A DECOMPOSITION METHOD FOR A SEMILINEAR BOUNDARY VALUE PROBLEM WITH A QUADRATIC NONLINEARITY

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The author adapts the decomposition method of Adomian to find a series solution of a one-dimensional boundary value problem for a semilinear heat equation with a quadratic nonlinearity. Local and global convergence results are obtained.

1. Introduction

In this work, we consider the following semilinear boundary value problem for $u(x,t)$ on the interval $0 < x < \pi$, $t > 0$:

$$\partial_t u = \partial_{xx} u + \gamma u^2, \tag{1.1}$$

$$u(0,t) = u(\pi,t) = 0, \tag{1.2}$$

$$u(x,0) = f(x), \tag{1.3}$$

where $f \in C([0,\pi])$.

The method of solution is based heavily on Adomian's method [1] of writing the solution as a series,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t), \tag{1.4}$$

and reducing the problem to one of iteratively solving a linear equation for u_n once the previous iterates have been determined. However, Adomian partitions (1.4) into a sequence of linear ODEs in either x or t whose solutions cannot generally be made to satisfy the boundary and initial conditions. Even when applied to an initial value problem with boundary conditions, the convergence of the solution depends sensitively on powers of f and its derivatives. In solving a similar problem in [1], Adomian must choose very specific initial data to guarantee local convergence in time. Our method arranges terms so that each linear problem is a PDE boundary value problem which is naturally solved with an expansion of eigenfunctions of ∂_{xx} or a similar operator. We are able to show local convergence for any initial data f and global convergence given a suitable bound

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on f. The method is similar to that of [2] which uses decomposition to derive special eigenfunction expansion solutions of (1.1) and (1.2).

In Section 2, we outline the method and construct a series solution of (1.1), (1.2), and (1.3), and in Section 3, we prove local and global existence results for the solution. In Section 4, we construct a series solution of (1.1) and (1.3) with Neumann boundary conditions in place of (1.2) and prove a local convergence result for the solution.

2. The method

To begin, (1.4) is substituted into (1.1) to obtain

$$\sum_{n=1}^{\infty} \partial_t u_n = \sum_{n=1}^{\infty} \partial_{xx} u_n + \gamma \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} u_{n-k} u_k. \tag{2.1}$$

Based on this, we seek solutions to the sequence of equations

$$\partial_t u_1 = \partial_{xx} u_1, \tag{2.2}$$

$$\partial_t u_n = \partial_{xx} u_n + \gamma \sum_{k=1}^{n-1} u_{n-k} u_k, \quad n \geq 2. \tag{2.3}$$

Notice that, once $u_1, u_2, \ldots, u_{n-1}$ are determined, (2.3) is a linear equation for u_n. The series (1.4) is constructed by solving this sequence of equations.

All of the u_n's are assumed to satisfy the boundary conditions

$$u_n(0,t) = u_n(\pi,t) = 0, \quad n \geq 1. \tag{2.4}$$

u_1 is assumed to satisfy the initial condition

$$u_1(x,0) = f(x), \tag{2.5}$$

and the remaining u_n's are assumed to be zero initially:

$$u_n(x,0) = 0, \quad n \geq 2. \tag{2.6}$$

The solution of (2.2), (2.4), and (2.5) is

$$u_1(x,t) = \int_0^{\pi} f(\xi) G(x,\xi,t)\, d\xi, \tag{2.7}$$

where G is Green's function given by

$$G(x,\xi,t) = \frac{2}{\pi} \sum_{j=1}^{\infty} e^{-j^2 t} \sin j\xi \sin jx. \tag{2.8}$$

Once $u_1, u_2, \ldots, u_{n-1}$ are known, (2.3), (2.4), and (2.6) can be solved for u_n to obtain

$$u_n(x,t) = \int_0^t \int_0^\pi h_n(\xi,\tau) G(x,\xi,t-\tau)\, d\xi\, d\tau, \tag{2.9}$$

for $n \geq 2$, where

$$h_n(x,t) = \gamma \sum_{k=1}^{n-1} u_{n-k} u_k. \tag{2.10}$$

The formulation (2.8) of G is useful for long-time estimates of the solution; for time-local estimates, we will use

$$G(x,\xi,t) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(x-\xi-2\pi n)^2}{4t} \right) - \exp\left(-\frac{(x+\xi-2\pi n)^2}{4t} \right) \right]. \tag{2.11}$$

(See [3].)

3. Convergence results

Lemma 3.1. *Let μ_n be a sequence of positive numbers defined by*

$$\mu_n = \kappa \sum_{k=1}^{n-1} \mu_{n-k} \mu_k, \tag{3.1}$$

where $\kappa > 0$. Then

$$\mu_n \leq (4\kappa\mu_1)^n. \tag{3.2}$$

Proof. We repeat the argument given in [2] using the method of generating functions. Define

$$\rho(z) = \kappa \sum_{n=1}^{\infty} \mu_n z^n \tag{3.3}$$

from which follows

$$\rho(z)^2 = \kappa^2 \sum_{n\geq 2} z^n \sum_{k=1}^{n-1} \mu_{n-k}\mu_k = \kappa \sum_{n\geq 2} z^n \mu_n = \rho(z) - \kappa\mu_1 z. \tag{3.4}$$

In other words, the generating function ρ satisfies the quadratic

$$\rho^2 - \rho + \kappa\mu_1 z = 0, \tag{3.5}$$

and so,

$$\rho = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\kappa\mu_1 z}. \tag{3.6}$$

The fact that $\mu_n \geq 0$ leads us to choose $\rho = 1/2 - (1/2)\sqrt{1-4\kappa\mu_1 z}$. Thus,

$$\begin{aligned}
\rho^{(n)}(z) &= \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)(1-4\kappa\mu_1 z)^{(1/2-n)}(-4\kappa\mu_1)^n \\
&= -\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\left(n-\frac{3}{2}\right)(1-4\kappa\mu_1 z)^{(1/2-n)}(4\kappa\mu_1)^n \\
&= -\frac{1}{2}(1-4\kappa\mu_1 z)^{(1/2-n)}(4\kappa\mu_1)^n\prod_{k=1}^{n}\left(k-\frac{3}{2}\right), \\
\mu_n &= \frac{\rho^{(n)}(0)}{n!} = -\frac{1}{2}(4\kappa\mu_1)^n\prod_{k=1}^{n}\left(1-\frac{3}{2k}\right) < (4\kappa\mu_1)^n.
\end{aligned} \tag{3.7}$$

□

In what follows, we adopt the following norm notation:

$$\|\cdot\|_\pi = \sup_{0<x<\pi}|\cdot|, \qquad \|\cdot\|_\delta = \sup_{\substack{0<x<\pi\\0<t<\delta}}|\cdot|, \qquad \|\cdot\| = \sup_{\substack{0<x<\pi\\t>0}}|\cdot|. \tag{3.8}$$

Our first theorem concerns local convergence of (1.4). Specifically, we show that the time interval over which the series converges to a solution of (1.1), (1.2), and (1.3) is inversely related to the size of the initial data and the coefficient of the nonlinear term.

Theorem 3.2. *With u_n defined by (2.7) and (2.9), the series (1.4) converges uniformly on $[0,\pi]\times[0,\delta]$ if $\delta < 1/(4\gamma\|f\|_\pi)$ and $f \in C([0,\pi])$.*

Proof. It is not hard to show from (2.9) and (2.11) that u_n can be expressed in the form

$$u_n(x,t) = \int_0^t\int_{-\infty}^{\infty}\widetilde{h}_n(\xi,\tau)H(x-\xi,t-\tau)d\xi\,d\tau, \tag{3.9}$$

where $\widetilde{h}_n$ is the odd, 2π-periodic extension of h_n and H is the usual heat kernel,

$$H(x,t) = \frac{\exp(-x^2/4t)}{\sqrt{4\pi t}}. \tag{3.10}$$

Then,

$$||u_n||_\delta \leq ||h_n||_\delta\int_0^\delta\int_{-\infty}^{\infty}H(x-\xi,\delta-\tau)d\xi\,d\tau = \delta||h_n||_\delta \tag{3.11}$$

for $n \geq 2$. Combining this with (2.10), we have

$$||u_n||_\delta \leq \gamma\delta\left\|\sum_{k=1}^{n-1}u_{n-k}u_k\right\|_\delta \leq \gamma\delta\sum_{k=1}^{n-1}||u_{n-k}||_\delta||u_k||_\delta \tag{3.12}$$

for $n \geq 2$. By Lemma 3.1,

$$||u_n||_\delta \leq (4\gamma\delta||u_1||_\delta)^n. \tag{3.13}$$

From (2.7) and (2.11), we can argue that

$$||u_1||_\delta \leq \|f\|_\pi \tag{3.14}$$

which, with (3.13), implies that

$$||u_n||_\delta \le (4\gamma\delta \| f \|_\pi)^n. \tag{3.15}$$

Thus (1.4) converges if $\delta < 1/(4\gamma\| f \|_\pi)$. □

Our next theorem gives conditions under which (1.4) converges globally.

Theorem 3.3. *With u_n defined by (2.7) and (2.9), the series (1.4) converges uniformly on $[0,\pi]\times[0,\infty)$ if $\| f \|_\pi < 3/(4\gamma\pi^2)$.*

Proof. Let

$$K(s) = \sup_{0<x<\pi} \int_0^\pi |G(x,\xi,s)|\, d\xi. \tag{3.16}$$

Then by (2.9),

$$||u_n|| \le ||h_n|| \int_0^\infty K(\tau) d\tau. \tag{3.17}$$

Notice from (2.8) that

$$K(s) \le \sup_{0<x<\pi} \frac{2}{\pi} \sum_{j=1}^\infty e^{-j^2 s} \int_0^\pi |\sin jx \sin j\xi| d\xi \le 2 \sum_{j=1}^\infty e^{-j^2 s}. \tag{3.18}$$

This means that

$$\int_0^\infty K(s) ds \le 2 \sum_{j=1}^\infty \frac{1}{j^2} = \frac{\pi^2}{3}. \tag{3.19}$$

Combining this with (3.17), we have

$$||u_n|| \le \frac{1}{3}\pi^2 ||h_n|| \le \frac{1}{3}\gamma\pi^2 \sum_{k=1}^{n-1} ||u_{n-k}||\,||u_k||. \tag{3.20}$$

Lemma 3.1 then implies that

$$||u_n|| \le \left(\frac{4}{3}\gamma\pi^2\right)^n ||u_1||^n \le \left(\frac{4}{3}\gamma\pi^2\right)^n \| f \|_\pi^n. \tag{3.21}$$

Thus (1.4) converges on $[0,\pi]\times[0,\infty)$ if $\| f \|_\pi < 3/(4\gamma\pi^2)$. □

4. Neumann boundary conditions

In this section, we replace (1.2) with Neumann boundary conditions, that is,

$$\partial_x u(0,t) = \partial_x u(\pi,t) = 0. \tag{4.1}$$

We now assume that the solution has the form

$$u(x,t) = u_0(t) + \sum_{n=1}^{\infty} u_n(x,t), \tag{4.2}$$

where u_0 is a purely time-dependent solution of (1.1), specifically,

$$u_0(t) = \frac{a_0}{1-\gamma a_0 t}, \tag{4.3}$$

where $a_0 = (1/\pi)\int_0^\pi f(x)dx$. (Notice that a global solution in time is not generally possible because of the blowup of u_0 in finite time.) Substituting (4.2) into (1.1) and using the fact that u_0 satisfies (1.1), we obtain

$$\sum_{n=1}^{\infty} \partial_t u_n = \sum_{n=1}^{\infty} \partial_{xx} u_n + 2\gamma u_0 \sum_{n=1}^{\infty} u_n + \gamma \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} u_{n-k} u_k, \tag{4.4}$$

leading to the sequence of linear equations

$$\partial_t u_1 = \partial_{xx} u_1 + 2\gamma u_0 u_1, \tag{4.5}$$

$$\partial_t u_n = \partial_{xx} u_n + 2\gamma u_0 u_n + \gamma \sum_{k=1}^{n-1} u_{n-k} u_k, \quad n \geq 2. \tag{4.6}$$

We assume (2.6) as before, but now (2.5) is replaced by

$$u_1(x,0) = f(x) - u_0(0) = f(x) - a_0 \tag{4.7}$$

and (2.4) by

$$\partial_x u_n(0,t) = \partial_x u_n(\pi,t) = 0. \tag{4.8}$$

Expressed in terms of a Green's function, the solutions of (4.5), (4.8), (4.7), and (4.6), (4.8), (2.6) are

$$u_1(x,t) = \int_0^\pi \frac{(f(\xi)-a_0)G(x,\xi,t)}{(1-\gamma a_0 t)^2} d\xi, \tag{4.9}$$

$$u_n(x,t) = \int_0^t \int_0^\pi h_n(\xi,\tau)\left(\frac{1-\gamma a_0 \tau}{1-\gamma a_0 t}\right)^2 G(x,\xi,t-\tau)d\xi\, d\tau, \quad n \geq 2, \tag{4.10}$$

where

$$G(x,\xi,t) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(x-\xi-2\pi n)^2}{4t}\right) + \exp\left(-\frac{(x+\xi-2\pi n)^2}{4t}\right)\right]. \tag{4.11}$$

THEOREM 4.1. *With u_n defined by (4.3), (4.9), and (4.10), the series (4.2) converges uniformly on $[0,\pi]\times[0,\delta]$ if $\|f-a_0\|_\pi < (1-\gamma a_0\delta)^4/(4\gamma\delta)$ and $f \in C([0,\pi])$.*

Proof. Equations (4.10) and (4.11) can be used to express u_n in the form

$$u_n(x,t) = \int_0^t \int_{-\infty}^{\infty} \widetilde{h}_n(\xi,\tau)\left(\frac{1-\gamma a_0\tau}{1-\gamma a_0 t}\right)^2 H(x-\xi,t-\tau)\,d\xi\,d\tau, \tag{4.12}$$

where $\widetilde{h}_n$ is the even, 2π-periodic extension of h_n and H is defined by (3.10). Then,

$$\begin{aligned}||u_n||_\delta &\le ||h_n||_\delta \int_0^\delta \left(\frac{1-\gamma a_0\tau}{1-\gamma a_0\delta}\right)^2 \int_{-\infty}^{\infty} H(x-\xi,\delta-\tau)\,d\xi\,d\tau \\ &= ||h_n||_\delta \int_0^\delta \left(\frac{1-\gamma a_0\tau}{1-\gamma a_0\delta}\right)^2 d\tau \le \frac{\delta ||h_n||_\delta}{(1-\gamma a_0\delta)^2}\end{aligned} \tag{4.13}$$

for $n \ge 2$. Combining this with (2.10), we have

$$||u_n||_\delta \le \frac{\gamma\delta}{(1-\gamma a_0\delta)^2}\left|\left|\sum_{k=1}^{n-1} u_{n-k}u_k\right|\right|_\delta \le \frac{\gamma\delta}{(1-\gamma a_0\delta)^2}\sum_{k=1}^{n-1} ||u_{n-k}||_\delta ||u_k||_\delta \tag{4.14}$$

for $n \ge 2$. By Lemma 3.1,

$$||u_n||_\delta \le \left(\frac{4\gamma\delta||u_1||_\delta}{(1-\gamma a_0\delta)^2}\right)^n. \tag{4.15}$$

From (4.9) and (4.11), we can argue that

$$||u_1||_\delta \le \frac{||f-a_0||_\pi}{(1-\gamma a_0\delta)^2} \tag{4.16}$$

which, with (4.15), implies that

$$||u_n||_\delta \le \left(\frac{4\gamma\delta||f-a_0||_\pi}{(1-\gamma a_0\delta)^4}\right)^n. \tag{4.17}$$

Thus (1.4) converges if

$$||f-a_0||_\pi < \frac{(1-\gamma a_0\delta)^4}{4\gamma\delta}. \tag{4.18}$$

□

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