starting by $(\omega_0'; \dots ; \omega_l')$, will cover the interval $[\sum_{n=0}^{l}(\omega_n'/2^{n+1}); \sum_{n=0}^{l}(\omega_n'/2^{n+1}) + \sum_{n=l+1}^{\infty}(1/2^{n+1})]$. We notice that

$\text{Leb}(C) = 1/2^{l+1}$, where Leb is the Lebesgue measure. Let us consider the shift θ defined by $(\theta \omega')_n = \omega'_{n+1}$. This shift leaves the Lebesgue measure of the cylinder C invariant if we write $\theta([Y_0 = \omega'_0; \dots; Y_l = \omega'_l]) = ([Y_1 = \omega'_0; \dots; Y_{l+1} = \omega'_l])$. Moreover, we have $\Phi(\theta(\omega')) = 2\Phi(\omega') \bmod 1$.

The random walk over $\mathbb{Z}$ described by $(\Omega = \{-1,1\}^{\mathbb{N}}, \mu^{\otimes \mathbb{N}}, \theta)$ is isomorphic to $(\Omega' = \{0,1\}^{\mathbb{N}}, \mu'^{\otimes \mathbb{N}}, \theta')$ which is isomorphic to the chaotic system $([0,1[, \beta[0,1[, \text{Leb}, f : x \mapsto 2x \bmod 1)$. As $\Phi \circ \theta' = f \circ \Phi$, the diagram

$$
\begin{array}{ccc}
(\{0,1\}^{\mathbb{N}}, \mu^{\otimes \mathbb{N}}) & \xrightarrow{\;\theta\;} & (\{0,1\}^{\mathbb{N}}, \mu^{\otimes \mathbb{N}}) \\
\Phi \downarrow & & \downarrow \Phi \\
([0,1[, \text{Leb}) & \xrightarrow{\;f\;} & ([0,1[, \text{Leb})
\end{array}
\tag{5.3}
$$

is commutative.

In [4], Biane proposes a noncommutative version of the Bernoulli process. Set $\Omega := \{+1, -1\}$, the probability space, and define the probability $\mathbb{P}(\{+1\}) = p$ and $\mathbb{P}(\{-1\}) = q$. The process $X : \Omega \to \mathbb{R}$ is defined as $X(+1) = +1$ and $X(-1) = -1$. By identifying $(1,0)$ with $(4p)^{-1/2}(1 + X)$ and $(0,1)$ with $(4q)^{-1/2}(1 - X)$, the space $L^2(\Omega, \mathbb{P})$ is isomorphic to $\mathbb{C}^2$. We notice that the algebra $L^{\infty}(\Omega, \mathbb{P})$, acting on L^2, can be identified with the algebra of diagonal matrices of $M_2(\mathbb{C})$. A natural noncommutative generalisation consists in lifting this commutative algebra into a noncommutative one, that is, $M_2(\mathbb{C})$. Notice that a subalgebra of $M_2(\mathbb{C})$ is also used in this framework.

5.2. Quantum graphs. Let B be a bistochastic matrix representing a directed graph, that is, two vertices x_i and x_j are linked if and only if $B_{ij} \neq 0$. The matrix B is said to be unistochastic if there exists a unitary matrix U such that $B_{ij} = |U_{ij}|^2$. In this case, we say that the graph can be quantised. The notion of quantum graph was introduced as a toy model for studying quantum chaos by Kottos and Smilansky [10, 11]. This notion was also studied by Tanner [17] and by Barra and Gaspard [2, 3]. In this paper, we will follow another approach leading to quantum graphs put forward by Pakoński et al. [15] concerning one-dimensional dynamical systems. They consider a one-dimensional mapping f acting on $I = [0,1]$ such that $f : I \to I$ is piecewise linear and f has to verify the following three conditions.

(1) There exists a Markov partition of the interval I into M equal cells $E_i := [(i-1)/M, i/M)$, $i = 1 \cdots M$, with M a positive integer, and f is linear on each cell E_i.

(2) For all $y \in I$,

$$
\sum_{x \in f^{-1}(y)} \frac{1}{f'(x)} = 1,
\tag{5.4}
$$

where f' is the right derivative (defined almost everywhere on I).

(3) The finite transfer matrix B describing the action of f on the cells E_i is unistochastic.
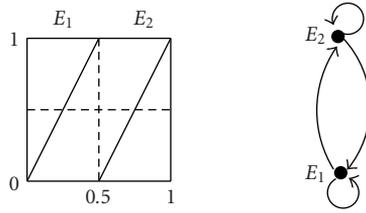
Figure 5.1

Remark 5.1. On each cell, f concides with the function $f_i : [(i-1)/M, i/M) \rightarrow I$, $x \mapsto c_i x + b_i$, where the c_i have to be nonzero integers and the b_i are rational. The unistochastic matrix B is an M-by-M matrix and $B_{ij} = 1/|c_i|$. Therefore, the probability of visiting the cell E_j from E_i is equal to $1/f'(x)$, with $x \in E_i$ and $f(x) \in E_j$.

Remark 5.2. The Kolmogorov-Sinai entropy of the Markov chain generated by the bistochastic matrix B [7] is

$$H_{\mathrm{KS}} = -\sum_{i=1}^{M} \tilde{p}_i \sum_{j=1}^{M} B_{ij} \log B_{ij}, \tag{5.5}$$

where $\tilde{p}$ is the normalised left eigenvector of B such that $\tilde{p}B = \tilde{p}$, with $\sum_{i=1}^{M} \tilde{p}_i = 1$. This equation gives the dynamical entropy of the system since the Markov partition on M equal cells is a generating partition of the system. As the transition matrix is bistochastic, all the components of $\tilde{p}$ are equal to $1/M$. Thus $H_{\mathrm{KS}} = 0$ if and only if all the $B_{ij} \in \{0,1\}$. This entails that $|f'(x)| = 1$, that is, the system is regular. With the conditions stated above, the converse is true.

Example 5.3 (chaotic system, the Bernoulli shift). The Bernoulli shift is described by $f : x \mapsto 2x \bmod 1$, where $x \in [0,1]$ (see Figure 5.1). This map is associated with the unistochastic matrix $B_2 = (1/2)\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

We recall the following result.

THEOREM 5.4 (see [15]). *With the assumptions described in Section 5.2 on the piecewise linear map f, to every periodic orbit of period n of the dynamical system described by f corresponds a unique periodic orbit of period n of the directed graph described by the associated unistochastic matrix, that is, by the $(2,1)$-De Bruijn graph.*

We denote by K such a bijection. That is, $K : \mathbb{PO} \rightarrow \mathbb{PO}[x \mapsto 2x \bmod 1]$, where $\mathbb{PO}[x \mapsto 2x \bmod 1]$ denotes the k-vector space of patterns associated with the periodic orbits of the classical system $x \mapsto 2x \bmod 1$.

Remark 5.5. Let B_2 be the unistochastic matrix associated with the chaotic map $x \mapsto 2x \bmod 1$. One of the possible quantisation of this chaotic map is the so-called Hadamard matrix, $U_{\mathrm{H}} = (1/\sqrt{2})\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$.
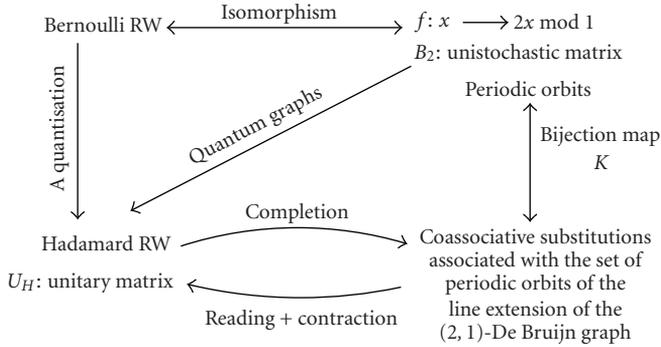
Figure 6.1

6. Summary

We now summarise our results in Figure 6.1.

Appendix

A. Link with language theory

In this appendix, we recall briefly the notion of grammar. We briefly show some points between grammar theory and the previous work.

Recall that a *grammar* with nonterminal symbols is a 4-tuple $G = (W, V, U, n_0)$, where W is a finite set (its elements are called nonterminal symbols), V is also a finite set such that $V \cap W = \varnothing$. The production set U is a subset of pairs $u = (\alpha \to \beta)$, where $\alpha \in S = W \cup V$ containing at least one symbol of W and $\beta \in S^*$. The finite set S plays the rôle of the alphabet and S^* is the set of all finite strings over the alphabet S including the empty one. The symbol n_0 belongs to W.

A grammar is called *right linear* if all the productions are of the form $n \to \alpha m$, where $n, m \in W$ and $\alpha \in V^*$. It is called *context-free* if all the productions are of the form $n \to \alpha$, where $n \in W$ and $\alpha \in V^*$. The *language $L(G)$ generated by G* is the set of all sentences generated by G.

In this paper, we use implicitly grammar theory. Indeed, embed $S = W \cup V := \{A, B, C, D\} \cup \{a, b, c, d\}$ into its free k-vector space. Consider the Fock space $F(G) := \bigoplus_n kS^{\otimes n}$. Define the concatenation product of two symbols of S, β, and α, by $\beta \otimes \alpha$. The space S^* is then the space of all the finite sentences $\beta_1 \otimes \cdots \otimes \beta_k$, $\beta_i \in S$. The *language $L(G)$ generated by G* is the k-vector space spanned by all the sentences generated by G. The element n_0 is then viewed as an element of kW. Now consider the production rules

$$U_M(\mathscr{F}) := \{A \longmapsto a \otimes A, \ A \longmapsto a \otimes B, \ B \longmapsto b \otimes C, \ B \longmapsto b \otimes D, \ C \longmapsto c \otimes A, \ C \longmapsto c \otimes B,$$
$$D \longmapsto d \otimes D, \ D \longmapsto d \otimes C\},$$

(A.1)

and the so-called *coassociative production rules*

$$U_c(\mathcal{F}) := \{A \longmapsto a \otimes A,\ A \longmapsto b \otimes C,\ B \longmapsto b \otimes D,\ B \longmapsto a \otimes B,\ C \longmapsto c \otimes A,\ C \longmapsto d \otimes C,$$
$$D \longmapsto d \otimes D,\ D \longmapsto c \otimes B\}.$$

(A.2)

Consider the nonterminal symbol START $n_0 := A + B + C + D$. Then, the language $L(G, U_M(\mathcal{F}), n_0)$ generated by n_0, that is, all the sentences of $L(G)$ obtained by applying the production rules $U_M(\mathcal{F})$ to the start symbol $n_0 := A + B + C + D$, is the same as the language $L(G, U_c(\mathcal{F}), n_0)$ generated by n_0, see Proposition 4.6. In addition, these grammars are right linear. Observe also that the grammar constructed on the set of periodic orbits or patterns of the graph $\mathcal{F}$ is context free. The production rules are still given by the coassociative coproduct of $\mathcal{F}$,

$$U_c(\text{Per orbs}) := \{A \longmapsto A \otimes A,\ A \longmapsto B \otimes C,\ B \longmapsto B \otimes D,\ B \longmapsto A \otimes B,\ C \longmapsto C \otimes A,$$
$$C \longmapsto D \otimes C,\ D \longmapsto D \otimes D,\ D \longmapsto C \otimes B\}.$$

(A.3)

The language of periodic orbits is the language generated by the sentence $A \otimes A + B \otimes C + D \otimes D$.

Acknowledgment

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