STRONG STABILITY OF WEIGHTED SUMS OF NA RANDOM VARIABLES

GAN SHIXIN

Received 5 July 2003 and in revised form 1 April 2004

We study the almost sure (strong) stability of weighted sums of NA random variables and obtain some new results which extend earlier results of Matula (1992), Chow and Teicher (1971), Jamison et al. (1965), and Petrov (1975).

1. Introduction and preliminaries

We start with definitions. Let $(\Omega,\mathcal{F},P)$ be a probability space. The random variables we deal with are all defined on $(\Omega,\mathcal{F},P)$. A random variable sequence $\{Y_n, n\ge 1\}$ is said to be strongly stable if there exist two constant sequences $\{b_n\}$ and $\{d_n\}$ with $0<b_n\uparrow\infty$ such that

$$b_n^{-1}Y_n - d_n \longrightarrow 0 \quad \text{a.s.} \tag{1.1}$$

A random variable sequence $\{X_n, n\ge 1\}$ is said to be stochastically dominated by a nonnegative random variable X (write $\{X_n\}\prec X$) if there exists a constant $c>0$ such that

$$P(|X_n|>t)\le cP(X>t) \quad \forall t>0,\ \forall n\ge 1. \tag{1.2}$$

A finite family of random variables $\{X_i, 1\le i\le n\}$ is said to be negatively associated (abbreviated to NA) if for any disjoint subsets A and B of $\{1,2,\dots,n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^A$ and g on $\mathbb{R}^B$,

$$\operatorname{Cov}(f(X_i, i\in A), g(X_j, j\in B))\le 0 \tag{1.3}$$

whenever the covariance exists. An infinite family of random variables $\{X_i, i\ge 1\}$ is NA if every finite subfamily is NA. This concept was introduced by Joag-Dev and Proschan [5]. They also pointed out and proved in their paper that a number of well-known multivariate distributions possess the NA property. Now people know that NA random variables have wide application in reliability theory and multivariate statistical analysis. Recently Su et al. [12] show that NA structure plays an important role in risk management. Because of these reasons, the notions of NA random variables have received more and more attention in recent years. A great number of papers for NA random variables have appeared in

International Journal of Mathematics and Mathematical Sciences 2005:6 (2005) 975–985
DOI: 10.1155/IJMMS.2005.975

the literature. We refer to Joag-Dev and Proschab [5] for fundamental properties, Newmann [8] for the central limit theorem, Matula [7] for the three-series theorem, Shao and Su [11] for the law of the iterated logarithm, Shao [10] for moment inequalities, Liu et al. [6] for the Hàjek-Rènyi inequality, and Barbour et al. [1] for Poison approximation.

The main purpose of this paper is to study the strong stability of weighted sums of NA random variables and try to obtain some new results which extend the corresponding results of Matula [7], Chow and Teicher [2], Jamison et al. [4], and Petrov [9]. For this goal, we need some lemmas. The following lemma is a simple extension of [7, Theorem 3].

Lemma 1.1. *Let $\{X_n, n \ge 1\}$ be a sequence of NA random variables with finite second moments and $\{b_n, n \ge 1\}$ a sequence of real numbers with $0 < b_n \uparrow \infty$. If $\sum_{n=1}^{\infty} \operatorname{Var}(X_n)/b_n^2 < \infty$, then $\sum_{n=1}^{\infty}(X_n - EX_n)/b_n$ converges a.s., and therefore $\sum_{i=1}^{n}(X_i - EX_i)/b_n \to 0$ a.s.*

Note that $\{X_n/b_n, n \ge 1\}$ is a sequence of NA random variables by [5, property P_6]. By using [7, Theorem 3], we can immediately obtain Lemma 1.1.

For a random variable X, write $X^{(c)} = XI(|X| \le c) + cI(X > c) - cI(X < -c)$ for some $c > 0$.

Lemma 1.2 (see [7, Theorem 4]). *Let $\{X_n, n \ge 1\}$ be a sequence of NA random variables. If, for some $c > 0$, the series $\sum_{n=1}^{\infty} EX_n^{(c)}$, $\sum_{n=1}^{\infty} \operatorname{Var}(X_n^{(c)})$, and $\sum_{n=1}^{\infty} P(|X_n| \ge c)$ are convergent, then $\sum_{n=1}^{\infty} X_n$ is convergent a.s.*

Lemma 1.3. *Let X be a random variable and X_0 a nonnegative random variable. If, for any $t > 0$, $P(|X| > t) \le cP(X_0 > t)$, then for all $p > 0$, $t > 0$,*

$$E|X|^p I(|X| \le t) \le c\left(t^p P(X_0 > t) + EX_0^p I(X_0 \le t)\right). \tag{1.4}$$

Proof. By the integral equality

$$p\int_0^t s^{p-1}P(|X| > s)\,ds = t^p P(|X| > t) + E|X|^p I(|X| \le t), \tag{1.5}$$

it follows that

$$\begin{aligned} E|X|^p I(|X| \le t) &\le p\int_0^t s^{p-1}P(|X| > s)\,ds \le cp\int_0^t s^{p-1}P(X_0 > s)\,ds \\ &= c\left(t^p P(X_0 > t) + EX_0^p I(X_0 \le t)\right). \end{aligned} \tag{1.6}$$

□

Lemma 1.4. *Let $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ be two sequences of positive numbers with $c_n = b_n/a_n$ and $b_n \uparrow \infty$. Let $\{X_n, n \ge 1\}$ be a sequence of mean zero variables with $\{X_n\} < X$, where X is a nonnegative random variable. Define $N(x) = \operatorname{Card}\{n : c_n \le x\}$, $x > 0$. If*

(1) $\sum_{i=1}^{\infty} P(|X_i| > c_i) < \infty$,
(2) $\sum_{n=1}^{\infty} \int_1^{\infty} P(|X_n| > sc_n)\,ds < \infty$, *or*
(1′) $EN(X) < \infty$,
(2′) $\int_1^{\infty} EN(X/s) < \infty$, *then*

$$b_n^{-1}\sum_{i=1}^{n} a_i EX_i^{(c_i)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{1.7}$$

Proof. Obviously (1′)⇒(1) and (2′)⇒(2). It suffices to show that under conditions (1) and (2) we have (1.7):

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{a_i\left|EX_i^{(c_i)}\right|}{b_i}&=\sum_{i=1}^{\infty}c_i^{-1}\left|E(X_i-c_i)I(X_i>c_i)+E(X_i+c_i)I(X_i<-c_i)\right|\\&\le\sum_{i=1}^{\infty}c_i^{-1}E(|X_i|+c_i)I(|X_i|>c_i)\\&=\sum_{i=1}^{\infty}c_i^{-1}E|X_i|I(|X_i|>c_i)+\sum_{i=1}^{\infty}P(|X_i|>c_i),\end{aligned}\tag{1.8}$$

$$\begin{aligned}\sum_{i=1}^{\infty}c_i^{-1}E|X_i|I(|X_i|>c_i)&\le\sum_{i=1}^{\infty}c_i^{-1}\left(c_iP(|X_i|>c_i)+\int_{c_i}^{\infty}P(|X_i|>t)\,dt\right)\\&=\sum_{i=1}^{\infty}P(|X_i|>c_i)+\sum_{i=1}^{\infty}\int_1^{\infty}P(|X_i|>sc_i)\,ds<\infty.\end{aligned}\tag{1.9}$$

Therefore, from (1.8) and (1.9), $\sum_{i=1}^{\infty}a_iEX_i^{(c_i)}/b_i$ converges. By Kronecker's lemma, we have (1.7). □

Corollary 1.5. *Let $\{X_n, n\ge 1\}$ be a sequence of mean zero random variables with $\{X_n\}<X\in L\log^+L$. If*
(1) $\sum_{n=1}^{\infty}P(|X_n|>n)<\infty$,
(2) $\sum_{n=1}^{\infty}\int_1^{\infty}P(|X_n|>sn)ds<\infty$, *then*

$$n^{-1}\sum_{i=1}^{n}EX_i^{(i)}\longrightarrow 0\quad \text{as } n\longrightarrow\infty.\tag{1.10}$$

Proof. Clearly $c_n=n$, $n\ge 1$. The first part of proof is like in Lemma 1.4. We only give the last part of proof:

$$\begin{aligned}&\sum_{i=1}^{\infty}i^{-1}E|X_i|I(|X_i|>i)\\&\quad\le\sum_{i=1}^{\infty}i^{-1}\left[iP(|X_i|>i)+\int_i^{\infty}P(|X_i|>t)\,dt\right]\end{aligned}\tag{1.11}$$

$$\begin{aligned}&\quad\le c\sum_{i=1}^{\infty}P(X>i)+c\sum_{i=1}^{\infty}i^{-1}\int_i^{\infty}P(X>t)dt\\&\quad\le cEX+c\sum_{i=1}^{\infty}i^{-1}\sum_{k=i}^{\infty}P(X>k)\le cEX+c\sum_{k=1}^{\infty}\sum_{i=1}^{k}i^{-1}P(X>k)\\&\quad\le cEX+c\sum_{k=1}^{\infty}(1+\log k)P(X>k)\le cEX+cEX\log^+X<\infty.\end{aligned}\tag{1.12}$$

This shows that $n^{-1}\sum_{i=1}^{n}EX_i^{(i)}\to 0$ as $n\to\infty$. □

Throughout this paper, the symbol c stands for a generic positive constant which may differ from one place to another.

2. Strong stability

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables and $\{g_n(x), n \geq 1\}$ a sequence of even functions, positive and nondecreasing in the interval $x > 0$. If one of the following conditions is satisfied for every $n \geq 1$:*

(1) $x/g_n(x) \uparrow$ *as* $0 < x \uparrow$,

(2) $x/g_n(x) \downarrow$ *and* $g_n(x)/x^2 \uparrow$ *as* $0 < x \uparrow$ *and also* $EX_n = 0$,

then for any positive real-number sequence $\{b_n, n \geq 1\}$ with $b_n \uparrow \infty$ satisfying

$$\sum_{n=1}^{\infty} \frac{Eg_n(X_n)}{g_n(b_n)} < \infty, \tag{2.1}$$

the series $\sum_{n=1}^{\infty} X_n/b_n$ converges almost surely, and therefore $\sum_{i=1}^{n} X_i/b_n \to 0$ a.s.

Proof. For each $n \geq 1$, put $Z_n = X_n/b_n$. Then $\{Z_n, n \geq 1\}$ remains a sequence of NA random variables by [5, property P_6]. Take $c = 1$ in Lemma 1.2. Since $g_n(x) \uparrow$ as $x > 0$, then $g_n(|X_n|) \geq g_n(b_n)$ on $\{|Z_n| \geq 1\}$. So

$$\sum_{n=1}^{\infty} P(|Z_n| \geq 1) \leq \sum_{n=1}^{\infty} \int \frac{g_n(|X_n|)I(|Z_n| \geq 1)}{g_n(b_n)} dP \leq \sum_{n=1}^{\infty} \frac{Eg_n(X_n)}{g_n(b_n)} < \infty. \tag{2.2}$$

We will suppose that the function $g_n(x)$ satisfies condition (1), then, in the interval $|x| \leq b_n$, we have $x^2/b_n^2 \leq g_n^2(x)/g_n^2(b_n) \leq g_n(x)/g_n(b_n)$. If, however, n is such that (2) is satisfied, then, in the same interval, we have $x^2/b_n^2 \leq g_n(x)/g_n(b_n)$, therefore

$$\begin{aligned} \sum_{n=1}^{\infty} E(Z_n^{(1)})^2 &= \sum_{n=1}^{\infty} \left(\int Z_n^2 I(|Z_n| \leq 1) dP + \int I(|Z_n| > 1) dP \right) \\ &\leq \sum_{n=1}^{\infty} \left(\int \frac{g_n(X_n)I(|X_n| \leq b_n)}{g_n(b_n)} dP + \int \frac{g_n(X_n)I(|X_n| > b_n)}{g_n(b_n)} dP \right) \\ &= \sum_{n=1}^{\infty} \frac{Eg_n(X_n)}{g_n(b_n)} < \infty. \end{aligned} \tag{2.3}$$

If condition (1) is satisfied, then

$$\begin{aligned} |EZ_n^{(1)}| &\leq \int \frac{|X_n| I(|X_n| \leq b_n)}{b_n} dP + \int I(|X_n| > b_n) dP \\ &\leq \int \frac{g_n(|X_n|)I(|X_n| \leq b_n)}{g_n(b_n)} dP + \int \frac{g_n(|X_n|)I(|X_n| > b_n)}{g_n(b_n)} dP \\ &= \frac{Eg_n(X_n)}{g_n(b_n)}. \end{aligned} \tag{2.4}$$

If condition (2) is satisfied, then

$$
\begin{aligned}
|EZ_n^{(1)}| &= \left| \int (Z_n - 1) I(Z_n > 1)\,dP + \int (Z_n + 1) I(Z_n < -1)\,dP \right| \\
&\le \int |Z_n| I(|Z_n| > 1)\,dP + \int I(|Z_n| > 1)\,dP \\
&\le \int \frac{g_n(|X_n|) I(|Z_n| > 1)}{g_n(b_n)}\,dP + \int \frac{g_n(|X_n|) I(|Z_n| > 1)}{g_n(b_n)}\,dP \\
&\le \frac{2Eg_n(X_n)}{g_n(b_n)}.
\end{aligned} \tag{2.5}
$$

Consequently, we have

$$
\sum_{n=1}^{\infty} |EZ_n^{(1)}| \le 2 \sum_{n=1}^{\infty} \frac{Eg_n(X_n)}{g_n(b_n)} < \infty. \tag{2.6}
$$

Thus (2.2), (2.3), (2.6), and Lemma 1.2 imply the convergence of $\sum_{n=1}^{\infty} Z_n = \sum_{n=1}^{\infty} X_n/b_n$ almost surely.

Take $g_n(x) = |x|^p$, $n \ge 1$, $1 \le p \le 2$, in Theorem 2.1. We can infer the following important special case which is a further extension of Lemma 1.1. □

Corollary 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of NA mean zero random variables and $\{b_n, n \ge 1\}$ a sequence of real numbers with $0 < b_n \uparrow \infty$, $1 \le p \le 2$. If $\sum_{n=1}^{\infty} E|X_n|^p/b_n^p < \infty$, then $\sum_{i=1}^{n} X_i/b_n \to 0$ a.s.*

Theorem 2.3. *Let $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ be two sequences of positive numbers with $c_n = b_n/a_n$ and $b_n \uparrow \infty$. Let $\{X_n, n \ge 1\}$ be a sequence of NA random variables which is stochastically dominated by a nonnegative random variable X. Set $N(x) = \text{Card}\{n : c_n \le x\}$, $x > 0$. $1 \le p \le 2$. If the following conditions are satisfied:*

(1) $EN(X) < \infty$,

(2) $\int_0^{\infty} t^{p-1} P(X > t) \int_t^{\infty} N(y)/y^{p+1}\,dy\,dt < \infty$,

then there exist $d_n \in \mathbb{R}$, $n = 1, 2, \ldots$ such that

$$
b_n^{-1} \sum_{i=1}^{n} a_i X_i - d_n \longrightarrow 0 \quad a.s. \tag{2.7}
$$

Proof. Let $S_n = \sum_{i=1}^{n} a_i X_i, T_n = \sum_{i=1}^{n} a_i X_i^{(c_i)}$, $n \ge 1$. Obviously we have

$$
\sum_{i=1}^{\infty} P\left(X_i \ne X_i^{(c_i)}\right) = \sum_{i=1}^{\infty} P(|X_i| > c_i) \le c \sum_{i=1}^{\infty} P(X > c_i) \le cEN(X) < \infty. \tag{2.8}
$$

By Borel-Cantelli lemma for any sequence $\{d_n\} \subset \mathbb{R}$, the sequences $\{b_n^{-1} T_n - d_n\}$ and $\{b_n^{-1} S_n - d_n\}$ converge on the same set and to the same limit. We will show that $b_n^{-1} \sum_{i=1}^{n} a_i (X_i^{(c_i)} - EX_i^{(c_i)}) \to 0$ a.s. which gives the theorem with $d_n = b_n^{-1} \sum_{i=1}^{n} a_i EX_i^{(c_i)}$. Now note that $\{a_i(X_i^{(c_i)} - EX_i^{(c_i)}), i \ge 1\}$ is a sequence of NA mean zero random variables by

[5, property P_6]. It follows from c_r-inequality and Lemma 1.3 that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{E\left|a_n\left(X_n^{(c_n)} - EX_n^{(c_n)}\right)\right|^p}{b_n^p}
&\le c\sum_{n=1}^{\infty} c_n^{-p} E|X_n|^p I(|X_n| \le c_n) + c\sum_{n=1}^{\infty} P(|X_n| > c_n) \\
&\le c\sum_{n=1}^{\infty} c_n^{-p}\left(c_n^p P(X > c_n) + EX^p I(X \le c_n)\right) + c\sum_{n=1}^{\infty} P(X > c_n) \\
&\le cEN(X) + c\sum_{n=1}^{\infty} pc_n^{-p}\int_0^{c_n} t^{p-1}P(X > t)dt,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} pc_n^{-p}\int_0^{c_n} t^{p-1}P(X > t)dt &= p\int_0^{\infty} t^{p-1}P(X > t)\sum_{\{n:c_n>t\}} c_n^{-p}dt \\
&\le p^2\int_0^{\infty} t^{p-1}P(X > t)\int_t^{\infty}\frac{N(y)}{y^{p+1}}dy\,dt.
\end{aligned} \tag{2.10}$$

The last inequality follows from the fact that

$$\begin{aligned}
\sum_{\{n:c_n>t\}} c_n^{-p} = \lim_{u\to\infty}\sum_{\{n:t<c_n<u\}} c_n^{-p} &= \lim_{u\to\infty}\int_t^u y^{-p}dN(y) \\
&= \lim_{u\to\infty}\left(u^{-p}N(u) - t^{-p}N(t) + \int_{t<y\le u} y^{-(p+1)}N(y)dy\right)
\end{aligned} \tag{2.11}$$

and $u^{-p}N(u) \le p\int_u^{\infty} y^{-(p+1)}N(y)dy \to 0$ as $u \to \infty$. Hence, $\sum_{n=1}^{\infty}|a_n(X_n^{(c_n)} - EX_n^{(c_n)})|^p/b_n^p < \infty$. By Corollary 2.2, it follows that $b_n^{-1}\sum_{i=1}^{n} a_i(X_i^{(c_i)} - EX_i^{(c_i)}) \to 0$ a.s. This completes the proof of Theorem 2.3. □

Remark 2.4. Heyde's [3, Theorem 2] extended [4, Theorem 2] of Jamsion et al. to more general weights, which are just the same weights considered in our paper. So Theorem 2.3 is an extension of [3, Theorem 2].

From Theorem 2.3 and Lemma 1.4, we have the following corollaries.

Corollary 2.5. *Let the conditions of Theorem 2.3 be fulfilled and $EX_n = 0, n \ge 1$, and $\int_1^{\infty} EN(X/s)ds < \infty$. Then $b_n^{-1}\sum_{i=1}^{n} a_iX_i \to 0$ a.s.*

Corollary 2.6. *Let $\{X_n, n \ge 1\}$ be a sequence of NA mean zero random variables and $\{X_n\} < X$.*

(1) *If $X \in L\log^+ L$, then $n^{-1}\sum_{i=1}^{n} X_i \to 0$ a.s.*
(2) *If $X \in L^r, 1 < r < 2$, then $n^{-1/r}\sum_{i=1}^{n} X_i \to 0$ a.s.*

Proof. (1) Take $a_n = 1$, $b_n = n$, $n \geq 1$, $p > 1$, in Theorem 2.3, then $c_n = n$, $N(x) = \text{Card}\{n : c_n \leq x\} \leq x$, $x \geq 0$. It is easy to show that conditions (1) and (2) in Theorem 2.3 hold true. Theorem 2.3 and Corollary 1.5 guarantee that $n^{-1}\sum_{i=1}^{n} X_i \to 0$ a.s.

(2) Take $a_n = 1$, $b_n = n^{1/r}$, $n \geq 1$, $p = 2$, in Theorem 2.3. Put $\delta = 2 - r > 0$, then $N(x) = \text{Card}\{n : n^{1/r} \leq x\} \leq x^r$, $x \geq 0$. It is easy to verify that the conditions in Corollary 2.5 hold true. By Corollary 2.5, we have $n^{-1/r}\sum_{i=1}^{n} X_i \to 0$ a.s. □

Theorem 2.7. *Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two sequences of positive numbers with $c_n = b_n/a_n$ and $b_n \uparrow \infty$. Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables which is stochastically dominated by a nonnegative random variable X. $N(x) = \text{Card}\{n : c_n \leq x\}$, $x > 0$. $1 \leq p \leq 2$. If the following conditions are satisfied:*

(1) $EN(X) < \infty$,

(2) $\int_1^\infty EN(X/s)ds < \infty$,

(3) $\max_{1 \leq j \leq n} c_j^p \sum_{j=n}^{\infty} c_j^{-p} = O(n)$,

then

$$b_n^{-1}\sum_{i=1}^{n} a_i X_i \longrightarrow 0 \quad a.s. \tag{2.12}$$

Proof.

$$\sum_{n=1}^{\infty} P\left(X_n \neq X_n^{(c_n)}\right) = \sum_{n=1}^{\infty} P\left(|X_n| > c_n\right) \leq c\sum_{n=1}^{\infty} P\left(X > c_n\right) \leq cEN(X) < \infty. \tag{2.13}$$

By Borel-Cantelli lemma, it suffices to show that $b_n^{-1}\sum_{i=1}^{n} a_i X_i^{(c_i)} \to 0$ a.s. From Lemma 1.4, we need only to show that $b_n^{-1}\sum_{i=1}^{n} a_i(X_i^{(c_i)} - EX_i^{(c_i)}) \to 0$ a.s. Note that $\{a_i(X_i^{(c_i)} - EX_i^{(c_i)}), i \geq 1\}$ is a sequence of NA mean zero random variables.

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{E\left|a_n\left(X_n^{(c_n)} - EX_n^{(c_n)}\right)\right|^p}{b_n^p} \\
&\quad \leq c\sum_{n=1}^{\infty} c_n^{-p} E\left|X_n^{(c_n)}\right|^p = c\sum_{n=1}^{\infty} c_n^{-p}\left(E|X_n|^p I(|X_n| \leq c_n) + c_n^p P(|X_n| > c_n)\right) \\
&\quad \leq c\sum_{n=1}^{\infty} c_n^{-p}\left(c_n^p P(X > c_n) + EX^p I(X \leq c_n)\right) + c\sum_{n=1}^{\infty} P(X > c_n) \\
&\quad \leq c\sum_{n=1}^{\infty} P(X > c_n) + c\sum_{n=1}^{\infty} \frac{EX^p I(X \leq c_n)}{c_n^p}.
\end{aligned} \tag{2.14}$$

Clearly

$$\sum_{n=1}^{\infty} P(X > c_n) \leq EN(X) < \infty. \tag{2.15}$$

Put $\varepsilon_n = \max_{1\le j\le n} c_j$, $\varepsilon_0 = 0$, then

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{EX^pI(X\le c_n)}{c_n^p} &\le \sum_{n=1}^{\infty}\frac{EX^pI(X\le \varepsilon_n)}{c_n^p} = \sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{EX^pI(\varepsilon_{j-1}<X\le\varepsilon_j)}{c_n^p}\\ &\le \sum_{j=1}^{\infty}P(\varepsilon_{j-1}<X\le\varepsilon_j)\varepsilon_j^p\sum_{n=j}^{\infty}\frac{1}{c_n^p}\le c\sum_{j=1}^{\infty}jP(\varepsilon_{j-1}<X\le\varepsilon_j)\\ &= c\sum_{j=1}^{\infty}\sum_{n=1}^{j}P(\varepsilon_{j-1}<X\le\varepsilon_j) = c\sum_{n=1}^{\infty}P(X>\varepsilon_{n-1})\\ &\le c\left(1+\sum_{n=1}^{\infty}P(X>c_n)\right)<\infty.\end{aligned} \tag{2.16}$$

By Corollary 2.2, we have $b_n^{-1}\sum_{i=1}^n a_i(X_i^{(c_i)} - EX_i^{(c_i)}) \to 0$ a.s. The proof is complete. □

Afterwards let $\alpha(x):\mathbb{R}_+\to\mathbb{R}_+$ be a positive, nonincreasing function with $a_n=\alpha(n)$, $b_n=\sum_{i=1}^n a_i$, $c_n = b_n/a_n$, $n\ge 1$, where

$$b_n \longrightarrow \infty, \tag{2.17}$$

$$0<\liminf_{n\to\infty} n^{-1}c_n\alpha(\log c_n)\le\limsup_{n\to\infty} n^{-1}c_n\alpha(\log c_n)<\infty, \tag{2.18}$$

$$x\alpha(\log^+ x) \text{ is nondecreasing for } x>0. \tag{2.19}$$

Theorem 2.8. *Let $\{X_n, n\ge 1\}$ be a sequence of identically distributed NA random variables. If $E|X_1|\alpha(\log^+|X_1|)<\infty$, then there exist $d_n\in\mathbb{R}$, $n=1,2,\dots$, such that $b_n^{-1}\sum_{i=1}^n a_iX_i - d_n\to 0$ a.s.*

Proof. Since $0<\alpha(x)\downarrow$, $b_n\uparrow\infty$, then $c_n\uparrow\infty$. By (2.18) we can choose $m_0\in\mathbb{N}$, $\gamma>0$, $\beta>0$, such that, for $n\ge m_0$,

$$\gamma n\le c_n\alpha(\log c_n)\le\beta n. \tag{2.20}$$

Hence, for $n\ge m\ge m_0$, we have $c_n\ge\gamma n(\alpha(\log c_m))^{-1}$ which guarantees that

$$\sum_{j=m}^{\infty}c_j^{-2}\le\frac{\alpha^2(\log c_m)}{\gamma^2 m},\quad m\ge m_0. \tag{2.21}$$

Note that $\{X_i^{(c_i)}, i\ge 1\}$ is a sequence of NA random variables by [5, property P_6]. For $m\ge m_0$,

$$\sum_{j=m}^{\infty}\frac{E\left(X_j^{(c_j)}\right)^2}{c_j^2} = \sum_{j=m}^{\infty}c_j^{-2}EX_j^2I(|X_j|\le c_j)+\sum_{j=m}^{\infty}P(|X_j|>c_j). \tag{2.22}$$

From (2.20) and (2.21), it follows that, for $m \geq m_0$,

$$\begin{aligned}\sum_{j=m}^{\infty} P(|X_j| > c_j) &\leq \sum_{j=m}^{\infty} P(|X_j|\alpha(\log^+|X_j|) \geq c_j\alpha(\log c_j)) \\ &\leq \sum_{j=m}^{\infty} P(|X_j|\alpha(\log^+|X_j|) \geq \gamma j) \\ &= \sum_{j=m}^{\infty} P(|X_1|\alpha(\log^+|X_1|) \geq \gamma j) < \infty,\end{aligned} \tag{2.23}$$

$$\begin{aligned}\sum_{j=m}^{\infty} c_j^{-2} EX_j^2 I(|X_j| \leq c_j) &= \sum_{j=m}^{\infty} c_j^{-2} EX_1^2 I(|X_1| \leq c_j) \\ &\leq \sum_{j=m}^{\infty} c_j^{-2}\left(\int_{\{|X_1|\leq c_{m-1}\}} X_1^2\,dP + \sum_{i=m}^{j}\int_{\{c_{i-1}<|X_1|\leq c_i\}} X_1^2\,dP\right) \\ &\leq O(1) + \sum_{j=m}^{\infty} c_j^{-2}\sum_{i=m}^{j}\int_{\{c_{i-1}<|X_1|\leq c_i\}} X_1^2\,dP \\ &\leq O(1) + \sum_{i=m}^{\infty} \gamma^{-2} i^{-1}\alpha^2(\log c_i)\int_{\{c_{i-1}<|X_1|\leq c_i\}} X_1^2\,dP \\ &\leq O(1) + \beta\gamma^{-2}\sum_{i=m}^{\infty}\alpha(\log c_i)\int_{\{c_{i-1}<|X_1|\leq c_i\}} |X_1|\,dP \\ &\leq O(1) + \beta\gamma^{-2}\sum_{i=m}^{\infty}\int_{\{c_{i-1}<|X_1|\leq c_i\}} |X_1|\alpha(\log^+|X_1|)\,dP < \infty.\end{aligned} \tag{2.24}$$

From (2.22)–(2.24), it follows that

$$\sum_{j=1}^{\infty}\frac{\operatorname{Var}\left(X_j^{(c_j)}\right)}{c_j^2} \leq \sum_{j=1}^{\infty}\frac{E\left(X_j^{(c_j)}\right)^2}{c_j^2} < \infty. \tag{2.25}$$

By Lemma 1.1, we have $b_n^{-1}\sum_{i=1}^{n} a_i(X_i^{(c_i)} - EX_i^{(c_i)}) \to 0$ a.s. On the other hand, by (2.23), we have

$$\begin{aligned}\sum_{i=1}^{\infty} P\left(X_i \neq X_i^{(c_i)}\right) &= \sum_{i=1}^{\infty} P(|X_i| > c_i) = \sum_{i=1}^{m_0-1} P(|X_i| > c_i) + \sum_{i=m_0}^{\infty} P(|X_i| > c_i) \\ &\leq m_0 - 1 + \sum_{i=m_0}^{\infty} P(|X_1| > c_i) < \infty.\end{aligned} \tag{2.26}$$

Take $d_n = b_n^{-1}\sum_{i=1}^{n} a_i EX_i^{(c_i)}$, $n \geq 1$. By Borel-Cantelli Lemma, we know that the theorem holds true. □

If $\{X_n, n \geq 1\}$ is not identically distributed, we have the following result.

Theorem 2.9. *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables (not necessarily identically distributed). $1 \leq p \leq 2$. If $\sum_{n=1}^{\infty} n^{-p} E|X_n \alpha(\log^+ |X_n|)|^p < \infty$, then there exist $\{d_n\} \subset \mathbb{R}$ such that $b_n^{-1} \sum_{i=1}^{n} a_i X_i - d_n \to 0$ a.s.*

Proof. Using the same notations and similar method of Theorem 2.8, we can easily get

$$\begin{aligned}
\sum_{i=1}^{\infty} P\left(X_i \neq X_i^{(c_i)}\right) &\leq m_0 - 1 + \sum_{i=m_0}^{\infty} P(|X_i| > c_i) \\
&\leq m_0 - 1 + \sum_{i=m_0}^{\infty} P(|X_i \alpha(\log^+ |X_i|)| > c_i \alpha(\log c_i)) \\
&\leq m_0 - 1 + \sum_{i=m_0}^{\infty} P(|X_i \alpha(\log^+ |X_i|)| \geq \gamma i) \\
&\leq m_0 - 1 + \gamma^{-p} \sum_{i=m_0}^{\infty} i^{-p} E|X_i \alpha(\log^+ |X_i|)|^p < \infty.
\end{aligned} \tag{2.27}$$

By Borel-Cantelli lemma for any sequence $\{d_n\} \subset \mathbb{R}$, the sequences $\{b_n^{-1} \sum_{i=1}^{n} a_i X_i^{(c_i)} - d_n\}$ and $\{b_n^{-1} \sum_{i=1}^{n} a_i X_i - d_n\}$ converge on the same set and to the same limit. We will show that $b_n^{-1} \sum_{i=1}^{n} a_i (X_i^{(c_i)} - EX_i^{(c_i)}) \to 0$ a.s. which gives the theorem with $d_n = b_n^{-1} \sum_{i=1}^{n} a_i EX_i^{(c_i)}$, $n \geq 1$. Now note that $\{a_i (X_i^{(c_i)} - EX_i^{(c_i)}), i \geq 1\}$ is a sequence of NA mean zero random variables.

$$\begin{aligned}
&\sum_{n=1}^{\infty} b_n^{-p} E\left|a_n\left(X_n^{(c_n)} - EX_n^{(c_n)}\right)\right|^p \\
&\quad \leq c \sum_{n=1}^{\infty} c_n^{-p} E|X_n^{(c_n)}|^p \\
&\quad \leq c(m_0 - 1) + c \sum_{n=m_0}^{\infty} c_n^{-p} (E|X_n|^p I(|X_n| \leq c_n) + c_n^p P(|X_n| > c_n)) \\
&\quad \leq c(m_0 - 1) + c \sum_{n=m_0}^{\infty} c_n^{-p} (E|X_n|^p I(|X_n| \leq c_n) + c \sum_{n=m_0}^{\infty} P(|X_n| > c_n)) \\
&\quad \leq c(m_0 - 1) + c\gamma^{-p} \sum_{n=m_0}^{\infty} n^{-p} (\alpha(\log c_n))^p E|X_n|^p I(|X_n| \leq c_n) + O(1) \\
&\quad \leq c(m_0 - 1) + c\gamma^{-p} \sum_{n=m_0}^{\infty} n^{-p} E|X_n \alpha(\log^+ |X_n|)|^p I(|X_n| \leq c_n) + O(1) < \infty.
\end{aligned} \tag{2.28}$$

By Corollary 2.2, we have $b_n^{-1} \sum_{i=1}^{n} a_i (X_i^{(c_i)} - EX_i^{(c_i)}) \to 0$ a.s. The proof is complete. □

From Lemma 1.4 and Theorem 2.9 we can get the following corollary.

Corollary 2.10. *Let the conditions of Theorem 2.9 be fulfilled and $EX_n = 0$, $n \geq 1$, and $\sum_{n=1}^{\infty} \int_1^{\infty} P(|X_n| > sc_n) ds < \infty$. Then $b_n^{-1} \sum_{i=1}^{n} a_i X_i \to 0$ a.s.*

Remark 2.11. Since an independent random variable sequence is a special NA sequence, Theorems 2.1, 2.3, 2.7, 2.8, and 2.9 all hold true for independent random variable sequences. So Theorems 2.1, 2.3, 2.7, 2.8, and 2.9 are extensions of corresponding results of Petrov [9, Chapter IX], Jamison et al. [4], Chow and Theicher [2], and Heyde [3].

Acknowledgment

This work was supported by the National Natural Science Foundation of China (Grant no. 10071058). The author thanks the referee for helpful comments.

References

[1] A. D. Barbour, L. Holst, and S. Janson, *Poisson Approximation*, Oxford Studies in Probability, vol. 2, The Clarendon Press; Oxford University Press, New York, 1992.

[2] Y. S. Chow and H. Teicher, *Almost certain summability of independent, identically distributed random variables*, Ann. Math. Statist. **42** (1971), 401–404.

[3] C. C. Heyde, *On almost sure convergence for sums of independent random variables.*, Sankhyā Ser. A **30** (1968), 353–358.

[4] B. Jamison, S. Orey, and W. Pruitt, *Convergence of weighted averages of independent random variables*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **4** (1965), 40–44.

[5] K. Joag-Dev and F. Proschan, *Negative association of random variables, with applications*, Ann. Statist. **11** (1983), no. 1, 286–295.

[6] J. Liu, S. Gan, and P. Chen, *The Hájeck-Rènyi inequality for the NA random variables and its application*, Statist. Probab. Lett. **43** (1999), no. 1, 99–105.

[7] P. Matuła, *A note on the almost sure convergence of sums of negatively dependent random variables*, Statist. Probab. Lett. **15** (1992), no. 3, 209–213.

[8] C. M. Newman, *Asymptotic independence and limit theorems for positively and negatively dependent random variables*, Inequalities in Statistics and Probability (Lincoln, Neb., 1982) (Y. L. Tong, ed.), IMS Lecture Notes Monogr. Ser., vol. 5, Inst. Math. Statist., California, 1984, pp. 127–140.

[9] V. V. Petrov, *Sums of Independent Random Variables*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 82, Springer, New York, 1975.

[10] Q.-M. Shao, *A comparison theorem on moment inequalities between negatively associated and independent random variables*, J. Theoret. Probab. **13** (2000), no. 2, 343–356.

[11] Q.-M. Shao and C. Su, *The law of the iterated logarithm for negatively associated random variables*, Stochastic Process. Appl. **83** (1999), no. 1, 139–148.

[12] C. Su, T. Jiang, Q. H. Tang, and H. Y. Liang, *The safety of negatively associated dependence structure*, Chinese J. Appl. Probab. Statist. **18** (2002), no. 4, 400–404.

Gan Shixin: School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
E-mail address: shixingan@sina.com