GENERALIZED $(\mathscr{F}, b, \phi, \rho, \theta)$-UNIVEX n-SET FUNCTIONS AND SEMIPARAMETRIC DUALITY MODELS IN MULTIOBJECTIVE FRACTIONAL SUBSET PROGRAMMING

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We construct a number of semiparametric duality models and establish appropriate duality results under various generalized $(\mathscr{F}, b, \phi, \rho, \theta)$-univexity assumptions for a multiobjective fractional subset programming problem.

1. Introduction

In this paper, we will present a number of semiparametric duality results under various generalized $(\mathscr{F}, b, \phi, \rho, \theta)$-univexity hypotheses for the following multiobjective fractional subset programming problem:

(P)

$$\text{Minimize} \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \dots, \frac{F_p(S)}{G_p(S)} \right) \quad \text{subject to } H_j(S) \leqq 0,\ j \in \underline{q},\ S \in \mathbb{A}^n, \tag{1.1}$$

where $\mathbb{A}^n$ is the n-fold product of the σ-algebra $\mathbb{A}$ of subsets of a given set $X, F_i, G_i, i \in \underline{p} \equiv \{1, 2, \dots, p\}$, and $H_j, j \in \underline{q}$, are real-valued functions defined on $\mathbb{A}^n$, and for each $i \in \underline{p}$, $G_i(S) > 0$ for all $S \in \mathbb{A}^n$ such that $H_j(S) \leqq 0$, $j \in \underline{q}$.

This paper is essentially a continuation of the investigation that was initiated in the companion paper [6] where some information about multiobjective fractional programming problems involving point-functions as well as n-set functions was presented, a fairly comprehensive list of references for multiobjective fractional subset programming problems was provided, a brief overview of the available results pertaining to multiobjective *fractional* subset programming problems was given, and numerous sets of semiparametric sufficient efficiency conditions under various generalized $(\mathscr{F}, b, \phi, \rho, \theta)$-univexity assumptions were established. These and some other related material that were discussed in [6] will not be repeated in the present paper. Making use of the semiparametric sufficient efficiency criteria developed in [6] in conjunction with a certain necessary efficiency result that will be recalled in the next section, here we will construct several semiparametric duality models for (P) with varying degrees of generality and, in each case, prove appropriate weak, strong, and strict converse duality theorems under a number of generalized $(\mathscr{F}, b, \phi, \rho, \theta)$-univexity conditions.

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The rest of this paper is organized as follows. In Section 3 we consider a simple dual problem and prove weak, strong, and strict converse duality theorems. In Section 4 we formulate another dual problem with a relatively more flexible structure that allows for a greater variety of generalized $(\mathcal{F},b,\phi,\rho,\theta)$-univexity conditions under which duality can be established. In Sections 5 and 6 we state and discuss two general duality models which are, in fact, two families of dual problems for (P), whose members can easily be identified by appropriate choices of certain sets and functions.

Evidently, all of these duality results are also applicable, when appropriately specialized, to the following three classes of problems with multiple, fractional, and conventional objective functions, which are particular cases of (P):

$(P1)$

$$\underset{S\in\mathbb{F}}{\text{Minimize}}\ \big(F_1(S),F_2(S),\dots,F_p(S)\big); \tag{1.2}$$

$(P2)$

$$\underset{S\in\mathbb{F}}{\text{Minimize}}\ \frac{F_1(S)}{G_1(S)}; \tag{1.3}$$

$(P3)$

$$\underset{S\in\mathbb{F}}{\text{Minimize}}\ F_1(S), \tag{1.4}$$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F}=\big\{S\in\mathbb{A}^n : H_j(S)\leqq 0,\ j\in\underline{q}\big\}. \tag{1.5}$$

Since in most cases the duality results established for (P) can easily be modified and restated for each one of the above problems, we will not explicitly state these results.

2. Preliminaries

In this section, we gather, for convenience of reference, a few basic definitions and auxiliary results which will be used frequently throughout the sequel.

Let $(X,\mathbb{A},\mu)$ be a finite atomless measure space with $L_1(X,\mathbb{A},\mu)$ separable, and let d be the pseudometric on $\mathbb{A}^n$ defined by

$$d(R,S)=\left[\sum_{i=1}^{n}\mu^2(R_i\triangle S_i)\right]^{1/2},\quad R=(R_1,\dots,R_n),\ S=(S_1,\dots,S_n)\in\mathbb{A}^n, \tag{2.1}$$

where $\triangle$ denotes symmetric difference; thus $(\mathbb{A}^n,d)$ is a pseudometric space. For $h\in L_1(X,\mathbb{A},\mu)$ and $T\in\mathbb{A}$ with characteristic function $\chi_T\in L_\infty(X,\mathbb{A},\mu)$, the integral $\int_T h\,d\mu$ will be denoted by $\langle h,\chi_T\rangle$.

We next define the notion of differentiability for n-set functions. It was originally introduced by Morris [3] for a set function, and subsequently extended by Corley [1] for n-set functions.

Definition 2.1. A function $F:\mathbb{A}\to\mathbb{R}$ is said to be *differentiable* at S^* if there exists $DF(S^*)\in L_1(X,\mathbb{A},\mu)$, called the *derivative* of F at S^*, such that for each $S\in\mathbb{A}$,

$$F(S)=F(S^*)+\langle DF(S^*),\chi_S-\chi_{S^*}\rangle+V_F(S,S^*), \tag{2.2}$$

where $V_F(S,S^*)$ is $o(d(S,S^*))$, that is, $\lim_{d(S,S^*)\to 0}V_F(S,S^*)/d(S,S^*)=0$.

Definition 2.2. A function $G:\mathbb{A}^n\to\mathbb{R}$ is said to have a *partial derivative* at $S^*=(S_1^*,\ldots,S_n^*)\in\mathbb{A}^n$ with respect to its ith argument if the function $F(S_i)=G(S_1^*,\ldots,S_{i-1}^*,S_i,S_{i+1}^*,\ldots,S_n^*)$ has derivative $DF(S_i^*)$, $i\in\underline{n}$; in that case, the ith partial derivative of G at S^* is defined to be $D_iG(S^*)=DF(S_i^*)$, $i\in\underline{n}$.

Definition 2.3. A function $G:\mathbb{A}^n\to\mathbb{R}$ is said to be *differentiable* at S^* if all the partial derivatives $D_iG(S^*), i\in\underline{n}$, exist and

$$G(S)=G(S^*)+\sum_{i=1}^{n}\langle D_iG(S^*),\chi_{S_i}-\chi_{S_i^*}\rangle+W_G(S,S^*), \tag{2.3}$$

where $W_G(S,S^*)$ is $o(d(S,S^*))$ for all $S\in\mathbb{A}^n$.

We next recall the definitions of the generalized $(\mathcal{F},b,\phi,\rho,\theta)$-univex n-set functions which will be used in the statements of our duality theorems. For more information about these and a number of other related classes of n-set functions, the reader is referred to [6]. We begin by defining a *sublinear function* which is an integral part of all the subsequent definitions.

Definition 2.4. A function $\mathcal{F}:\mathbb{R}^n\to\mathbb{R}$ is said to be *sublinear (superlinear)* if $\mathcal{F}(x+y)\leqq(\geqq)\mathcal{F}(x)+\mathcal{F}(y)$ for all $x,y\in\mathbb{R}^n$, and $\mathcal{F}(ax)=a\mathcal{F}(x)$ for all $x\in\mathbb{R}^n$ and $a\in\mathbb{R}_+\equiv[0,\infty)$.

Let $S,S^*\in\mathbb{A}^n$, and assume that the function $F:\mathbb{A}^n\to\mathbb{R}$ is differentiable at S^*.

Definition 2.5. The function F is said to be (*strictly*) $(\mathcal{F},b,\phi,\rho,\theta)$-*univex* at S^* if there exist a sublinear function $\mathcal{F}(S,S^*;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$, a function $b:\mathbb{A}^n\times\mathbb{A}^n\to\mathbb{R}$ with positive values, a function $\theta:\mathbb{A}^n\times\mathbb{A}^n\to\mathbb{A}^n\times\mathbb{A}^n$ such that $S\neq S^*\Rightarrow\theta(S,S^*)\neq(0,0)$, a function $\phi:\mathbb{R}\to\mathbb{R}$, and a real number ρ such that for each $S\in\mathbb{A}^n$,

$$\phi(F(S)-F(S^*))(>)\geqq\mathcal{F}(S,S^*;b(S,S^*)DF(S^*))+\rho d^2(\theta(S,S^*)). \tag{2.4}$$

Definition 2.6. The function F is said to be (*strictly*) $(\mathcal{F},b,\phi,\rho,\theta)$-*pseudounivex* at S^* if there exist a sublinear function $\mathcal{F}(S,S^*;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$, a function $b:\mathbb{A}^n\times\mathbb{A}^n\to\mathbb{R}$ with positive values, a function $\theta:\mathbb{A}^n\times\mathbb{A}^n\to\mathbb{A}^n\times\mathbb{A}^n$ such that $S\neq S^*\Rightarrow\theta(S,S^*)\neq(0,0)$, a function $\phi:\mathbb{R}\to\mathbb{R}$, and a real number ρ such that for each $S\in\mathbb{A}^n$ $(S\neq S^*)$,

$$\mathcal{F}(S,S^*;b(S,S^*)DF(S^*))\geqq-\rho d^2(\theta(S,S^*))\Longrightarrow\phi(F(S)-F(S^*))(>)\geqq 0. \tag{2.5}$$

Definition 2.7. The function F is said to be (*prestrictly*) $(\mathcal{F},b,\phi,\rho,\theta)$-*quasiunivex* at S^* if there exist a sublinear function $\mathcal{F}(S,S^*;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$, a function $b:\mathbb{A}^n\times\mathbb{A}^n\to\mathbb{R}$ with positive values, a function $\theta:\mathbb{A}^n\times\mathbb{A}^n\to\mathbb{A}^n\times\mathbb{A}^n$ such that $S\neq S^*\Rightarrow\theta(S,S^*)\neq(0,0)$, a function $\phi:\mathbb{R}\to\mathbb{R}$, and a real number ρ such that for each $S\in\mathbb{A}^n$,

$$\phi(F(S)-F(S^*))(<)\leqq 0\Longrightarrow\mathcal{F}(S,S^*;b(S,S^*)DF(S^*))\leqq-\rho d^2(\theta(S,S^*)). \tag{2.6}$$

From the above definitions it is clear that if F is $(\mathscr{F},b,\phi,\rho,\theta)$-univex at S^*, then it is both $(\mathscr{F},b,\phi,\rho,\theta)$-pseudounivex and $(\mathscr{F},b,\phi,\rho,\theta)$-quasiunivex at S^*, if F is $(\mathscr{F},b,\phi,\rho,\theta)$-quasiunivex at S^*, then it is prestrictly $(\mathscr{F},b,\phi,\rho,\theta)$-quasiunivex at S^*, and if F is strictly $(\mathscr{F},b,\phi,\rho,\theta)$-pseudounivex at S^*, then it is $(\mathscr{F},b,\phi,\rho,\theta)$-quasiunivex at S^*.

In the proofs of the duality theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, $(\mathscr{F},b,\phi,\rho,\theta)$-quasiunivexity can be defined in the following equivalent way: F is said to be $(\mathscr{F},b,\phi,\rho,\theta)$-quasiunivex at S^* if for each $S \in \mathbb{A}^n$,

$$\mathscr{F}(S,S^*;b(S,S^*)DF(S^*)) > -\rho d^2(\theta(S,S^*)) \Longrightarrow \phi(F(S)-F(S^*)) > 0. \tag{2.7}$$

Needless to say, the new classes of generalized convex n-set functions specified in Definitions 2.5, 2.6, and 2.7 contain a variety of special cases; in particular, they subsume all the previously defined types of generalized n-set functions. This can easily be seen by appropriate choices of $\mathscr{F}$, b, ϕ, ρ, and θ.

In the sequel we will also need a consistent notation for vector inequalities. For all $a,b \in \mathbb{R}^m$, the following order notation will be used: $a \geqq b$ if and only if $a_i \geqq b_i$ for all $i \in \underline{m}$; $a \geqslant b$ if and only if $a_i \geqq b_i$ for all $i \in \underline{m}$, but $a \neq b$; $a > b$ if and only if $a_i > b_i$ for all $i \in \underline{m}$; $a \ngeqslant b$ is the negation of $a \geqslant b$.

Throughout the sequel we will deal exclusively with the efficient solutions of (P). An $x^* \in \mathscr{X}$ is said to be an *efficient solution* of (P) if there is no other $x \in \mathscr{X}$ such that $\varphi(x) \leqslant \varphi(x^*)$, where φ is the objective function of (P).

Next, we recall a set of parametric necessary efficiency conditions for (P).

Theorem 2.8 [5]. *Assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at $S^* \in \mathbb{A}^n$, and that for each $i \in \underline{p}$, there exist $\hat{S}^i \in \mathbb{A}^n$ such that*

$$H_j(S^*) + \sum_{k=1}^{n} \langle D_k H_j(S^*), \chi_{\hat{S}_k} - \chi_{S_k^*} \rangle < 0, \quad j \in \underline{q}, \tag{2.8}$$

and for each $\ell \in \underline{p} \setminus \{i\}$,

$$\sum_{k=1}^{n} \langle D_k F_\ell(S^*) - \lambda_\ell^* D_k G_\ell(S^*), \chi_{\hat{S}_k} - \chi_{S_k^*} \rangle < 0. \tag{2.9}$$

If S^* is an efficient solution of (P) and $\lambda_i^* = \varphi(S^*)$, $i \in \underline{p}$, then there exist $u^* \in U = \{u \in \mathbb{R}^p : u > 0, \sum_{i=1}^{p} u_i = 1\}$ and $v^* \in \mathbb{R}_+^q$ such that

$$\begin{aligned} \sum_{k=1}^{n} \Bigg\langle \sum_{i=1}^{p} u_i^* [D_k F_i(S^*) - \lambda_i^* D_k G_i(S^*)] \\ + \sum_{j=1}^{q} v_j^* D_k H_j(S^*), \chi_{S_k} - \chi_{S_k^*} \Bigg\rangle \geqq 0, \quad \forall S \in \mathbb{A}^n, \end{aligned} \tag{2.10}$$

$v_j^* H_j(S^*) = 0$, $j \in \underline{q}$.

The above theorem contains two sets of parameters u_i^* and $\lambda_i^*, i \in \underline{p}$, which were introduced as a consequence of our indirect approach in [5] requiring two intermediate auxiliary problems. It is possible to eliminate one of these two sets of parameters and thus obtain a semiparametric version of Theorem 2.8. Indeed, this can be accomplished by simply replacing λ_i^* by $F_i(S^*)/G_i(S^*)$, $i \in \underline{p}$, and redefining u^* and v^*. For future reference, we state this in the next theorem.

Theorem 2.9. *Assume that $F_i, G_i, i \in \underline{p}$, and $H_j, j \in \underline{q}$, are differentiable at $S^* \in \mathbb{A}^n$, and that for each $i \in \underline{p}$, there exist $\hat{S}^i \in \mathbb{A}^n$ such that*

$$H_j(S^*) + \sum_{k=1}^{n} \langle D_k H_j(S^*), \chi_{\hat{S}_k} - \chi_{S_k^*} \rangle < 0, \quad j \in \underline{q}, \tag{2.11}$$

and for each $\ell \in \underline{p} \setminus \{i\}$,

$$\sum_{k=1}^{n} \langle G_i(S^*) D_k F_\ell(S^*) - F_i(S^*) D_k G_\ell(S^*), \chi_{\hat{S}_k} - \chi_{S_k^*} \rangle < 0. \tag{2.12}$$

If S^* is an efficient solution of (P), then there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that

$$\begin{aligned} \sum_{k=1}^{n} \Big\langle \sum_{i=1}^{p} u_i^* [G_i(S^*) D_k F_i(S^*) - F_i(S^*) D_k G_i(S^*)] \\ + \sum_{j=1}^{q} v_j^* D_k H_j(S^*), \chi_{S_k} - \chi_{S_k^*} \Big\rangle \geqq 0, \quad \forall S \in \mathbb{A}^n, \end{aligned} \tag{2.13}$$

$v_j^* H_j(S^*) = 0,\ j \in \underline{q}$.

For simplicity, we will henceforth refer to an efficient solution S^* of (P) satisfying (2.11) and (2.12) for some $\hat{S}^i, i \in \underline{p}$, as a *normal* efficient solution.

The form and contents of the necessary efficiency conditions given in Theorem 2.9 in conjunction with the sufficient efficiency results established in [6] provide clear guidelines for constructing various types of semiparametric duality models for (P).

3. Duality model I

In this section, we discuss a duality model for (P) with a somewhat restricted constraint structure that allows only certain types of generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity conditions for establishing duality. More general duality models will be presented in subsequent sections.

In the remainder of this paper, we assume that the functions $F_i, G_i, i \in \underline{p}$, and $H_j, j \in \underline{q}$, are differentiable on $\mathbb{A}^n$ and that $F_i(T) \geqq 0$ and $G_i(T) > 0$ for each $i \in \underline{p}$ and for all T such that (T, u, v) is a feasible solution of the dual problem under consideration.

Consider the following problem:
(DI)

$$\text{Minimize}\left(\frac{F_1(T)}{G_1(T)},\ldots,\frac{F_p(T)}{G_p(T)}\right) \tag{3.1}$$

subject to

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p}u_i\left[G_i(T)DF_i(T)-F_i(T)DG_i(T)\right]+\sum_{j=1}^{q}v_jDH_j(T)\right)\geqq 0 \quad \forall S\in\mathbb{A}^n, \tag{3.2}$$

$$\sum_{j=1}^{q}v_jH_j(T)\geqq 0, \tag{3.3}$$

$$T\in\mathbb{A}^n,\quad u\in U,\quad v\in\mathbb{R}_+^q, \tag{3.4}$$

where $\mathcal{F}(S,T;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$ is a sublinear function.

The following two theorems show that (DI) is a dual problem for (P).

Theorem 3.1 (weak duality). *Let S and (T,u,v) be arbitrary feasible solutions of (P) and (DI), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

(a) (i) *for each $i\in\underline{p}$, F_i is $(\mathcal{F},b,\bar{\phi},\bar{\rho}_i,\theta)$-univex at T, and $-G_i$ is $(\mathcal{F},b,\bar{\phi},\hat{\rho}_i,\theta)$-univex at T, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a)\geqq 0\Rightarrow a\geqq 0$;*

(ii) *for each $j\in\underline{q}$, H_j is $(\mathcal{F},b,\tilde{\phi},\tilde{\rho},\theta)$-univex at T, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0)=0$;*

(iii) *$\rho^*+\sum_{j\in J_+}v_j\tilde{\rho}_j\geqq 0$, where $\rho^*=\sum_{i=1}^{p}u_i[G_i(T)\bar{\rho}_i+F_i(T)\hat{\rho}_i]$;*

(b) (i) *for each $i\in\underline{p}$, F_i is $(\mathcal{F},b,\bar{\phi},\bar{\rho}_i,\theta)$-univex at T, and $-G_i$ is $(\mathcal{F},\bar{b},\bar{\phi},\hat{\rho}_i,\theta)$-univex at T, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a)\geqq 0\Rightarrow a\geqq 0$;*

(ii) *the function $T\to\sum_{j=1}^{q}v_jH_j(T)$ is $(\mathcal{F},b,\tilde{\phi},\tilde{\rho},\theta)$-quasiunivex at T, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0)=0$;*

(iii) *$\rho^*+\tilde{\rho}\geqq 0$;*

(c) (i) *the Lagrangian-type function*

$$T\longrightarrow\sum_{i=1}^{p}u_i\left[G_i(S)F_i(T)-F_i(S)G_i(T)\right]+\sum_{j=1}^{q}v_jH_j(T), \tag{3.5}$$

where S is fixed in $\mathbb{A}^n$, is $(\mathcal{F},b,\bar{\phi},0,\theta)$-pseudounivex at T and $\bar{\phi}(a)\geqq 0\Rightarrow a\geqq 0$. Then $\varphi(S)\nleq\xi(T,u,v)$, where $\xi=(\xi_1,\ldots,\xi_p)$ is the objective function of (DI).

Proof. (a) From (i) and (ii) it follows that

$$\bar{\phi}(F_i(S)-F_i(T))\geqq\mathcal{F}(S,T;b(S,T)DF_i(T))+\bar{\rho}_id^2(\theta(S,T)),\quad i\in\underline{p}, \tag{3.6}$$

$$\bar{\phi}(-G_i(S)+G_i(T))\geqq\mathcal{F}(S,T;-b(S,T)DG_i(T))+\hat{\rho}_id^2(\theta(S,T)),\quad i\in\underline{p}, \tag{3.7}$$

$$\tilde{\phi}(H_j(S)-H_j(T))\geqq\mathcal{F}(S,T;b(S,T)DH_j(T))+\tilde{\rho}_jd^2(\theta(S,T)),\quad j\in\underline{q}. \tag{3.8}$$

Multiplying (3.6) by $u_iG_i(T)$ and (3.7) by $u_iF_i(T)$, $i \in \underline{p}$, adding the resulting inequalities, and then using the superlinearity of $\bar{\phi}$ and sublinearity of $\mathscr{F}(S,T;\cdot)$, we obtain

$$\begin{aligned}\bar{\phi}\Bigg(\sum_{i=1}^{p}u_i\big[G_i(T)F_i(S)-F_i(T)G_i(S)\big]-\sum_{i=1}^{p}u_i\big[G_i(T)F_i(T)-F_i(T)G_i(T)\big]\Bigg)\\ \geqq \mathscr{F}\Bigg(S,T;b(S,T)\sum_{i=1}^{p}u_i\big[G_i(T)DF_i(T)-F_i(T)DG_i(T)\big]\Bigg)\\ +\sum_{i=1}^{p}u_i\big[G_i(T)\bar{\rho}_i+F_i(T)\hat{\rho}_i\big]d^2(\theta(S,T)).\end{aligned}\tag{3.9}$$

Likewise, from (3.8) we deduce that

$$\tilde{\phi}\Bigg(\sum_{j=1}^{q}v_j\big[H_j(S)-H_j(T)\big]\Bigg)\geqq \mathscr{F}\Bigg(S,T;b(S,T)\sum_{j=1}^{q}v_jDH_j(T)\Bigg)+\sum_{j=1}^{q}\tilde{\rho}_jd^2(\theta(S,T)).\tag{3.10}$$

Since $v \geqq 0$, $S \in \mathbb{F}$, and (3.3) holds, it is clear that

$$\sum_{j=1}^{q}v_j\big[H_j(S)-H_j(T)\big]\leqq 0,\tag{3.11}$$

which implies, in view of the properties of $\tilde{\phi}$, that the left-hand side of (3.10) is less than or equal to zero, that is,

$$0\geqq \mathscr{F}\Bigg(S,T;b(S,T)\sum_{j=1}^{q}v_jDH_j(T)\Bigg)+\sum_{j=1}^{q}\tilde{\rho}_jd^2(\theta(S,T)).\tag{3.12}$$

From the sublinearity of $\mathscr{F}(S,T;\cdot)$ and (3.2) it follows that

$$\begin{aligned}\mathscr{F}\Bigg(S,T;b(S,T)\sum_{i=1}^{p}u_i\big[G_i(T)DF_i(T)-F_i(T)DG_i(T)\big]\Bigg)\\ +\mathscr{F}\Bigg(S,T;b(S,T)\sum_{j=1}^{q}v_jDH_j(T)\Bigg)\geqq 0.\end{aligned}\tag{3.13}$$

Now adding (3.9) and (3.12), and then using (3.13) and (iii), we obtain

$$\bar{\phi}\Bigg(\sum_{i=1}^{p}u_i\big[G_i(T)F_i(S)-F_i(T)G_i(S)\big]-\sum_{i=1}^{p}u_i\big[G_i(T)F_i(T)-F_i(T)G_i(T)\big]\Bigg)\geqq 0.\tag{3.14}$$

But $\bar{\phi}(a)\geqq 0 \Rightarrow a \geqq 0$, and so (3.14) yields

$$\sum_{i=1}^{p}u_i\big[G_i(T)F_i(S)-F_i(T)G_i(S)\big]\geqq 0.\tag{3.15}$$

Since $u > 0$, (3.15) implies that

$$(G_1(T)F_1(S) - F_1(T)G_1(S), \dots, G_p(T)F_p(S) - F_p(T)G_p(S)) \nleq (0, \dots, 0), \tag{3.16}$$

which in turn implies that

$$\varphi(S) = \left(\frac{F_1(S)}{G_1(S)}, \dots, \frac{F_p(S)}{G_p(S)}\right) \nleq \left(\frac{F_1(T)}{G_1(T)}, \dots, \frac{F_p(T)}{G_p(T)}\right) = \xi(T, u, v). \tag{3.17}$$

(b) Since for each $j \in \underline{q}$, $v_j H_j(S) \leqq 0$, it follows from (3.3) that

$$\sum_{j=1}^{q} v_j H_j(S) \leqq 0 \leqq \sum_{j=1}^{q} v_j H_j(T), \tag{3.18}$$

and so using the properties of $\tilde{\phi}$, we obtain

$$\tilde{\phi}\left(\sum_{j=1}^{q} v_j H_j(S) - \sum_{j=1}^{q} v_j H_j(T)\right) \leqq 0, \tag{3.19}$$

which in view of (ii) implies that

$$\mathscr{F}\left(S, T; b(S, T) \sum_{j=1}^{q} v_j DH_j(T)\right) \leqq -\tilde{\rho} d^2(\theta(S, T)). \tag{3.20}$$

Now combining (3.9), (3.13), and (3.20), and using (iii), we obtain (3.15). Therefore, the rest of the proof is identical to that of part (a).

(c) From the $(\mathscr{F}, b, \bar{\phi}, 0, \theta)$-pseudounivexity assumption and (3.2) it follows that

$$\begin{aligned} \bar{\phi}\bigg(&\sum_{i=1}^{p} u_i[G_i(T)F_i(S) - F_i(T)G_i(S)] + \sum_{j=1}^{q} v_j H_j(S) \\ &- \left\{\sum_{i=1}^{p} u_i[G_i(T)F_i(T) - F_i(T)G_i(T)] + \sum_{j=1}^{q} v_j H_j(T)\right\}\bigg) \geqq 0. \end{aligned} \tag{3.21}$$

In view of the properties of $\bar{\phi}$, this inequality becomes

$$\sum_{i=1}^{p} u_i[G_i(T)F_i(S) - F_i(T)G_i(S)] + \sum_{j=1}^{q} v_j H_j(S) - \sum_{j=1}^{q} v_j H_j(T) \geqq 0, \tag{3.22}$$

which because of (3.3), primal feasibility of S, and nonnegativity of v, reduces to (3.15), and so the rest of the proof is identical to that of part (a). □

Theorem 3.2 (strong duality). *Let S^* be a regular efficient solution of (P), let $\mathscr{F}(S, S^*; DF(S^*)) = \sum_{k=1}^{n} \langle D_k F(S^*), \chi_{S_k} - \chi_{S_k^*}\rangle$ for any differentiable function $F : \mathbb{A}^n \to \mathbb{R}$ and $S \in \mathbb{A}^n$, and assume that any one of the three sets of hypotheses specified in Theorem 3.1 holds for all feasible solutions of (DI). Then there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that (S^*, u^*, v^*) is an efficient solution of (DI) and $\varphi(S^*) = \xi(S^*, u^*, v^*)$.*

Proof. By Theorem 2.9, there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that (S^*,u^*,v^*) is a feasible solution of (DI). If it were not an efficient solution, then there would exist a feasible solution $(\hat{T},\hat{u},\hat{v})$ such that $\xi(\hat{T},\hat{u},\hat{v}) \geqslant \xi(S^*,u^*,v^*) = \varphi(S^*)$, which contradicts the weak duality relation established in Theorem 5.1. Therefore, (S^*,u^*,v^*) is an efficient solution of (DI). □

We also have the following converse duality result for (P) and (DI).

Theorem 3.3 (strict converse duality). *Let S^* and $\mathcal{F}(S,S^*;\cdot)$ be as in Theorem 3.2, let $(\tilde{S},\tilde{u},\tilde{v})$ be a feasible solution of (DI) such that*

$$\sum_{i=1}^{p} \tilde{u}_i\big[G_i(\tilde{S})F_i(S^*) - F_i(\tilde{S})G_i(S^*)\big] \leqq 0. \tag{3.23}$$

Furthermore, assume that any one of the following three sets of hypotheses is satisfied:

(a) *the assumptions specified in part (a) of Theorem 3.1 are satisfied for the feasible solution $(\tilde{S},\tilde{u},\tilde{v})$ of (DI); F_i is strictly $(\mathcal{F},b,\bar{\phi},\bar{\rho}_i,\theta)$-univex at $\tilde{S}$ for at least one index $i \in \underline{p}$ with the corresponding component $\tilde{u}_i$ of $\tilde{u}$ positive, and $\bar{\phi}(a) > 0 \Rightarrow a > 0$, or $-G_i$ is strictly $(\mathcal{F},b,\bar{\phi},\hat{\rho}_i,\theta)$-univex at $\tilde{S}$ for at least one index $i \in \underline{p}$ with $\tilde{u}_i$ positive, and $\bar{\phi}(a) > 0 \Rightarrow a > 0$, or H_j is strictly $(\mathcal{F},b,\tilde{\phi},\tilde{\rho}_j,\theta)$-univex at $\tilde{S}$ for at least one index $j \in \underline{q}$ with $\tilde{v}_j$ positive, and $\tilde{\phi}(a) > 0 \Rightarrow a > 0$, or $\sum_{i=1}^{p} \tilde{u}_i[G_i(\tilde{S})\bar{\rho}_i + F_i(\tilde{S})\hat{\rho}_i] + \sum_{j=1}^{q} \tilde{v}_j\tilde{\rho}_j > 0$;*

(b) *the assumptions specified in part (b) of Theorem 3.1 are satisfied for the feasible solution $(\tilde{S},\tilde{u},\tilde{v})$ of (DI), F_i and $\bar{\phi}$ or $-G_i$ and $\bar{\phi}$ satisfy the requirements described in part (a), or the function $R \to \sum_{j=1}^{q} \tilde{v}_jH_j(R)$ is strictly $(\mathcal{F},b,\tilde{\phi},\tilde{\rho},\theta)$-pseudounivex at $\tilde{S}$, or $\sum_{i=1}^{p} \tilde{u}_i[G_i(\tilde{S})\bar{\rho}_i + F_i(\tilde{S})\hat{\rho}_i] + \tilde{\rho} > 0$;*

(c) *the assumptions specified in part (c) of Theorem 3.1 are satisfied for the feasible solution $(\tilde{S},\tilde{u},\tilde{v})$ of (DI), and the function*

$$R \longrightarrow \sum_{i=1}^{p} \tilde{u}_i\big[G_i(\tilde{S})F_i(R) - F_i(\tilde{S})G_i(R)\big] + \sum_{j=1}^{q} \tilde{v}_jH_j(R) \tag{3.24}$$

is strictly $(\mathcal{F},b,\bar{\phi},0,\theta)$-pseudounivex at $\tilde{S}$, and $\bar{\phi}(a) > 0 \Rightarrow a > 0$.

Then $\tilde{S} = S^*$, that is, $\tilde{S}$ is an efficient solution of (P).

Proof. (a) Suppose to the contrary that $\tilde{S} \neq S^*$. Proceeding as in the proof of part (a) of Theorem 5.1, we arrive at the strict inequality

$$\sum_{i=1}^{p} \tilde{u}_i\big[G_i(\tilde{S})F_i(S^*) - F_i(\tilde{S})G_i(S^*)\big] > -\sum_{i=1}^{p} \tilde{u}_i\big[G_i(\tilde{S})F_i(\tilde{S}) - F_i(\tilde{S})G_i(\tilde{S})\big] = 0, \tag{3.25}$$

in contradiction to (3.23). Hence we conclude that $\tilde{S} = S^*$.

(b) and (c) The proofs are similar to that of part (a). □

4. Duality model II

In this section, we consider a slightly different version of (DI) that allows for a greater variety of generalized $(\mathscr{F},b,\phi,\rho,\theta)$-univexity conditions under which duality can be established. This duality model has the form

(DII)

$$\text{Maximize}\left(\frac{F_1(T)}{G_1(T)},\ldots,\frac{F_p(T)}{G_p(T)}\right) \tag{4.1}$$

subject to

$$\mathscr{F}\left(S,T;\sum_{i=1}^{p}u_i\left[G_i(T)DF_i(T)-F_i(T)DG_i(T)\right]+\sum_{j=1}^{q}v_jDH_j(T)\right)\geqq 0\quad \forall S\in\mathbb{A}^n, \tag{4.2}$$

$$v_jH_j(T)\geq 0,\quad j\in\underline{q}, \tag{4.3}$$

$$T\in\mathbb{A}^n,\quad u\in U_0,\quad v\in\mathbb{R}_+^q, \tag{4.4}$$

where $\mathscr{F}(S,T;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$ is a sublinear function, and $U_0=\{u\in\mathbb{R}^q: u\geqq 0, \sum_{i=1}^{p}u_i=1\}$.

We next show that (DII) is a dual problem for (P) by establishing weak and strong duality theorems. As demonstrated below, this can be accomplished under numerous sets of generalized $(\mathscr{F},b,\phi,\rho,\theta)$-univexity conditions. Here we use the functions $f_i(\cdot,S)$, $i\in\underline{p}$, $f(\cdot,S,u)$, and $h(\cdot,v):\mathbb{A}^n\to\mathbb{R}$, which are defined, for fixed S, u, and v, as follows:

$$\begin{aligned} f_i(T,S,u)&=G_i(S)F_i(T)-F_i(S)G_i(T),\quad i\in\underline{p},\\ f(T,S,u)&=\sum_{i=1}^{p}u_i\left[G_i(S)F_i(T)-F_i(S)G_i(T)\right],\\ h(T,v)&=\sum_{j=1}^{q}v_jH_j(T). \end{aligned} \tag{4.5}$$

For given $u^*\in U_0$ and $v^*\in\mathbb{R}_+^q$, let $I_+(u^*)=\{i\in\underline{p}:u_i^*>0\}$ and $J_+(v^*)=\{j\in\underline{q}:v_j^*>0\}$.

Theorem 4.1 (weak duality). *Let S and (T,u,v), with $u>0$, be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following six sets of hypotheses is satisfied:*

(a) (i) *$f(\cdot,T,u)$ is $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-pseudounivex at T, and $\bar{\phi}(a)\geqq 0\Rightarrow a\geqq 0$;*
(ii) *for each $j\in J_+\equiv J_+(v)$, H_j is $(\mathscr{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at T, $\tilde{\phi}_j$ is increasing, and $\tilde{\phi}_j(0)=0$;*
(iii) *$\bar{\rho}+\sum_{j\in J_+}v_j\tilde{\rho}_j\geqq 0$;*

(b) (i) *$f(\cdot,T,u)$ is $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-pseudounivex at T, and $\bar{\phi}(a)\geqq 0\Rightarrow a\geqq 0$;*
(ii) *$h(\cdot,v)$ is $(\mathscr{F},b,\tilde{\phi},\tilde{\rho},\theta)$-quasiunivex at T, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0)=0$;*
(iii) *$\bar{\rho}+\tilde{\rho}\geqq 0$;*

(c) (i) $f(\cdot,T,u)$ *is prestrictly* $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$*-quasiunivex at* T*, and* $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$*;*
(ii) *for each* $j \in J_+$*,* H_j *is* $(\mathscr{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$*-quasiunivex at* T*,* $\tilde{\phi}_j$ *is increasing, and* $\tilde{\phi}_j(0) = 0$*;*
(iii) $\bar{\rho} + \sum_{j\in J_+} v_j\tilde{\rho}_j > 0$*;*

(d) (i) $f(\cdot,T,u)$ *is prestrictly* $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$*-quasiunivex at* T*, and* $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$*;*
(ii) $h(\cdot,v)$ *is* $(\mathscr{F},b,\tilde{\phi},\tilde{\rho},\theta)$*-quasiunivex at* T*,* $\tilde{\phi}$ *is increasing, and* $\tilde{\phi}(0) = 0$*;*
(iii) $\bar{\rho} + \tilde{\rho} > 0$*;*

(e) (i) $f(\cdot,T,u)$ *is prestrictly* $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$*-quasiunivex at* T*, and* $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$*;*
(ii) *for each* $j \in J_+$*,* H_j *is strictly* $(\mathscr{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$*-pseudounivex at* T*,* $\tilde{\phi}_j$ *is increasing, and* $\tilde{\phi}_j(0) = 0$*;*
(iii) $\bar{\rho} + \sum_{j\in J_+} v_j\tilde{\rho}_j \geqq 0$*;*

(f) (i) $f(\cdot,T,u)$ *is prestrictly* $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$*-quasiunivex at* T*, and* $\bar{\phi}(a) \geqq 0 \Rightarrow a \geqq 0$*;*
(ii) $h(\cdot,v)$ *is strictly* $(\mathscr{F},\tilde{b},\tilde{\phi},\tilde{\rho},\theta)$*-pseudounivex at* T*,* $\tilde{\phi}$ *is increasing, and* $\tilde{\phi}(0) = 0$*;*
(iii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then $\varphi(S) \nleq \psi(T,u,v)$*, where* $\psi = (\psi_1,\dots,\psi_p)$ *is the objective function of (DII).*

Proof. (a) From the primal feasibility of S and (4.3) it is clear that for each $j \in J_+$, $H_j(S) \leqq H_j(T)$ and so using the properties of $\tilde{\phi}_j$, we obtain $\tilde{\phi}_j(H_j(S) - H_j(T)) \leqq 0$, which by virtue of (ii) implies that for each $j \in J_+$,

$$\mathscr{F}(S,T;b(S,T)DH_j(T)) \leqq -\tilde{\rho}_j d^2(\theta(S,T)). \tag{4.6}$$

Since $v \geqq 0$, $v_j = 0$ for each $j \in \underline{q} \setminus J_+$, and $\mathscr{F}(S,T;\cdot)$ is sublinear, these inequalities can be combined as follows:

$$\mathscr{F}\left(S,T;b(S,T)\sum_{j=1}^{q} v_j DH_j(T)\right) \leqq -\sum_{j\in J_+} v_j\tilde{\rho}_j d^2(\theta(S,T)). \tag{4.7}$$

From (3.13) and (4.7) we see that

$$\begin{aligned}\mathscr{F}\Bigg(S,T;b(S,T)\sum_{i=1}^{p} u_i\big[G_i(T)DF_i(T) - F_i(T)DG_i(T)\big]\Bigg)\\ \geqq \sum_{j\in J_+} v_j\tilde{\rho}_j d^2(\theta(S,T)) \geqq -\bar{\rho} d^2(\theta(S,T)),\end{aligned} \tag{4.8}$$

where the second inequality follows from (iii). In view of (i), (4.8) implies that $\bar{\phi}(f(S,T,u) - f(T,T,u)) \geqq 0$, which because of the properties of $\bar{\phi}$, reduces to $f(S,T,u) - f(T,T,u) \geqq 0$. But $f(T,T,u) = 0$ and hence $f(S,T,u) \geqq 0$, which is precisely (3.15). Therefore, the rest of the proof is identical to that of part (a) of Theorem 3.1.

(b)–(f) The proofs are similar to that of part (a). □

Theorem 4.2 (weak duality). *Let* S *and* (T,u,v) *be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following six sets of hypotheses is satisfied:*

(a) (i) *for each* $i \in I_+ \equiv I_+(u)$*,* $f_i(\cdot,T)$ *is strictly* $(\mathscr{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$*-pseudounivex at* T*,* $\bar{\phi}_i$ *is increasing, and* $\bar{\phi}_i(0) = 0$*;*

(ii) *for each* $j \in J_+ \equiv J_+(v)$, H_j *is* $(\mathscr{F}, b, \tilde{\phi}_j, \tilde{\rho}_j, \theta)$*-quasiunivex at* T, $\tilde{\phi}_j$ *is increasing, and* $\tilde{\phi}_j(0) = 0$*;*

(iii) $\rho^\circ + \sum_{j \in J_+} v_j \tilde{\rho}_j \geqq 0$, *where* $\rho^\circ = \sum_{i \in I_+} u_i \bar{\rho}_i$*;*

(b) (i) *for each* $i \in I_+$, $f_i(\cdot, T)$ *is strictly* $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$*-pseudounivex at* T, $\bar{\phi}_i$ *is increasing, and* $\bar{\phi}_i(0) = 0$*;*

(ii) $h(\cdot, v)$ *is* $(\mathscr{F}, b, \tilde{\phi}, \tilde{\rho}, \theta)$*-quasiunivex at* T, $\tilde{\phi}$ *is increasing, and* $\tilde{\phi}(0) = 0$*;*

(iii) $\rho^\circ + \tilde{\rho} \geqq 0$*;*

(c) (i) *for each* $i \in I_+$, $f_i(\cdot, T)$ *is* $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$*-quasiunivex at* T, $\bar{\phi}_i$ *is increasing, and* $\bar{\phi}_i(0) = 0$*;*

(ii) *for each* $j \in J_+$, H_j *is strictly* $(\mathscr{F}, b, \tilde{\phi}_j, \tilde{\rho}_j, \theta)$*-pseudounivex at* T, $\tilde{\phi}_j$ *is increasing, and* $\tilde{\phi}_j(0) = 0$*;*

(iii) $\rho^\circ + \sum_{j \in J_+} v_j \tilde{\rho}_j \geqq 0$*;*

(d) (i) *for each* $i \in I_+$, $f_i(\cdot, T)$ *is* $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$*-quasiunivex at* T, $\bar{\phi}_i$ *is increasing, and* $\bar{\phi}_i(0) = 0$*;*

(ii) $h(\cdot, v)$ *is strictly* $(\mathscr{F}, b, \tilde{\phi}, \tilde{\rho}, \theta)$*-pseudounivex at* T, $\tilde{\phi}$ *is increasing, and* $\tilde{\phi}(0) = 0$*;*

(iii) $\rho^\circ + \tilde{\rho} \geqq 0$*;*

(e) (i) *for each* $i \in I_+$, $f_i(\cdot, T)$ *is* $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$*-quasiunivex at* T, $\bar{\phi}_i$ *is increasing, and* $\bar{\phi}_i(0) = 0$*;*

(ii) *for each* $j \in J_+$, H_j *is* $(\mathscr{F}, b, \tilde{\phi}_j, \tilde{\rho}_j, \theta)$*-quasiunivex at* T, $\tilde{\phi}_j$ *is increasing, and* $\tilde{\phi}_j(0) = 0$*;*

(iii) $\rho^\circ + \sum_{j \in J_+} v_j \tilde{\rho}_j > 0$*;*

(f) (i) *for each* $i \in I_+$, $f_i(\cdot, T)$ *is* $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$*-quasiunivex at* T, $\bar{\phi}_i$ *is increasing, and* $\bar{\phi}_i(0) = 0$*;*

(ii) $h(\cdot, v)$ *is* $(\mathscr{F}, b, \tilde{\phi}, \tilde{\rho}, \theta)$*-quasiunivex at* T, $\tilde{\phi}$ *is increasing, and* $\tilde{\phi}(0) = 0$*;*

(iii) $\rho^\circ + \tilde{\rho} > 0$*;*

Then $\varphi(S) \nleq \psi(T, u, v)$.

Proof. (a) Suppose to the contrary that $\varphi(S) \leqslant \psi(T, u, v)$. This implies that for each $i \in \underline{p}$, $F_i(S)/G_i(S) \leqq F_i(T)/G_i(T)$, and so $G_i(T)F_i(S) - F_i(T)G_i(S) \leqq 0$, with strict inequality holding for at least one index $\ell \in \underline{p}$. Hence for each $i \in \underline{p}$,

$$G_i(T)F_i(S) - F_i(T)G_i(S) \leqq 0 = G_i(T)F_i(T) - F_i(T)G_i(T), \tag{4.9}$$

which in view of the properties of $\bar{\phi}_i$ can be expressed as $\bar{\phi}_i(f_i(S, T) - f_i(T, T)) \leqq 0$. By virtue of (i), these inequalities imply that for each $i \in I_+$,

$$\mathscr{F}\left(S, T; b(S, T)\left[G_i(T)DF_i(T) - F_i(T)DG_i(T)\right]\right) < -\bar{\rho}_i d^2\left(\theta(S, T)\right). \tag{4.10}$$

Inasmuch as $u \geqq 0$, $u_i = 0$ for each $i \in \underline{p} \setminus I_+$, $\sum_{i \in I_+} u_i = 1$, and $\mathscr{F}(S, T; \cdot)$ is sublinear, these inequalities yield

$$\mathscr{F}\left(S, T; b(S, T)\sum_{i=1}^{p} u_i\left[G_i(T)DF_i(T) - F_i(T)DG_i(T)\right]\right) < -\sum_{i \in I_+} u_i \bar{\rho}_i d^2\left(\theta(S, T)\right). \tag{4.11}$$

From (3.13), (4.11), and (iii) we deduce that

$$\mathscr{F}\left(S,T;b(S,T)\sum_{j=1}^{q}v_jDH_j(T)\right)>\rho^\circ d^2(\theta(S,T))\geqq-\sum_{j\in J_+}v_j\tilde{\rho}_jd^2(\theta(S,T)). \tag{4.12}$$

But this contradicts (4.7), which is valid for the present case because of our hypotheses set forth in (ii). Hence $\varphi(S)\nleq\psi(T,u,v)$.

(b)–(f) The proofs are similar to that of part (a). □

The next theorem may be viewed as a variant of Theorem 4.2; its proof is similar to that of Theorem 4.2 and hence omitted.

Theorem 4.3 (weak duality). *Let S and (T,u,v) be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following four sets of hypotheses is satisfied:*

(a) (i) *for each $i\in I_{1+}\neq\emptyset$, $f_i(\cdot,T)$ is strictly $(\mathscr{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at T, for each $i\in I_{2+}$, $f_i(\cdot,T)$ is $(\mathscr{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at T, and for each $i\in I_+\equiv I_+(u)$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0)=0$, where $\{I_{1+},I_{2+}\}$ is a partition of I_+;*

(ii) *for each $j\in J_+\equiv J_+(v)$, H_j is $(\mathscr{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at T, $\tilde{\phi}_j$ is increasing and $\tilde{\phi}_j(0)=0$;*

(iii) *$\rho^\circ+\sum_{j\in J_+}v_j\tilde{\rho}_j\geqq0$, where $\rho^\circ=\sum_{i\in I_+}u_i\bar{\rho}_i$;*

(b) (i) *for each $i\in I_{1+}\neq\emptyset$, $f_i(\cdot,T)$ is strictly $(\mathscr{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at T, for each $i\in I_{2+}$, $f_i(\cdot,T)$ is $(\mathscr{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at T, and for each $i\in I_+$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0)=0$, where $\{I_{1+},I_{2+}\}$ is a partition of I_+;*

(ii) *$h(\cdot,v)$ is $(\mathscr{F},b,\tilde{\phi},\tilde{\rho},\theta)$-quasiunivex at T, $\tilde{\phi}$ is increasing, and $\tilde{\phi}(0)=0$;*

(iii) *$\rho^\circ+\tilde{\rho}\geqq0$;*

(c) (i) *for each $i\in I_+$, $f_i(\cdot,T)$ is $(\mathscr{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at T, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *for each $j\in J_{1+}\neq\emptyset$, H_j is strictly $(\mathscr{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-pseudounivex at T, for each $j\in J_{2+}$, H_j is $(\mathscr{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at T, and for each $j\in J_+$, $\tilde{\phi}_j$ is increasing and $\tilde{\phi}_j(0)=0$, where $\{J_{1+},J_{2+}\}$ is a partition of J_+;*

(iii) *$\rho^\circ+\sum_{j\in J_+}v_j\tilde{\rho}_j\geqq0$;*

(d) (i) *for each $i\in I_{1+}$, $f_i(\cdot,T)$ is strictly $(\mathscr{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at T, for each $i\in I_{2+}$, $f_i(\cdot,T)$ is $(\mathscr{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at T, and for each $i\in I_+$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0)=0$, where $\{I_{1+},I_{2+}\}$ is a partition of I_+;*

(ii) *for each $j\in J_{1+}$, H_j is strictly $(\mathscr{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-pseudounivex at T, for each $j\in J_{2+}$, H_j is $(\mathscr{F},b,\tilde{\phi}_j,\tilde{\rho}_j,\theta)$-quasiunivex at T, and for each $j\in J_+$, $\tilde{\phi}_j$ is increasing and $\tilde{\phi}_j(0)=0$, where $\{J_{1+},J_{2+}\}$ is a partition of J_+;*

(iii) *$\rho^\circ+\sum_{j\in J_+}v_j\tilde{\rho}_j\geqq0$;*

(iv) *$I_{1+}\neq\emptyset$, $J_{1+}\neq\emptyset$, or $\rho^\circ+\sum_{j\in J_+}v_j\tilde{\rho}_j>0$.*

Then $\varphi(x)\nleq\psi(T,u,v)$.

Theorem 4.4 (strong duality). *Let S^* be a regular efficient solution of (P), let $\mathscr{F}(S,S^*;DF(S^*))=\sum_{k=1}^{n}\langle D_kF(S^*),\chi_{S_k}-\chi_{S_k^*}\rangle$ for any differentiable function $F:\mathbb{A}^n\to\mathbb{R}$ and $S\in\mathbb{A}^n$,* *and assume that any one of the sixteen sets of hypotheses specified in Theorems 4.1, 4.2, and 4.3 holds for all feasible solutions of (DII). Then there exist $u^* \in U$ and $v^* \in \mathbb{R}^q_+$ such that (S^*, u^*, v^*) is an efficient solution of (DII) and $\phi(S^*) = \psi(S^*, u^*, v^*)$.*

Proof. By Theorem 2.9, there exist $u^* \in U$ and $v^* \in \mathbb{R}^q_+$ such that (S^*, u^*, v^*) is a feasible solution of (DII) and $\varphi(S^*) = \psi(S^*, u^*, v^*)$. That (S^*, u^*, v^*) is efficient for (DII) follows from (the corresponding part of) Theorems 4.1, 4.2, and 4.3. □

The proofs of the next two theorems are similar to that of Theorem 3.3.

Theorem 4.5 (strict converse duality). *Let S^* and $\mathcal{F}(S, S^*; \cdot)$ be as in Theorem 4.4, let $(\tilde{S}, \tilde{u}, \tilde{v})$ be a feasible solution of (DII) such that $f(S^*, \tilde{S}, \tilde{u}) \leqq 0$, and assume that either one of the two sets of hypotheses specified in parts (a) and (b) of Theorem 4.1 is satisfied for the feasible solution $(\tilde{S}, \tilde{u}, \tilde{v})$ of (DII). Assume, furthermore, that $f(\cdot, \tilde{S}, \tilde{u})$ is strictly $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-pseudounivex at $\tilde{S}$ and that $\bar{\phi}(a) > 0 \Rightarrow a > 0$. Then $\tilde{S} = S^*$, that is, $\tilde{S}$ is an efficient solution of (P).*

Theorem 4.6 (strict converse duality). *Let S^* and $\mathcal{F}(S, S^*; \cdot)$ be as in Theorem 4.4, let $(\tilde{S}, \tilde{u}, \tilde{v})$ be a feasible solution of (DII) such that $f(S^*, \tilde{S}, \tilde{u}) \leqq 0$, and assume that any one of the four sets of hypotheses specified in parts (c)–(f) of Theorem 4.1 is satisfied for the feasible solution $(\tilde{S}, \tilde{u}, \tilde{v})$ of (DII). Assume, furthermore, that $f(\cdot, \tilde{S}, \tilde{u})$ is $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-quasiunivex at $\tilde{S}$ and that $\bar{\phi}(a) > 0 \Rightarrow a > 0$. Then $\tilde{S} = S^*$, that is, $\tilde{S}$ is an efficient solution of (P).*

5. Duality model III

In this section, we formulate a more general duality model for (P) with a more flexible structure which will allow us to establish duality under various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity hypotheses that can be imposed on certain combinations of the problem functions. This will be accomplished by utilizing a partitioning scheme which was originally proposed in [2] for the purpose of constructing generalized dual problems for nonlinear programs with point-functions. Prior to formulating this duality model, we need to introduce some additional notation.

Let $\{J_0, J_1, \ldots, J_m\}$ be a partition of the index set $\underline{q}$; thus $J_r \subset \underline{q}$ for each $r \in \{0, 1, \ldots, m\}$, $J_r \cap J_s = \emptyset$ for each $r, s \in \{0, 1, \ldots, m\}$ with $r \neq s$, and $\cup_{r=0}^m J_r = \underline{q}$. In addition, we will make use of the real-valued functions $\Pi_i(\cdot, T, v)$, $\Pi(\cdot, T, u, v)$, and $\Lambda_t(\cdot, v)$ defined, for fixed T, u, and v, on $\mathbb{A}^n$ by

$$\Pi_i(R, T, v) = G_i(T)\left[F_i(R) + \sum_{j \in J_0} v_j H_j(R)\right] - \left[F_i(T) + \Lambda_0(T, v)\right] G_i(R), \quad i \in \underline{p},$$

$$\Pi(R, T, u, v) = \sum_{i=1}^{p} u_i \left\{ G_i(T)\left[F_i(R) + \sum_{j \in J_0} v_j H_j(R)\right] - \left[F_i(T) + \Lambda_0(T, v)\right] G_i(R) \right\}, \tag{5.1}$$

$$\Lambda_t(R, v) = \sum_{j \in J_t} v_j H_j(R), \quad t \in \underline{m} \cup \{0\}.$$

Consider the following problem:
$(DIII)$

$$\text{Maximize}\left(\frac{F_1(T)+\sum_{j\in J_0}v_jH_j(T)}{G_1(T)},\dots,\frac{F_p(T)+\sum_{j\in J_0}v_jH_j(T)}{G_p(T)}\right) \tag{5.2}$$

subject to

$$\begin{aligned}\mathscr{F}\Bigg(S,T;\sum_{i=1}^{p}u_i\Bigg\{G_i(T)\Bigg[DF_i(T)+\sum_{j\in J_0}v_jDH_j(T)\Bigg]-\big[F_i(T)+\Lambda_0(T,v)\big]DG_i(T)\Bigg\}\\ +\sum_{j\in \underline{q}\setminus J_0}v_jDH_j(T)\Bigg)\geqq 0\quad \forall S\in\mathbb{A}^n,\end{aligned} \tag{5.3}$$

$$\sum_{j\in J_t}v_jH_j(T)\geqq 0,\quad t\in\underline{m}, \tag{5.4}$$

$$T\in\mathbb{A}^n,\quad u\in U_0,\quad v\in\mathbb{R}^q_+, \tag{5.5}$$

where $\mathscr{F}(S,T;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$ is a sublinear function.

We next show that $(DIII)$ is a dual problem for (P) by proving weak and strong duality theorems.

Theorem 5.1 (weak duality). *Let S and (T,u,v), with $u>0$, be arbitrary feasible solutions of (P) and $(DIII)$, respectively, and assume that any one of the following four sets of hypotheses is satisfied:*

(a) (i) $\Pi(\cdot,T,u,v)$ *is* $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$ *-pseudounivex at T, and* $\bar{\phi}(a)\geqq 0\Rightarrow a\geqq 0$;

(ii) *for each $t\in\underline{m}$, $\Lambda_t(\cdot,v)$ is $(\mathscr{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at T, $\tilde{\phi}_t$ is increasing, and* $\tilde{\phi}_t(0)=0$;

(iii) $\bar{\rho}+\sum_{t=1}^m\tilde{\rho}_t\geqq 0$;

(b) (i) $\Pi(\cdot,T,u,v)$ *is prestrictly* $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$*-quasiunivex at T, and* $\bar{\phi}(a)\geqq 0\Rightarrow a\geqq 0$;

(ii) *for each $t\in\underline{m}$, $\Lambda_t(\cdot,v)$ is strictly $(\mathscr{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-pseudounivex at T, $\tilde{\phi}_t$ is increasing, and* $\tilde{\phi}_t(0)=0$;

(iii) $\bar{\rho}+\sum_{t=1}^m\tilde{\rho}_t\geqq 0$;

(c) (i) $\Pi(\cdot,T,u,v)$ *is prestrictly* $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$*-quasiunivex at T, and* $\bar{\phi}(a)\geqq 0\Rightarrow a\geqq 0$;

(ii) *for each $t\in\underline{m}$, $\Lambda_t(\cdot,v)$ is $(\mathscr{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at T, $\tilde{\phi}_t$ is increasing, and* $\tilde{\phi}_t(0)=0$;

(iii) $\bar{\rho}+\sum_{t=1}^m\tilde{\rho}_t>0$;

(d) (i) $\Pi(\cdot,T,u,v)$ *is prestrictly* $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$*-quasiunivex at T, and* $\bar{\phi}(a)\geqq 0\Rightarrow a\geqq 0$;

(ii) *for each $t\in\underline{m_1}$, $\Lambda_t(\cdot,v)$ is $(\mathscr{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at T, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0)=0$, for each $t\in\underline{m_2}\neq\emptyset$, $\Lambda_t(\cdot,v)$ is strictly $(\mathscr{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-pseudounivex at T, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0)=0$, where $\{\underline{m_1},\underline{m_2}\}$ is a partition of $\underline{m}$;*

(iii) $\bar{\rho}+\sum_{t=1}^m\tilde{\rho}_t\geqq 0$.

Then $\varphi(S)\nleq\omega(T,u,v)$, where $\omega=(\omega_1,\dots,\omega_p)$ is the objective function of $(DIII)$.

Proof. (a) From the sublinearity of $\mathcal{F}(S,T;\cdot)$ and (5.3) it follows that

$$\mathcal{F}\left(S,T;b(S,T)\sum_{i=1}^{p}u_i\left\{G_i(T)\left[DF_i(T)+\sum_{j\in J_0}v_jH_j(T)\right]-[F_i(T)+\Lambda_0(T,v)]DG_i(T)\right\}\right)$$
$$+\mathcal{F}\left(S,T;b(S,T)\sum_{t=1}^{m}\sum_{j\in J_t}v_jDH_j(T)\right)\geqq 0. \tag{5.6}$$

Since $S\in\mathbb{F}$ and $v\geqq 0$, it is clear from (5.4) that for each $t\in\underline{m}$,

$$\Lambda_t(S,v)=\sum_{j\in J_t}v_jH_j(S)\leqq\sum_{j\in J_t}v_jH_j(T)=\Lambda_t(T,v), \tag{5.7}$$

and so using the properties of $\tilde{\phi}_t$, we get $\phi_t(\Lambda_t(S,v)-\Lambda_t(T,v))\leqq 0$, which in view of (ii) implies that for each $t\in\underline{m}$,

$$\mathcal{F}\left(S,T;b(S,T)\sum_{j\in J_t}v_jDH_j(T)\right)\leqq-\tilde{\rho}_t d^2(\theta(S,T)). \tag{5.8}$$

Adding these inequalities and using the sublinearity of $\mathcal{F}(S,T;\cdot)$, we obtain

$$\mathcal{F}\left(S,T;b(S,T)\sum_{t=1}^{m}\sum_{j\in J_t}v_jDH_j(T)\right)\leqq-\sum_{t=1}^{m}\tilde{\rho}_t d^2(\theta(S,T)). \tag{5.9}$$

From (5.6) and (5.9) we deduce that

$$\mathcal{F}\left(S,T;b(S,T)\sum_{i=1}^{p}u_i\left\{G_i(T)\left[DF_i(T)+\sum_{j\in J_0}v_jH_j(T)\right]-[F_i(T)+\Lambda_0(T,v)]DG_i(T)\right\}\right)$$
$$\geqq\sum_{t=1}^{m}\tilde{\rho}_t d^2(\theta(S,T))\geqq-\bar{\rho}d^2(\theta(S,T)), \tag{5.10}$$

where the second inequality follows from (iii). Because of (i), this inequality implies that $\bar{\phi}(\Pi(S,T,u,v)-\Pi(T,T,u,v))\geqq 0$. But $\bar{\phi}(a)\geq 0\Rightarrow a\geq 0$, and so we get $\Pi(S,T,u,v)\geqq\Pi(T,T,u,v)=0$. Inasmuch as $S\in\mathbb{F}$, $u>0$, $v\geqq 0$, and $G_i(T)>0$, $i\in\underline{p}$, this inequality yields

$$\sum_{i=1}^{p}u_i\{G_i(T)F_i(S)-[F_i(T)+\Lambda_0(T,v)]G_i(S)\}\geqq 0. \tag{5.11}$$

Since $u>0$, this inequality implies that

$$\begin{aligned}&(G_1(T)F_1(S)-[F_1(T)+\Lambda_0(T,v)]G_1(S),\dots,G_p(T)F_p(S)\\&\quad-[F_p(T)+\Lambda_0(T,v)]G_p(S))\nleq(0,\dots,0),\end{aligned} \tag{5.12}$$

which in turn implies that

$$\varphi(S)=\left(\frac{F_1(S)}{G_1(S)},\dots,\frac{F_p(S)}{G_p(S)}\right)\nleq\left(\frac{F_1(T)+\Lambda_0(T,v)}{G_1(T)},\dots,\frac{F_p(T)+\Lambda_0(T,v)}{G_p(T)}\right)=\omega(T,u,v).\tag{5.13}$$

(b)–(d) The proofs are similar to that of part (a). □

Theorem 5.2 (weak duality). *Let S and (T,u,v) be arbitrary feasible solutions of (P) and (DIII), respectively, and assume that any one of the following six sets of hypotheses is satisfied:*

(a) (i) *for each $i\in I_+\equiv I_+(u)$, $\Pi_i(\cdot,T,v,)$ is strictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at T, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *for each $t\in \underline{m}$, $\Lambda_t(\cdot,v)$ is $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at T, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0)=0$;*

(iii) $\sum_{i\in I_+}u_i\bar{\rho}_i+\sum_{t=1}^m\tilde{\rho}_t\geqq 0$;

(b) (i) *for each $i\in I_+$, $\Pi_i(\cdot,T,v)$ is prestrictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at T, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *for each $t\in \underline{m}$, $\Lambda_t(\cdot,v)$ is strictly $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-pseudounivex at T, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0)=0$;*

(iii) $\sum_{i\in I_+}u_i\bar{\rho}_i+\sum_{t=1}^m\tilde{\rho}_t\geqq 0$;

(c) (i) *for each $i\in I_+$, $\Pi_i(\cdot,T,v)$ is prestrictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at T, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *for each $t\in \underline{m}$, $\Lambda_t(\cdot,v)$ is $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at T, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0)=0$;*

(iii) $\sum_{i\in I_+}u_i\bar{\rho}_i+\sum_{t=1}^m\tilde{\rho}_t> 0$;

(d) (i) *for each $i\in I_{1+}\neq\emptyset$, $\Pi_i(\cdot,T,v)$ is strictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at T, for each $i\in I_{2+}$, $\Pi_i(\cdot,T,v)$ is prestrictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at T, and for each $i\in I_+$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0)=0$, where $\{I_{1+},I_{2+}\}$ is a partition of I_+;*

(ii) *for each $t\in \underline{m}$, $\Lambda_t(\cdot,v)$ is $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at T, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0)=0$;*

(iii) $\sum_{i\in I_+}u_i\bar{\rho}_i+\sum_{t=1}^m\tilde{\rho}_t\geqq 0$;

(e) (i) *for each $i\in I_+$, $\Pi_i(\cdot,T,v)$ is prestrictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at T, $\bar{\phi}_i$ is increasing, and $\bar{\phi}_i(0)=0$;*

(ii) *for each $t\in \underline{m_1}\neq\emptyset$, $\Lambda_t(\cdot,v)$ is strictly $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-pseudounivex at T, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0)=0$, and for each $t\in\underline{m_2}$, $\Lambda_t(\cdot,v)$ is $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at T, and for each $t\in\underline{m}$, $\tilde{\phi}_t$ is increasing and $\tilde{\phi}_t(0)=0$, where $\{\underline{m_1},\underline{m_2}\}$ is a partition of $\underline{m}$;*

(iii) $\sum_{i\in I_+}u_i\bar{\rho}_i+\sum_{t=1}^m\tilde{\rho}_t\geqq 0$;

(f) (i) *for each $i\in I_{1+}$, $\Pi_i(\cdot,T,v)$ is strictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at T, for each $i\in I_{2+}$, $\Pi_i(\cdot,T,v)$ is prestrictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-quasiunivex at T, and for each $i\in I_+$, $\bar{\phi}_i$ is increasing and $\bar{\phi}_i(0)=0$, where $\{I_{1+},I_{2+}\}$ is a partition of I_+;*

(ii) *for each* $t \in \underline{m_1}$, $\Lambda_t(\cdot,v)$ *is strictly* $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$*-pseudounivex at* T, *for each* $t \in \underline{m_2}$, $\Lambda_t(\cdot,v)$ *is* $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$*-quasiunivex at* T, *and for each* $t \in \underline{m}$, $\tilde{\phi}_t$ *is increasing and* $\tilde{\phi}_t(0) = 0$, *where* $\{\underline{m_1},\underline{m_2}\}$ *is a partition of* $\underline{m}$;

(iii) $\sum_{i\in I_+} u_i\bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geqq 0$;

(iv) $I_{1+} \neq \emptyset$, $\underline{m_1} \neq \emptyset$, $\sum_{i\in I_+} u_i\bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t > 0$.

Then $\varphi(S) \nleq \omega(T,u,v)$.

Proof. (a) Suppose to the contrary that $\varphi(S) \leqslant \omega(T,u,v)$. This implies that for each $i \in \underline{p}$, $G_i(T)F_i(S) - [F_i(T) + \Lambda_0(T,v)]G_i(S) \leqq 0$, with strict inequality holding for at least one index $\ell \in \underline{p}$. Using these inequalities along with the primal feasibility of S and nonnegativity of v, we see that

$$\begin{aligned}\Pi_i(S,T,v) &= G_i(T)\left[F_i(S) + \sum_{j\in J_0} v_j H_j(S)\right] - [F_i(T) + \Lambda_0(T,v)]G_i(S)\\ &\leqq G_i(T)F_i(S) - [F_i(T) + \Lambda_0(T,v)]G_i(S) \leqq 0\\ &= \Pi_i(T,T,v).\end{aligned} \tag{5.14}$$

It follows from the properties of $\bar{\phi}_i$ that for each $i \in \underline{p}$, $\bar{\phi}_i(\Pi_i(S,T,u,v) - \Pi_i(T,T,u,v)) \leqq 0$, which by (i) implies that for each $i \in \underline{p}$,

$$\begin{aligned}&\mathcal{F}\Bigg(S,T;b(S,T)\bigg\{G_i(T)\bigg[DF_i(T) + \sum_{j\in J_0} v_j DH_j(T)\bigg]\\ &\qquad - [F_i(T) + \Lambda_0(T,v)]DG_i(T)\bigg\}\Bigg) < -\bar{\rho}_i d^2(\theta(S,T)).\end{aligned} \tag{5.15}$$

Because $u \geqq 0$, $u \neq 0$, and $\mathcal{F}(S,T;\cdot)$ is sublinear, these inequalities yield

$$\begin{aligned}&\mathcal{F}\Bigg(S,T;b(S,T)\sum_{i=1}^{p} u_i\bigg\{G_i(T)\bigg[DF_i(T) + \sum_{j\in J_0} v_j DH_j(T)\bigg]\\ &\qquad - [F_i(T) + \Lambda_0(T,v)]DG_i(T)\bigg\}\Bigg) < -\sum_{i=1}^{p} u_i\bar{\rho}_i d^2(\theta(S,T)).\end{aligned} \tag{5.16}$$

Comparing this inequality with (5.6), we observe that

$$\mathcal{F}\left(S,T;\sum_{t=1}^{m}\sum_{j\in J_t} v_j DH_j(T)\right) > \sum_{i=1}^{p} u_i\bar{\rho}_i d^2(\theta(S,T)) \geqq -\sum_{t=1}^{m} \tilde{\rho}_t d^2(\theta(S,T)), \tag{5.17}$$

where the second inequality follows from (iii). Obviously, this inequality contradicts (5.9), which is valid for the present case because of (ii). Hence we must have $\varphi(S) \nleq \omega(T,u,v)$.

(b)–(f) The proofs are similar to that of part (a). □

Theorem 5.3 (strong duality). *Let S^* be a regular efficient solution of (P), let $\mathscr{F}(S,S^*;DF(S^*)) = \sum_{k=1}^{n}\langle D_kF(S^*),\chi_{S_k}-\chi_{S_k^*}\rangle$ for any differentiable function $F:\mathbb{A}^n\to\mathbb{R}$ and $S\in\mathbb{A}^n$, and assume that any one of the ten sets of hypotheses specified in Theorems 5.1 and 5.2 holds for all feasible solutions of (DIII). Then there exist $u^*\in U$ and $v^*\in\mathbb{R}_+^q$ such that (S^*,u^*,v^*) is an efficient solution of (DIII) and $\varphi(S^*)=\omega(S^*,u^*,v^*)$.*

Proof. By Theorem 2.9, there exist $u^*\in U$ and $\bar{v}\in\mathbb{R}_+^q$ such that

$$\left\langle \sum_{i=1}^{p} u_i^*\left[G_i(S^*)D_kF_i(S^*)-F_i(S^*)D_kG_i(S^*)\right] + \sum_{j=1}^{q}\bar{v}_jD_kH_j(S^*),\chi_{S_k}-\chi_{S_k^*}\right\rangle \geqq 0 \quad \forall S_k\in\mathbb{A},\ k\in\underline{n}, \tag{5.18}$$

$$\bar{v}_jH_j(S^*)=0,\quad j\in\underline{q}. \tag{5.19}$$

Now if we let $v_j^*=\bar{v}_j/G_i(S^*)$ for each $j\in J_0$, $v_j^*=\bar{v}_j$ for each $j\in\underline{q}\setminus J_0$, and observe that $\Lambda_0(S^*,v^*)=0$, then (5.18) and (5.19) can be rewritten as follows:

$$\left\langle \sum_{i=1}^{p} u_i^*\left\{G_i(S^*)\left[D_kF_i(S^*)+\sum_{j\in J_0}v_j^*D_kH_j(S^*)\right]-\left[F_i(S^*)+\Lambda_0(S^*,v^*)\right]D_kG_i(S^*)\right\} + \sum_{j\in\underline{q}\setminus J_0}v_j^*D_kH_j(S^*),\chi_{S_k}-\chi_{S_k^*}\right\rangle \geqq 0 \quad \forall S_k\in\mathbb{A},\ k\in\underline{n}, \tag{5.20}$$

$$\sum_{j\in J_t}v_j^*H_j(S^*)=0,\quad t\in\underline{m}. \tag{5.21}$$

From (5.20) and (5.21) it is clear that (S^*,u^*,v^*) is a feasible solution of $(DIII)$. Proceeding as in the proof of Theorem 3.3, it can easily be verified that it is an efficient solution of $(DIII)$. □

We next show that certain modifications in Theorem 5.1 lead to a number of strict converse duality results for (P) and $(DIII)$.

Theorem 5.4 (strict converse duality). *Let S^* and $\mathscr{F}$ be as in Theorem 5.3, let $(\tilde{S},\tilde{u},\tilde{v})$ be a feasible solution of (DIII) such that*

$$\sum_{i=1}^{p}\tilde{u}_i\{G_i(\tilde{S})F_i(S^*)-[F_i(\tilde{S})+\Lambda_0(\tilde{S},\tilde{v})]G_i(S^*)\}\leqq 0, \tag{5.22}$$

and assume that any one of the four sets of hypotheses specified in Theorem 5.1 is satisfied for the feasible solution $(\tilde{S},\tilde{u},\tilde{v})$ of (DIII). Assume furthermore that any one of the following corresponding conditions holds:

(a) *$\Pi(\cdot,\tilde{S},\tilde{u},\tilde{v})$ is strictly $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-pseudounivex at $\tilde{S}$, and $\bar{\phi}(a)>0\Rightarrow a>0$;*

(b) *$\Pi(\cdot,\tilde{S},\tilde{u},\tilde{v})$ is $(\mathscr{F},b,\bar{\phi},\bar{\rho},\theta)$-quasiunivex at $\tilde{S}$, and $\bar{\phi}(a)>0\Rightarrow a>0$;*

(c) $\Pi(\cdot,\widetilde{S},\widetilde{u},\widetilde{v})$ *is* $(\mathcal{F},b,\bar{\phi},\bar{\rho},\theta)$*-quasiunivex at* $\widetilde{S}$*, and* $\bar{\phi}(a)>0\Rightarrow a>0$*;*
(d) $\Pi(\cdot,\widetilde{S},\widetilde{u},\widetilde{v})$ *is* $(\mathcal{F},b,\bar{\phi},\bar{\rho},\theta)$*-quasiunivex at* $\widetilde{S}$*, and* $\bar{\phi}(a)>0\Rightarrow a>0$.
Then $\widetilde{S}=S^*$.

Proof. The proof is similar to that of Theorem 3.3. □

Evidently, $(DIII)$ contains a number of important special cases which can easily be identified by appropriate choices of the partitioning sets $J_0,J_1,\ldots,J_m$, and the sublinear function $\mathcal{F}(S,T;\cdot)$. We conclude this section by briefly looking at a few of these special cases. In each case, we specify the required conditions for duality by specializing part (a) of Theorem 5.2.

If we let $J_0=\underline{q}$, then $(DIII)$ takes the following form:
$(DIIIa)$

$$\text{Maximize}\left(\frac{F_1(T)+\sum_{j=1}^{q}v_jH_j(T)}{G_1(T)},\ldots,\frac{F_p(T)+\sum_{j=1}^{q}v_jH_j(T)}{G_p(T)}\right) \tag{5.23}$$

subject to (5.5) and

$$\begin{aligned}\mathcal{F}\Bigg(S,T;\sum_{i=1}^{p}u_i\Bigg\{G_i(T)\Bigg[DF_i(T)+\sum_{j=1}^{q}v_jDH_j(T)\Bigg]\\ -\Bigg[F_i(T)+\sum_{j=1}^{q}v_jH_j(T)\Bigg]DG_i(T)\Bigg\}\Bigg)\geqq 0\quad \forall S\in\mathbb{A}^n.\end{aligned} \tag{5.24}$$

$(DIIIa)$ is dual to (P) if for each $i\in I_+$, the function

$$R\longrightarrow G_i(T)\Bigg[F_i(R)+\sum_{j=1}^{q}v_jH_j(R)\Bigg]-\Bigg[F_i(T)+\sum_{j=1}^{q}v_jH_j(T)\Bigg]G_i(R) \tag{5.25}$$

is strictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at T, $\bar{\phi}_i$ is increasing, $\bar{\phi}_i(0)=0$, and $\sum_{i\in I_+}u_i\bar{\rho}_i\geqq 0$.

If we let $J_1=\underline{q}$, then $(DIII)$ becomes
$(DIIIb)$

$$\text{Maximize}\left(\frac{F_1(T)}{G_1(T)},\ldots,\frac{F_p(T)}{G_p(T)}\right) \tag{5.26}$$

subject to (5.5) and

$$\begin{aligned}\mathcal{F}\Bigg(S,T;\sum_{i=1}^{p}u_i\big[G_i(T)DF_i(T)-F_i(T)DG_i(T)\big]+\sum_{j=1}^{q}v_jDH_j(T)\Bigg)\geqq 0\quad\forall S\in\mathbb{A}^n,\\ \sum_{j=1}^{q}v_jH_j(T)\geqq 0.\end{aligned} \tag{5.27}$$

$(DIIIb)$ is dual to (P) if for each $i \in I_+$, the function $R \to G_i(T)F_i(R) - F_i(T)G_i(R)$ is strictly $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at T, $\bar{\phi}_i$ is increasing, $\bar{\phi}_i(0) = 0$, the function $R \to \sum_{j=1}^{q} v_j H_j(R)$ is $(\mathscr{F}, b, \tilde{\phi}, \tilde{\rho}, \theta)$-quasiunivex at T, $\tilde{\phi}$ is increasing, $\tilde{\phi}(0) = 0$, and $\sum_{i\in I_+} u_i \bar{\rho}_i + \tilde{\rho} \geqq 0$.

If we choose $J_0 = \emptyset$, then we obtain the following special case of $(DIII)$:
$(DIIIc)$

$$\text{Maximize } \left(\frac{F_1(T)}{G_1(T)}, \ldots, \frac{F_p(T)}{G_p(T)} \right) \tag{5.28}$$

subject to (5.5) and

$$\begin{gathered} \mathscr{F}\left(S, T; \sum_{i=1}^{p} u_i \left[G_i(T) DF_i(T) - F_i(T) DG_i(T) \right] + \sum_{j=1}^{q} v_j DH_j(T) \right) \geqq 0 \quad \forall S \in \mathbb{A}^n, \\ \sum_{j\in J_t} v_j H_j(T) \geqq 0, \quad t \in \underline{m}. \end{gathered} \tag{5.29}$$

$(DIIIc)$ is dual to (P) if for each $i \in I_+$, the function $R \to G_i(T)F_i(R) - F_i(T)G_i(R)$ is strictly $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at T, $\bar{\phi}_i$ is increasing, $\bar{\phi}_i(0) = 0$, for each $t \in \underline{m}$, $\Lambda_t(\cdot, v)$ is $(\mathscr{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasiunivex at T, $\tilde{\phi}_t$ is increasing, $\tilde{\phi}_t(0) = 0$, and $\sum_{i\in I_+} u_i \bar{\rho}_i + \sum_{t=1}^{m} \tilde{\rho}_t \geqq 0$.

If we set $m = q$, $J_t = \{t\}$, $t \in \underline{q}$, then $(DIII)$ reduces to the following problem:
$(DIIId)$

$$\text{Maximize } \left(\frac{F_1(T)}{G_1(T)}, \ldots, \frac{F_p(T)}{G_p(T)} \right) \tag{5.30}$$

subject to (5.5) and

$$\begin{gathered} \mathscr{F}\left(S, T; \sum_{i=1}^{p} u_i \left[G_i(T) DF_i(T) - F_i(T) DG_i(T) \right] + \sum_{j=1}^{q} v_j DH_j(T) \right) \geqq 0 \quad \forall S \in \mathbb{A}^n, \\ v_j H_j(T) \geqq 0, \quad j \in \underline{q}. \end{gathered} \tag{5.31}$$

$(DIIId)$ is dual to (P) if for each $i \in I_+$, the function $R \to G_i(T)F_i(R) - F_i(T)G_i(R)$ is strictly $(\mathscr{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at T, $\bar{\phi}_i$ is increasing, $\bar{\phi}_i(0) = 0$, for each $j \in \underline{q}$, $R \to v_j H_j(R)$ is $(\mathscr{F}, b, \tilde{\phi}_j, \tilde{\rho}_j, \theta)$-quasiunivex at T, $\tilde{\phi}_j$ is increasing, $\tilde{\phi}_j(0) = 0$, and $\sum_{i\in I_+} u_i \bar{\rho}_i + \sum_{j=1}^{q} \tilde{\rho}_j \geqq 0$.

If we let $J_t = \emptyset$, $t = 2, 3, \ldots, m$, then we get the following special case of $(DIII)$:
$(DIIIe)$

$$\text{Maximize } \left(\frac{F_1(T) + \sum_{j\in J_0} v_j H_j(T)}{G_1(T)}, \ldots, \frac{F_p(T) + \sum_{j\in J_0} v_j H_j(T)}{G_p(T)} \right) \tag{5.32}$$

subject to (5.5) and

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p}u_i\left\{G_i(T)\left[DF_i(T)+\sum_{j\in J_0}v_jDH_j(T)\right]\right.\right.$$
$$\left.\left.-\left[F_i(T)+\Lambda_0(T,v)\right]DG_i(T)\right\}+\sum_{j\in J_1}v_jDH_j(T)\right)\geqq 0 \quad \forall S\in\mathbb{A}^n, \tag{5.33}$$

$$\sum_{j\in J_1}v_jH_j(T)\geqq 0. \tag{5.34}$$

$(DIIIe)$ is dual to (P) if for each $i\in I_+$, the function

$$R\longrightarrow G_i(T)\left[F_i(R)+\sum_{j\in J_0}v_jH_j(R)\right]-\left[F_i(T)+\sum_{j\in J_0}v_jH_j(T)\right]G_i(R) \tag{5.35}$$

is strictly $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-pseudounivex at T, $\bar{\phi}_i$ is increasing, $\bar{\phi}_i(0)=0$, the function $R\to\sum_{j\in J_1}v_jH_j(R)$ is $(\mathcal{F},b,\tilde{\phi},\tilde{\rho},\theta)$-quasiunivex at T, $\tilde{\phi}$ is increasing, $\tilde{\phi}(0)=0$, and $\sum_{i\in I_+}u_i\bar{\rho}_i+\tilde{\rho}\geqq 0$.

In a similar manner, one can obtain a vast number of duality theorems for (P) by specializing the other nine sets of conditions for $(DIIIa)$–$(DIIIe)$ and other special cases of $(DIII)$.

6. Duality model IV

In this section, we present another general duality model for (P) that is different from $(DIII)$ in that here in constructing the constraints we not only use a partition of the index set $\underline{q}$, but also a partition of the set $\underline{p}$. A parametric point-function version of this dual problem was considered earlier in [4].

Let $\{I_0,I_1,\ldots,I_k\}$ be a partition of $\underline{p}$ such that $K\equiv\{0,1,\ldots,k\}\subset M\equiv\{0,1,\ldots,m\}$, and let the function $\Omega_t(\cdot,S,u,v):\mathbb{A}^n\to\mathbb{R}$ be defined, for fixed S, u, and v, by

$$\Omega_t(T,S,u,v)=\sum_{i\in I_t}u_i\left[G_i(S)F_i(T)-F_i(S)G_i(T)\right]+\sum_{j\in J_t}v_jH_j(T),\quad t\in K. \tag{6.1}$$

Consider the following problem:
(DIV)

$$\text{Maximize}\left(\frac{F_1(T)}{G_1(T)},\ldots,\frac{F_p(T)}{G_p(T)}\right) \tag{6.2}$$

subject to

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p}u_i[G_i(T)DF_i(T)-F_i(T)DG_i(T)]+\sum_{j=1}^{q}v_jDH_j(T)\right)\geqq 0 \quad \forall S\in\mathbb{A}^n, \tag{6.3}$$

$$\sum_{j\in J_t}v_jH_j(T)\geqq 0, \quad t\in M, \tag{6.4}$$

$$T\in\mathbb{A}^n, \quad u\in U, \quad v\in\mathbb{R}_+^q, \tag{6.5}$$

where $\mathcal{F}(S,T;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$ is a sublinear function.

The next two theorems show that (DIV) is a dual problem for (P).

Theorem 6.1 (weak duality). *Let S and (T,u,v) be arbitrary feasible solutions of (P) and (DIV), respectively, and assume that any one of the following six sets of hypotheses is satisfied:*

(a) (i) *for each $t\in K$, $\Omega_t(\cdot,T,u,v)$ is strictly $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-pseudounivex at T, ϕ_t is increasing, and $\phi_t(0)=0$;*
 (ii) *for each $t\in M\setminus K$, $\Lambda_t(\cdot,v)$ is $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-quasiunivex at T, ϕ_t is increasing, and $\phi_t(0)=0$;*
 (iii) $\sum_{t\in M}\rho_t\geqq 0$;

(b) (i) *for each $t\in K$, $\Omega_t(\cdot,T,u,v)$ is prestrictly $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-quasiunivex at T, ϕ_t is increasing, and $\phi_t(0)=0$;*
 (ii) *for each $t\in M\setminus K$, $\Lambda_t(\cdot,v)$ is strictly $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-pseudounivex at T, ϕ_t is increasing, and $\phi_t(0)=0$;*
 (iii) $\sum_{t\in M}\rho_t\geqq 0$;

(c) (i) *for each $t\in K$, $\Omega_t(\cdot,T,u,v)$ is prestrictly $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-quasiunivex at T, ϕ_t is increasing, and $\phi_t(0)=0$;*
 (ii) *for each $t\in M\setminus K$, $\Lambda_t(\cdot,v)$ is $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-quasiunivex at T, ϕ_t is increasing, and $\phi_t(0)=0$;*
 (iii) $\sum_{t\in M}\rho_t>0$;

(d) (i) *for each $t\in K_1\neq\emptyset$, $\Omega_t(\cdot,T,u,v)$ is strictly $(\mathcal{F},b,\bar{\phi}_t,\bar{\rho}_t,\theta)$-pseudounivex at T, for each $t\in K_2$, $\Omega_t(\cdot,T,u,v)$ is prestrictly $(\mathcal{F},b,\bar{\phi}_t,\bar{\rho}_t,\theta)$-quasiunivex at T, and for each $t\in K$, $\bar{\phi}_t$ is increasing and $\bar{\phi}_t(0)=0$, where $\{K_1,K_2\}$ is a partition of K;*
 (ii) *for each $t\in M\setminus K$, $\Lambda_t(\cdot,v)$ is $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-quasiunivex at T, ϕ_t is increasing, and $\phi_t(0)=0$;*
 (iii) $\sum_{t\in M}\rho_t\geqq 0$;

(e) (i) *for each $t\in K$, $\Omega_t(\cdot,T,u,v)$ is prestrictly $(\mathcal{F},b,\phi_t,\rho_t,\theta)$-quasiunivex at T, ϕ_t is increasing, and $\phi_t(0)=0$;*
 (ii) *for each $t\in(M\setminus K)_1\neq\emptyset$, $\Lambda_t(\cdot,v)$ is strictly $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-pseudounivex at T, for each $t\in(M\setminus K)_2$, $\Lambda_t(\cdot,v)$ is $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-quasiunivex at T, and for each $t\in M\setminus K$, $\tilde{\phi}_t$ is increasing and $\tilde{\phi}_t(0)=0$, where $\{(M\setminus K)_1,(M\setminus K)_2\}$ is a partition of $M\setminus K$;*
 (iii) $\sum_{t\in M}\rho_t\geqq 0$;

(f) (i) *for each* $t \in K_1$, $\Omega_t(\cdot, T, u, v)$ *is strictly* $(\mathscr{F}, b, \bar{\phi}_t, \bar{\rho}_t, \theta)$*-pseudounivex at* T, *for each* $t \in K_2$, $\Omega_t(\cdot, T, u, v)$ *is prestrictly* $(\mathscr{F}, b, \bar{\phi}_t, \bar{\rho}_t, \theta)$*-quasiunivex at* T, *and for each* $t \in K$, $\bar{\phi}_t$ *is increasing and* $\bar{\phi}_t(0) = 0$, *where* $\{K_1, K_2\}$ *is a partition of* K;

(ii) *for each* $t \in (M \setminus K)_1$, $\Lambda_t(\cdot, v)$ *is strictly* $(\mathscr{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-pseudounivex at* T, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$, *and for each* $t \in (M \setminus K)_2$, $\Lambda_t(\cdot, v)$ *is* $(\mathscr{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-quasiunivex at* T, *and for each* $t \in M \setminus K$, $\tilde{\phi}_t$ *is increasing and* $\tilde{\phi}_t(0) = 0$, *where* $\{(M \setminus K)_1, (M \setminus K)_2\}$ *is a partition of* $M \setminus K$;

(iii) $\sum_{t \in M} \rho_t \geqq 0$;

(iv) $K_1 \neq \emptyset$, $(M \setminus K)_1 \neq \emptyset$, *or* $\sum_{t \in M} \rho_t > 0$.

Then $\varphi(S) \nleq \delta(T, u, v)$, *where* $\delta = (\delta_1, \ldots, \delta_p)$ *is the objective function of (DIV).*

Proof. (a) Suppose to the contrary that $\varphi(S) \leqslant \delta(T, u, v)$. This implies that $G_i(T)F_i(S) - F_i(T)G_i(S) \leqq 0$ for each $i \in \underline{p}$, and $G_\ell(T)F_\ell(S) - F_\ell(T)G_\ell(S) < 0$ for some $\ell \in \underline{p}$. From these inequalities, nonnegativity of v, primal feasibility of S, and (6.4) we deduce that $\Omega_t(S, T, u, v) \leqq \Omega_t(T, T, u, v)$ for each $t \in K$, with strict inequality holding for at least one $t \in K$ since $u > 0$. It follows from the properties of $\bar{\phi}_t$ that for each $t \in K, \phi_t(\Omega_t(S, T, u, v) - \Omega_t(T, T, u, v)) \leqq 0$, which in view of (i) implies that

$$\mathscr{F}\left(S, T; b(S, T)\left\{\sum_{i \in I_t} u_i[G_i(T)DF_i(T) - F_i(T)DG_i(T)] + \sum_{j \in J_t} v_j DH_j(T)\right\}\right) < -\rho_t d^2(\theta(S, T)). \tag{6.6}$$

Adding these inequalities and using the sublinearity of $\mathscr{F}(S, T; \cdot)$, we obtain

$$\mathscr{F}\left(S, T; b(S, T)\left\{\sum_{i=1}^{p} u_i[G_i(T)DF_i(T) - F_i(T)DG_i(T)] + \sum_{t \in K} \sum_{j \in J_t} v_j DH_j(T)\right\}\right) < -\sum_{t \in K} \rho_t d^2(\theta(S, T)). \tag{6.7}$$

Since for each $t \in M \setminus K$, $\Lambda_t(S, v) \leqq 0 \leqq \Lambda_t(T, v)$, it follows from the properties of ϕ_t that $\phi_t(\Lambda_t(S, v) - \Lambda_t(T, v)) \leqq 0$, which by (ii) implies that

$$\mathscr{F}\left(S, T; b(S, T) \sum_{j \in J_t} v_j DH_j(T)\right) \leqq -\rho_t d^2(\theta(S, T)). \tag{6.8}$$

Adding these inequalities and using the sublinearity of $\mathscr{F}(S, T; \cdot)$, we obtain

$$\mathscr{F}\left(S, T; b(S, T) \sum_{t \in M \setminus K} \sum_{j \in J_t} v_j DH_j(T)\right) \leqq -\sum_{t \in M \setminus K} \rho_t d^2(\theta(S, T)). \tag{6.9}$$

Now combining (6.7) and (6.9), and using the sublinearity of $\mathcal{F}(S,T;\cdot)$ and (iii), we get

$$\mathcal{F}\Bigg(S,T;b(S,T)\sum_{i=1}^{p}u_i\big[G_i(T)DF_i(T)-F_i(T)DG_i(T)\big] \\ +\sum_{j=1}^{q}v_jDH_j(T)\Bigg) < -\sum_{t\in M}\rho_t d^2(\theta(S,T)) \leqq 0, \tag{6.10}$$

which contradicts (6.3). Therefore, $\varphi(S) \nleq \delta(T,u,v)$.

(b)–(f) The proofs are similar to that of part (a). □

Theorem 6.2 (strong duality). *Let S^* be a regular efficient solution of (P), let $\mathcal{F}(S,S^*;DF(S^*)) = \sum_{k=1}^{n}\langle D_kF(S^*),\chi_{S_k}-\chi_{S_k^*}\rangle$ for any differentiable function $F:\mathbb{A}^n\to\mathbb{R}$ and $S\in\mathbb{A}^n$, and assume that any one of the six sets of hypotheses specified in Theorem 6.1 holds for all feasible solutions of (DIV). Then there exist $u^*\in U$ and $v^*\in\mathbb{R}_+^q$ such that (S^*,u^*,v^*) is an efficient solution of (DIV) and $\varphi(S^*)=\delta(S^*,u^*,v^*)$.*

Proof. The proof is similar to that of Theorem 4.4. □

Various special cases of (DIV) can easily be generated by appropriate choices of the partitioning sets $I_0,I_1,\dots,I_k$, J_0, and $\mathcal{F}(S,T;\cdot)$.

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