BILINEAR MULTIPLIERS AND TRANSFERENCE

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We give de Leeuw-type transference theorems for bilinear multipliers. In particular, it is shown that bilinear multipliers arising from regulated functions $m(\xi,\eta)$ in $\mathbb{R}\times\mathbb{R}$ can be transferred to bilinear multipliers acting on $\mathbb{T}\times\mathbb{T}$ and $\mathbb{Z}\times\mathbb{Z}$. The results follow from the description of bilinear multipliers on the discrete real line acting on L^p-spaces.

1. Introduction

Let (p_1,p_2,p_3) be such that $0 < p_1,p_2,p_3 \leq \infty$, $1/p_1+1/p_2 = 1/p_3$ and let $m(\xi,\eta)$ be a bounded measurable function on $\mathbb{R}^2$. It is said to be a bilinear (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ if

$$\mathcal{C}_m(f,g)(x) = \int_{\mathbb{R}^2} \widehat{f}(\xi)\widehat{g}(\eta)m(\xi,\eta)e^{2\pi ix(\xi+\eta)}d\xi\, d\eta \tag{1.1}$$

(defined for Schwarzt test functions f and g in $\mathcal{S}$) extends to a bounded bilinear operator from $L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})$ into $L^{p_3}(\mathbb{R})$.

The theory of these multipliers has been tremendously developed after the results proved by Lacey and Thiele (see [16, 18, 17]) which establish that $m(\xi,\nu) = \operatorname{sign}(\xi+\alpha\nu)$ is a (p_1,p_2)-multiplier for each triple (p_1,p_2,p_3) such that $1 < p_1,p_2 \leq \infty$, $p_3 > 2/3$, and each $\alpha \in \mathbb{R}\setminus\{0,1\}$.

The study of such multipliers was started by Coifman and Meyer (see [3, 4, 19]) for smooth symbols and new results for nonsmooth symbols, extending the ones given by the bilinear Hilbert transform, have been achieved by Gilbert and Nahmod (see [8, 9, 10]) and also by Muscalu et al. (see [20]).

We refer the reader also to [7, 12, 11, 15] for new results on bilinear multipliers and related topics.

In a recent paper (see [7]), Fan and Sato have shown certain de Leeuw-type theorems for transferring multilinear operators on Lebesgue and Hardy spaces from $\mathbb{R}^n$ to $\mathbb{T}^n$. Here we will consider bilinear multipliers on Lebesgue spaces $L^p(\mathbb{R})$ and get a characterization which allows us to transfer not only to the bilinear multipliers on $\mathbb{T}$ but also on $\mathbb{Z}$. Our approach will follow closely the ideas in the original paper by de Leeuw (see [6]) and will

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provide an alternative proof of some results in [7], whose proof follows, in the multilinear case, the approach used by Stein and Weiss (see [21, page 260]).

We start by setting up natural analogous versions of bilinear multipliers in the periodic and discrete cases. Let $m = (m_{k,k'})$ be a bounded sequence and let $\widetilde{m}$ be a periodic function on $\mathbb{T}\times\mathbb{T}$. Define for $\theta \in [-1/2, 1/2]$,

$$\mathcal{P}_m(f,g)(\theta) = \sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} \hat{f}(k)\hat{g}(k')m_{k,k'}e^{2\pi i\theta(k+k')} \tag{1.2}$$

for functions f, g defined on $\mathbb{T}$, and for $k \in \mathbb{Z}$,

$$\mathcal{D}_{\widetilde{m}}(a,b)(k) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} P(t)Q(s)\widetilde{m}(t,s)e^{2\pi ik(t+s)}\,dt\,ds \tag{1.3}$$

for sequences $a = (a(n))_{n\in\mathbb{Z}}$ and $b = (b(n))_{n\in\mathbb{Z}}$, where $P(t) = \sum_{n\in\mathbb{Z}} a(n)e^{2\pi int}$ and $Q(t) = \sum_{n\in\mathbb{Z}} b(n)e^{2\pi int}$.

Now we say that m (resp., $\widetilde{m}$) is a bilinear (p_1, p_2)-multiplier on $\mathbb{Z}\times\mathbb{Z}$ (resp., $\mathbb{T}\times\mathbb{T}$) if $\mathcal{P}_m$ (resp., $\mathcal{D}_{\widetilde{m}}$) defines a bounded bilinear operator from $L^{p_1}(\mathbb{T})\times L^{p_2}(\mathbb{T})$ into $L^{p_3}(\mathbb{T})$ (resp., $\ell^{p_1}(\mathbb{Z})\times\ell^{p_2}(\mathbb{Z})$ into $\ell^{p_3}(\mathbb{Z})$), where $1/p_1 + 1/p_2 = 1/p_3$.

Of course we can see these three cases as instances of the general bilinear multiplier acting on different groups. Let G be a locally compact abelian group and $\hat{G}$ its dual group with Haar measure μ. Let $1 \le p_1, p_2 \le \infty$ and let m be a bounded measurable function on $\hat{G}\times\hat{G}$. We say that m is a (p_1, p_2)-multiplier on $\hat{G}\times\hat{G}$ if the operator

$$T_m(f,g)(x) = \int_{\hat{G}}\int_{\hat{G}} \mathcal{F}f(\gamma_1)\mathcal{F}g(\gamma_2)m(\gamma_1,\gamma_2)\gamma_1(-x)\gamma_2(-x)\,d\mu(\gamma_1)\,d\mu(\gamma_2) \tag{1.4}$$

(defined for simple functions f and g) extends to a bounded bilinear operator from $L^{p_1}(G)\times L^{p_2}(G)$ to $L^{p_3}(G)$, where $1/p_1 + 1/p_2 = 1/p_3$. The reader is referred to [14] for the general theory in the linear case.

The first transference results on linear multipliers were given by de Leeuw (see [6]). He showed, among other things, that if m is regulated (all its points are Lebesgue points) and m is a p-multiplier on $\mathbb{R}$, then $(m(\varepsilon k))_k$ is a uniformly bounded p-multiplier for all $\varepsilon > 0$ on $\mathbb{Z}$ (see [21, page 264] for the converse of this result for continuous multipliers). Transference results of similar nature are presented in [1].

A general transference method was considered by [5] (see also the generalization given by [13]), but we will not consider these approaches in our bilinear generalization in the paper.

In [7], the multilinear version of the continuous result was shown, namely that for any continuous function $m(\xi,\eta)$, one has that m is a (p_1, p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ if and only if $m(\varepsilon k, \varepsilon k')_{k,k'}$ is a uniformly bounded multiplier on $\mathbb{Z}\times\mathbb{Z}$ for $\varepsilon > 0$. An extension of the result to Lorentz spaces was achieved in [2].

We will first characterize the boundedness of bilinear multipliers on $\mathbb{R}\times\mathbb{R}$ by the existence of a constant K such that

$$\left|\sum_{t\in\mathbb{R}}\sum_{s\in\mathbb{R}} m(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\})\right| \le K\|\widehat{\mu}\|_{B_{p_1}}\|\widehat{\nu}\|_{B_{p_2}}\|\widehat{\lambda}\|_{B_{p_3'}} \tag{1.5}$$

for all measures μ, ν, λ of finite supports.

This allows us to transfer from the continuous $\mathcal{C}_m$ to the discrete case $\mathcal{D}_{\tilde{m}}$ recovering some of the Fan-Sato results in [7].

We also obtain the transference from the continuous case $\mathcal{C}_m$ to the periodic case $\mathcal{P}_m$. Our main result establishes that m is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ if and only if $D_\varepsilon m = m_{\varepsilon\cdot,\varepsilon\cdot}\chi_{[-1/2,1/2]\times[-1/2,1/2]}$ (extended by periodicity) are uniformly bounded (p_1,p_2)-multipliers on $\mathbb{T}\times\mathbb{T}$.

The reader should be aware that the results of the paper can be stated for multilinear multipliers, with the condition $1/p=\sum_{i=1}^{n}(1/p_i)$, by considering the corresponding multilinear notions, for instance, for $m(\xi_1,\dots,\xi_n)$, one has

$$\mathcal{C}_m(f_1,\dots,f_n)(x)=\int_{\mathbb{R}^n}\widehat{f_1}(\xi_1)\cdots\widehat{f_n}(\xi_n)m(\xi_1,\dots,\xi_n)e^{2\pi i x(\xi_1+\cdots+\xi_n)}d\xi_1\cdots d\xi_n, \tag{1.6}$$

and similar modifications for $\mathcal{P}_m$ and $\mathcal{D}_{\tilde{m}}$. We simply do the bilinear case for the sake of simplicity.

Throughout the paper, $1\le p_1,p_2,p_3\le\infty$ and $1/p_3=1/p_1+1/p_2$. For a given finite Borel measure μ on $\mathbb{R}$, we write $\widehat{\mu}(\xi)=\int_{\mathbb{R}}e^{-2\pi i\xi t}d\mu(t)$ and, for an almost periodic function g, we denote $\|g\|_{B_p}=\lim_{T\to\infty}((1/2T)\int_{-T}^{T}|g(t)|^p dt)^{1/p}$. We will use the notations $D_\varepsilon m(x,y)=m(\varepsilon x,\varepsilon y)$ and $\phi_\varepsilon(x)=(1/\varepsilon)\phi(x/\varepsilon)$.

2. Bilinear multipliers on $\mathbb{R}\times\mathbb{R}$

We start by reformulating the condition of (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ using duality. The proof is straightforward and is left to the reader.

Lemma 2.1. *Let $m(\xi,\eta)$ be a bounded measurable function on $\mathbb{R}\times\mathbb{R}$. Then m is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ if and only if there exists a constant K so that*

$$\left|\int_{\mathbb{R}^2}\phi(\xi)\psi(\eta)\nu(\xi+\eta)m(\xi,\eta)d\xi d\eta\right| \le K\|\widehat{\phi}\|_{p_1}\|\widehat{\psi}\|_{p_2}\|\widehat{\nu}\|_{p_3'} \tag{2.1}$$

for all $\phi,\psi,\nu\in\mathcal{S}$.

Now we present some behavior of multipliers on $\mathbb{R}\times\mathbb{R}$ with respect to convolution and dilation operators to be used later on.

Lemma 2.2. *Let $m(\xi,\eta)$ be a bounded measurable function on $\mathbb{R}\times\mathbb{R}$. If $\Phi\in L^1(\mathbb{R}^2)$ and m is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$, then $\Phi * m$ is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ and $\|\mathcal{C}_{\Phi*m}\|\le\|\Phi\|_1\|\mathcal{C}_m\|$, where $\|\mathcal{C}_m\|$ stands for the norm of the corresponding bilinear map from $L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})$ into $L^{p_3}(\mathbb{R})$.*

Proof. Let $f_s(x)=f(x+s)$ for any $s\in\mathbb{R}$ and function f. Then for any $s,t\in\mathbb{R}$ and $\phi,\psi,\nu\in\mathcal{S}$ with $\|\widehat{\phi}\|_{p_1}=\|\widehat{\psi}\|_{p_2}=\|\widehat{\nu}\|_{p_3'}=1$, we have

$$\left|\int_{\mathbb{R}^2}\phi_s(\xi)\psi_t(\eta)\nu_{t+s}(\xi+\eta)m(\xi,\eta)d\xi\,d\eta\right|\leq K. \tag{2.2}$$

Now

$$\begin{aligned}\int_{\mathbb{R}^2}&\phi(\xi)\psi(\eta)\nu(\xi+\eta)\Phi*m(\xi,\eta)d\xi\,d\eta\\&=\int_{\mathbb{R}^2}\phi(\xi)\psi(\eta)\nu(\xi+\eta)\left(\int_{\mathbb{R}^2}m(\xi-s,\eta-t)\Phi(s,t)ds\,dt\right)d\xi\,d\eta\\&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\phi(\xi+s)\psi(\eta+t)\nu(\xi+\eta+s+t)m(\xi,\eta)\Phi(s,t)d\xi d\eta\,ds\,dt.\end{aligned} \tag{2.3}$$

And the result follows by Lemma 2.1. □

Lemma 2.3. *Let $\varepsilon>0$ and $m(\xi,\eta)$ be a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$. Then $m(\varepsilon\xi,\varepsilon\eta)$ is also a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ and $\|\mathcal{C}_{m(\varepsilon\cdot,\varepsilon\cdot)}\|\leq\|\mathcal{C}_m\|$.*

Proof. For $\phi,\psi,\nu\in\mathcal{S}$ and $\|\widehat{\phi}\|_{p_1}=\|\widehat{\psi}\|_{p_2}=\|\widehat{\nu}\|_{p_3'}=1$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}&\phi(\xi)\psi(\eta)\nu(\xi+\eta)m(\varepsilon\xi,\varepsilon\eta)d\xi\,d\eta\\&=\int_{\mathbb{R}^2}\frac{1}{\varepsilon^{1/p_1'}}\phi\left(\frac{\xi}{\varepsilon}\right)\frac{1}{\varepsilon^{1/p_2'}}\psi\left(\frac{\eta}{\varepsilon}\right)\frac{1}{\varepsilon^{1/p_3}}\nu\left(\frac{\xi+\eta}{\varepsilon}\right)m(\xi,\eta)d\xi\,d\eta.\end{aligned} \tag{2.4}$$

The proof is finished invoking Lemma 2.1 again. □

Theorem 2.4. *Let $m(\xi,\eta)$ be a bounded continuous function on $\mathbb{R}\times\mathbb{R}$. The following are equivalent:*
(i) *m is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$;*
(ii) *there exists a constant K so that*

$$\left|\sum_{t\in\mathbb{R}}\sum_{s\in\mathbb{R}}m(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\})\right|\leq K\|\widehat{\mu}\|_{B_{p_1}}\|\widehat{\nu}\|_{B_{p_2}}\|\widehat{\lambda}\|_{B_{p_3'}} \tag{2.5}$$

for all measures μ, ν, λ supported on a finite number of points.

Proof. (i)⇒(ii). Assume that m is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$. Denote by ϕ the Gaussian function $\phi(x)=e^{-x^2/2}$. Then for any $\alpha>0$ and $a\in\mathbb{R}$,

$$\left(\frac{1}{\varepsilon}\right)^{\alpha}\phi^{\alpha}\left(\frac{\xi-a}{\varepsilon}\right)=\delta_a*(\phi_\varepsilon)^{\alpha}(\xi). \tag{2.6}$$

Now choose $0 < \alpha,\beta,\gamma$ such that $\alpha+\beta+\gamma = 2$, and $\mu = \delta_a$, $\nu = \delta_b$, and $\lambda = \delta_c$ for $a,b,c \in \mathbb{R}$. It is easily checked that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{1}{\varepsilon^2}\phi^\alpha\left(\frac{\xi-a}{\varepsilon}\right)\phi^\beta\left(\frac{\eta-b}{\varepsilon}\right)\phi^\gamma\left(\frac{\xi+\eta-c}{\varepsilon}\right)m(\xi,\eta)d\xi\,d\eta \\
&\quad= \int_{\mathbb{R}^2} \phi^\alpha(\xi)\phi^\beta(\eta)\phi^\gamma\left(\xi+\eta+\frac{a+b-c}{\varepsilon}\right)m(a+\varepsilon\xi,b+\varepsilon\eta)d\xi\,d\eta \\
&\quad= \int_{\mathbb{R}^2} \mu*(\phi_\varepsilon)^\alpha(\xi)\nu*(\phi_\varepsilon)^\beta(\eta)\lambda*(\phi_\varepsilon)^\gamma(\xi+\eta)m(\xi,\eta)d\xi\,d\eta.
\end{aligned} \tag{2.7}$$

Since

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\phi^\alpha(\xi)\phi^\beta(\eta)\phi^\gamma\left(\xi+\eta+\frac{a+b-c}{\varepsilon}\right)m(a+\varepsilon\xi,b+\varepsilon\eta) \\
&\quad= \delta_c(a+b)\phi^\alpha(\xi)\phi^\beta(\eta)\phi^\gamma(\xi+\eta)m(a,b),
\end{aligned} \tag{2.8}$$

the Lebesgue convergence theorem implies that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2} \frac{1}{\varepsilon^2}\phi^\alpha\left(\frac{\xi-a}{\varepsilon}\right)\phi^\beta\left(\frac{\eta-b}{\varepsilon}\right)\phi^\gamma\left(\frac{\xi+\eta-c}{\varepsilon}\right)m(\xi,\eta)d\xi\,d\eta \\
&\quad= Cm(a,b)\delta_c(a+b) = Cm(a,b)\mu(\{a\})\nu(\{b\})\lambda(\{a+b\}),
\end{aligned} \tag{2.9}$$

where $C = \int_{\mathbb{R}^2}\phi^\alpha(\xi)\phi^\beta(\eta)\phi^\gamma(\xi+\eta)d\xi d\eta$.

Therefore we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2} \mu*(\phi_\varepsilon)^\alpha(\xi)\nu*(\phi_\varepsilon)^\beta(\eta)\lambda*(\phi_\varepsilon)^\gamma(\xi+\eta)m(\xi,\eta)d\xi\,d\eta \\
&\quad= C\sum_{t\in\mathbb{R}}\sum_{s\in\mathbb{R}} m(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{(t+s)\})
\end{aligned} \tag{2.10}$$

for all measures μ, ν, λ having their supports on finite sets of points.

On the other hand, from (i) and Lemma 2.1, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \mu*(\phi_\varepsilon)^\alpha(\xi)\nu*(\phi_\varepsilon)^\beta(\eta)\lambda*(\phi_\varepsilon)^\gamma(\xi+\eta)m(\xi,\eta)d\xi\,d\eta\right| \\
&\quad\le K\|\hat{\mu}\widehat{(\phi_\varepsilon)^\alpha}\|_{p_1}\left\|\hat{\nu}\widehat{(\phi_\varepsilon)^\beta}\right\|_{p_2}\left\|\hat{\lambda}\widehat{(\phi_\varepsilon)^\gamma}\right\|_{p_3'}.
\end{aligned} \tag{2.11}$$

We now choose $\alpha = 1/p_1'$, $\beta = 1/p_2'$, and $\gamma = 1/p_3$. Since $(\phi_\varepsilon)^\alpha = \varepsilon^{1-\alpha}/\alpha^{1/2}\phi_{\varepsilon\alpha^{-1/2}}$, we get $\widehat{(\phi_\varepsilon)^\alpha}(\xi) = C_\alpha\varepsilon^{1/p_1}e^{-\varepsilon^2\xi^2/2\alpha}$, $\widehat{(\phi_\varepsilon)^\beta}(\xi) = C_\beta\varepsilon^{1/p_2}e^{-\varepsilon^2\xi^2/2\beta}$, and $\widehat{(\phi_\varepsilon)^\gamma}(\xi) = C_\gamma\varepsilon^{1/p_3}e^{-\varepsilon^2\xi^2/2\gamma}$ for some constants C_α, C_β, and C_γ.

Now taking into account that $\int_{\mathbb{R}} e^{-\varepsilon^2 p_1\xi^2/2\alpha}d\xi = C_\alpha'\varepsilon^{-1}$, we have

$$\left\|\hat{\mu}\widehat{(\phi_\varepsilon)^\alpha}\right\|_{p_1} = C\left(\frac{1}{A(\varepsilon)}\int_{\mathbb{R}}|\hat{\mu}(\xi)|^{p_1}\varepsilon^{-p_1\varepsilon^2\xi^2/2\alpha}d\xi\right)^{1/p_1}, \tag{2.12}$$

for $A(\varepsilon) = \int_{\mathbb{R}} e^{-\varepsilon^2 p_1\xi^2/2\alpha}d\xi$. Hence $C\|\hat{\mu}\|_{B_{p_1}} = \lim_{\varepsilon\to 0}\|\hat{\mu}\hat{\phi}_\varepsilon^\alpha\|_{p_1}$.

Applying a similar procedure for ν and λ, we finish this implication.

(ii)⇒(i). From (ii) we can get that the inequality holds for all finite measures μ, ν, λ, with countable supports. We take ϕ, ψ, and ρ such that $\widehat{\phi}$, $\widehat{\psi}$, and $\widehat{\rho}$ have compact support contained in $[-N/2, N/2]$ for N big enough. Now consider μ_N, ν_N, and λ_N the measures with support in $(1/N)\mathbb{Z}$ whose Fourier transform coincides with the periodic extensions of $\widehat{\phi}$, $\widehat{\psi}$, and $\widehat{\rho}$. In particular, we have

$$\mu_N\left(\left\{\frac{n}{N}\right\}\right) = \frac{1}{N}\phi\left(\frac{n}{N}\right), \qquad \nu_N\left(\left\{\frac{n}{N}\right\}\right) = \frac{1}{N}\psi\left(\frac{n}{N}\right), \qquad \lambda_N\left(\left\{\frac{n}{N}\right\}\right) = \frac{1}{N}\rho\left(\frac{n}{N}\right). \tag{2.13}$$

Therefore we have

$$\begin{aligned}\lim_{N\to\infty} N &\sum_{(t,s)\in\mathbb{R}\times\mathbb{R}} m(t,s)\mu_N(\{t\})\nu_N(\{s\})\lambda_N(\{t+s\}) \\ &= \lim_{N\to\infty} \sum_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} m\left(\frac{n}{N},\frac{m}{N}\right)\phi\left(\frac{n}{N}\right)\psi\left(\frac{m}{N}\right)\rho\left(\frac{n+m}{N}\right)\frac{1}{N^2} \\ &= \int_{\mathbb{R}^2} m(\xi,\nu)\phi(\xi)\psi(\eta)\rho(\xi+\eta)\,d\xi\,d\eta.\end{aligned} \tag{2.14}$$

Now observe that $\|\widehat{\mu}_N\|_{B_{p_1}} = ((1/2N)\int_{-N}^{N}|\widehat{\phi}(\xi)|^{p_1}d\xi)^{1/p_1} = (1/2N)^{1/p_1}\|\widehat{\phi}\|_{p_1}$ and the same for the others.

Using that $\|\widehat{\mu}_N\|_{B_{p_1}} \cdot \|\widehat{\nu}_N\|_{B_{p_2}}\|\widehat{\lambda}_N\|_{B_{p_3'}} = 1/2N$ and passing to the limit, we get the result. □

Remark 2.5. We point out that condition (ii) in Theorem 2.4 is simply a way to say that m defines a multiplier on $\mathbb{D}\times\mathbb{D}$ where $\mathbb{D}$ is the group $\mathbb{R}$ with the discrete topology (see [6]).

Recall that a function m is called regulated if

$$\lim_{\varepsilon\to 0}\frac{1}{4\varepsilon^2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} m(x-s,y-t)\,ds\,dt = m(x,y) \tag{2.15}$$

for all $(x,y)\in\mathbb{R}^2$.

Theorem 2.6. *Let $m(\xi,\eta)$ be a bounded regulated function on $\mathbb{R}\times\mathbb{R}$. Then m is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ if and only if there exists a constant K so that*

$$\left|\sum_{t\in\mathbb{R}}\sum_{s\in\mathbb{R}} m(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\})\right| \le K\|\widehat{\mu}\|_{B_{p_1}}\|\widehat{\nu}\|_{B_{p_2}}\|\widehat{\lambda}\|_{B_{p_3'}} \tag{2.16}$$

for all measures μ, ν, λ having their supports on finite sets of points.

Proof. Assume that m is a (p_1,p_2)-multiplier. Let $\Phi(s,t) = (1/4)\chi_{[-1,1]}(s)\chi_{[-1,1]}(t)$ and $\Phi_\varepsilon(\xi,\eta) = (1/\varepsilon^2)\Phi(\xi/\varepsilon,\eta/\varepsilon)$ for $\varepsilon > 0$. Now Lemma 2.2, Theorem 2.4, and the fact that $m(x,y) = \lim_{\varepsilon\to 0} m * \Phi_\varepsilon(x,y)$ give the direct implication.

Conversely, assume (2.16) for μ, ν, λ having finite supports. Then

$$\begin{aligned}
&\sum_{t\in\mathbb{R}}\sum_{s\in\mathbb{R}}(m*\Phi_\varepsilon)(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\}) \\
&\quad=\int_{\mathbb{R}^2}\left(\sum_{t\in\mathbb{R}}\sum_{s\in\mathbb{R}}m(t-u,s-v)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\})\right)\Phi_\varepsilon(u,v)\,du\,dv \\
&\quad=\int_{\mathbb{R}^2}\left(\sum_{t\in\mathbb{R}}\sum_{s\in\mathbb{R}}m(t,s)\mu(\{t+u\})\nu(\{s+v\})\lambda(\{t+s+u+v\})\right)\Phi_\varepsilon(u,v)\,du\,dv.
\end{aligned} \tag{2.17}$$

This shows that $m*\Phi_\varepsilon$ verifies (2.16) with a uniform constant for all $\varepsilon>0$. Now apply Theorem 2.4 to get that $m*\Phi_\varepsilon$ are (p_1,p_2)-multipliers with uniform norm.

Finally we have that for $\phi,\psi,\nu\in\mathcal{S}$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\phi(\xi)\psi(\eta)\nu(\xi+\eta)m(\xi,\eta)\,d\xi\,d\eta\right| \\
&\quad=\left|\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2}\phi(\xi)\psi(\eta)\nu(\xi+\eta)(m*\Phi_\varepsilon)(\xi,\eta)\,d\xi\,d\eta\right| \\
&\quad\le C\|\widehat{\phi}\|_{p_1}\|\widehat{\psi}\|_{p_2}\|\widehat{\nu}\|_{p_3'}.
\end{aligned} \tag{2.18}$$

The result now follows from Lemma 2.1. □

3. Transference theorems

We mention the formulations for (p_1,p_2)-multipliers on the groups $\mathbb{T}$ and $\mathbb{Z}$ which follow directly from duality.

Lemma 3.1. *Let $\widetilde{m}(t,s)$ be a bounded measurable function on $\mathbb{T}\times\mathbb{T}$. Then m is a (p_1,p_2)-multiplier on $\mathbb{T}\times\mathbb{T}$ if and only if there exists a constant K so that*

$$\left|\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}P_a(t)P_b(s)P_c(t+s)\widetilde{m}(t,s)\,dt\,ds\right|\le K\|a\|_{p_1}\|b\|_{p_2}\|c\|_{p_3'} \tag{3.1}$$

for all finite sequences $(a(n))_n$, $(b(n))_n$, $(c(n))_n$, where $P_a(t)=\sum_n a(n)e^{2\pi int}$.

Lemma 3.2. *Let $(m_{k,k'})$ be a bounded sequence on $\mathbb{Z}\times\mathbb{Z}$. Then m is a (p_1,p_2)-multiplier on $\mathbb{Z}\times\mathbb{Z}$ if and only if there exists a constant K so that*

$$\left|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}m_{k,k'}\widehat{P}(k)\widehat{Q}(k')\widehat{R}(k+k')\right|\le K\|P\|_{p_1}\|Q\|_{p_2}\|R\|_{p_3'} \tag{3.2}$$

for all trigonometric polynomials P, Q, and R.

Theorem 3.3 (see [7, Theorem 1]). *Let $m(\xi,\eta)$ be a regulated bounded function on $\mathbb{R}\times\mathbb{R}$. If $m(\xi,\eta)$ is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$, then $(m(k,k'))_{k,k'}$ is a (p_1,p_2)-multiplier on $\mathbb{Z}\times\mathbb{Z}$.*

Proof. According to Lemma 3.2, we have to show that there exists a constant K so that

$$\left|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} m(k,k')\widehat{P}(k)\widehat{Q}(k')\widehat{R}(k+k')\right| \le K\|P\|_{p_1}\|Q\|_{p_2}\|R\|_{p_3'} \tag{3.3}$$

for all trigonometric polynomials P, Q, and R.

This follows by selecting the measures μ, ν, λ in Theorem 2.6 such that $\widehat{\mu} = P$, $\widehat{\nu} = Q$, and $\widehat{\lambda} = R$. □

Theorem 3.4. *Let $m(\xi,\eta)$ be a bounded regulated function on $\mathbb{R}\times\mathbb{R}$. The following are equivalent:*

(i) *$m(\xi,\eta)$ is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$;*

(ii) *$m(\varepsilon\cdot,\varepsilon\cdot)\chi_{[-1/2\varepsilon,1/2\varepsilon]}\chi_{[-1/2\varepsilon,1/2\varepsilon]}$ (extended by periodicity) are uniformly bounded (p_1,p_2)-multipliers on $\mathbb{T}\times\mathbb{T}$.*

Proof. (i)⇒(ii). Using Lemma 3.1, it suffices to show that for any finite sequences $(a(n))_n$, $(b(n))_n$, and $(c(n))_n$ with $\|a\|_{p_1} = \|b\|_{p_2} = \|c\|_{p_3'} = 1$, there exists a constant $K > 0$ such that

$$\left|\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} m(\xi,\eta)P_a(\xi)P_b(\eta)P_c(\xi+\eta)d\xi\, d\eta\right| \le K, \tag{3.4}$$

where $P_a(\xi) = \sum_n a(n)e^{2\pi in\xi}$.

Since $P_a(x)\chi_{[-1/2,1/2]}(x) = \widehat{\phi}_a(x)$, where $\phi_a(x) = \sum_n a(n)(\sin(\pi(x-n))/\pi(x-n))$, and $P_c(x)\chi_{[-1,1]}(x) = \widehat{\psi}_c(x)$, where $\psi_c(x) = \sum_n c(n)(\sin(2\pi(x-n))/\pi(x-n))$, we can write

$$\begin{aligned}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} & m(\xi,\eta)P_a(\xi)P_b(\eta)P_c(\xi+\eta)d\xi\, d\eta \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} m(\xi,\eta)\widehat{\phi}_a(\xi)\widehat{\phi}_b(\eta)\widehat{\psi}_c(\xi+\eta)d\xi\, d\eta.\end{aligned} \tag{3.5}$$

Using now the assumption and Shanon's sampling theorem, one gets $\|\psi_a\|_{L^p(\mathbb{R})} \le C_1\|\phi_a\|_{L^p(\mathbb{R})} \le C_2\|a\|_{\ell_p} \le C_3\|\psi_a\|_{L^p(\mathbb{R})}$ for some constants C_i for $i = 1,2,3$. Hence the desired inequality follows.

Now we apply Lemma 2.3 to get the result for each ε.

(ii)⇒(i). We take ϕ and ψ such that $\operatorname{supp}\phi$ and $\operatorname{supp}\psi$ are contained in $[-1/4,1/4]$. For a fixed $u \in [-1/2,1/2]$, consider the periodic extensions of the functions $\widehat{\phi}(\xi)e^{2\pi iu\xi}$, $\widehat{\psi}(\eta)e^{2\pi iu\eta}$ to be denoted $\widetilde{P}_u$ and $\widetilde{Q}_u$, respectively.

If $a^u(n) = \int_{-1/2}^{1/2}\widetilde{P}_u(\xi)e^{-i2\pi n\xi}d\xi$, $b^u(n) = \int_{-1/2}^{1/2}\widetilde{Q}_u(\xi)e^{-i2\pi n\xi}d\xi$ for all $n\in\mathbb{Z}$, we have that if $x = k+u$ for some $k\in\mathbb{Z}$ and $u\in[-1/2,1/2)$,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} & m(\xi,\eta)\widehat{\phi}(\xi)\widehat{\psi}(\eta)e^{2\pi ix(\xi+\eta)}d\xi\, d\eta \\ &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} m(\xi,\eta)\widetilde{P}_u(\xi)\widetilde{Q}_u(\eta)e^{2\pi ik(\xi+\eta)}d\xi\, d\eta.\end{aligned} \tag{3.6}$$

Let $\widetilde{m}(\xi,\eta) = m(\xi,\eta)\chi_{[-1/2,1/2]}(\xi)\chi_{[-1/2,1/2]}(\eta)$. Hence for $x = u+k$,

$$\mathscr{C}_m(\phi,\psi)(x) = \mathscr{D}_{\widetilde{m}}(a^u,b^u)(k). \tag{3.7}$$

Now

$$\begin{aligned}
\int_{\mathbb{R}} |\mathscr{C}_m(\phi,\psi)(x)|^{p_3} dx
&= \sum_k \int_{-1/2}^{1/2} |\mathscr{C}_m(\phi,\psi)(k+u)|^{p_3} du \\
&= \int_{-1/2}^{1/2} \sum_k |\mathscr{D}_{\tilde{m}}(a^u,b^u)(k)|^{p_3} du \\
&\le ||\mathscr{D}_{\tilde{m}}||^{p_3} \int_{-1/2}^{1/2} \left(\sum_k |a^u(k)|^{p_1}\right)^{p_3/p_1} \left(\sum_k |b^u(k)|^{p_2}\right)^{p_3/p_2} du \\
&\le ||\mathscr{D}_{\tilde{m}}||^{p_3} \left(\int_{-1/2}^{1/2} \sum_k |a^u(k)|^{p_1} du\right)^{p_3/p_1} \left(\int_{-1/2}^{1/2} \sum_k |b^u(k)|^{p_2} du\right)^{p_3/p_2} \\
&= ||\mathscr{D}_{\tilde{m}}||^{p_3} \left(\int_{-1/2}^{1/2} \sum_k |\phi(u+k)|^{p_1} du\right)^{p_3/p_1} \left(\int_{-1/2}^{1/2} \sum_k |\psi(u+k)|^{p_2} du\right)^{p_3/p_2} \\
&= ||\mathscr{D}_{\tilde{m}}||^{p_3} ||\phi||_{p_1}^{p_3} ||\psi||_{p_2}^{p_3}.
\end{aligned} \tag{3.8}$$

In the general case if ϕ, ψ are such that $\widehat{\phi}$, $\widehat{\psi}$ have compact support, then there exists $\varepsilon > 0$ so that $\widehat{\phi}_\varepsilon$, $\widehat{\psi}_\varepsilon$ have their support in $[-1/4, 1/4]$. Now observe that

$$\mathscr{C}_m(\phi,\psi)(x) = \varepsilon^2 C_{m(\varepsilon\cdot,\varepsilon\cdot)}(\phi_\varepsilon,\psi_\varepsilon)(\varepsilon x). \tag{3.9}$$

Applying the previous case and the assumption, we obtain

$$\begin{aligned}
||\mathscr{C}_m(\phi,\psi)||_{p_3} &= \varepsilon^{2-1/p_3} ||C_{m(\varepsilon\cdot,\varepsilon\cdot)}(\phi_\varepsilon,\psi_\varepsilon)||_{p_3} \\
&\le K\varepsilon^{2-1/p_3} ||\phi_\varepsilon||_{p_1} ||\psi_\varepsilon||_{p_2} \\
&= K\varepsilon^{2-1/p_3} \|\phi\|_{p_1} \varepsilon^{-1/p_1'} \|\psi\|_{p_1} \varepsilon^{-1/p_2'} \\
&= K \|\phi\|_{p_1} \|\psi\|_{p_1}.
\end{aligned} \tag{3.10}$$

□

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