ON π-s-IMAGES OF METRIC SPACES

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We establish the characterizations of metric spaces under compact-covering (resp., pseudo-sequence-covering, sequence-covering) π-s-maps by means of cfp-covers (resp., sfp-covers, cs-covers) and σ-strong networks.

1. Introduction and definitions

In 1966, Michael [11] introduced the concept of compact-covering maps. Since many important kinds of maps are compact-covering, such as closed maps on paracompact spaces, much work has been done to seek the characterizations of metric spaces under various compact-covering maps, for example, compact-covering (open) s-maps, pseudo-sequence-covering (quotient) s-maps, sequence-covering (quotient) s-maps, and compact-covering (quotient) s-maps, see [3, 9, 12, 15, 16]. π-map is another important map which was introduced by Ponomarev [13] in 1960 and correspondingly, many spaces, including developable spaces, weak Cauchy spaces, g-developable spaces, and semimetrizable spaces, were characterized as the images of metric spaces under certain quotient π-maps, see [1, 4, 6, 7].

The purpose of this paper is to establish the characterizations of metric spaces under compact-covering (resp., pseudo-sequence-covering, sequence-covering) π-s-maps by means of cfp-covers (resp., sfp-covers, cs-covers) and σ-strong networks.

In this paper, all spaces are Hausdroff, and all maps are continuous and surjective. $\mathbb{N}$ denotes the set of all natural numbers. ω denotes $\mathbb{N} \cup \{0\}$. $\tau(X)$ denotes a topology on X. For a collection $\mathcal{P}$ of subsets of a space X and a map $f : X \to Y$, denote $\{f(P) : P \in \mathcal{P}\}$ by $f(\mathcal{P})$. For the usual product space $\prod_{i\in\mathbb{N}} X_i$, π_i denotes the projective $\prod_{i\in\mathbb{N}} X_i$ onto X_i. For a sequence $\{x_n\}$ in X, denote $\langle x_n\rangle = \{x_n : n \in \mathbb{N}\}$.

Definition 1.1. Let $f : X \to Y$ be a map.

(1) f is called a compact-covering map [11] if each compact subset of Y is the image of some compact subset of X.

(2) f is called a sequence-covering map [14] if whenever $\{y_n\}$ is a convergent sequence in Y, then there exists a convergent sequence $\{x_n\}$ in X such that each $x_n \in f^{-1}(y_n)$.

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(3) f is called a pseudo-sequence-covering map [3] if each convergent sequence (including its limit point) of Y is the image of some compact subset of X.
(4) f is called an s-map, if $f^{-1}(y)$ is separable in X for any $y \in Y$.
(5) f is called a π-map [13], if (X,d) is a metric space, and for each $y \in Y$ and its open neighborhood V in Y, $d(f^{-1}(y), M \setminus f^{-1}(V)) > 0$.
(6) f is called a π-s-map, if f is both π-map and s-map.
It is easy to check that compact maps on metric spaces are π-s-maps.

Definition 1.2. Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space X such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in \mathbb{N}$.
(1) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is called a σ-strong network [5] for X if for each $x \in X$, $\langle \mathrm{st}(x, \mathcal{P}_n)\rangle$ is a local network of x in X. If every $\mathcal{P}_n$ satisfies property P, then $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is called a σ-strong network consisting of P-covers.
(2) $\{\mathcal{P}_n\}$ is called a weak development for X if for each $x \in X$, $\langle \mathrm{st}(x,\mathcal{P}_n)\rangle$ is a weak neighborhood base of x in X.

Definition 1.3 [2]. Let X be a space.
(1) Let $\{x_n\}$ be a convergent sequence in X, and $P \subset X$. $\{x_n\}$ is eventually in P if whenever $\{x_n\}$ converges to x, then $\{x\} \bigcup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$.
(2) Let $x \in P \subset X$. P is called a sequential neighborhood of x in X if whenever a sequence $\{x_n\}$ in X converges to X, then $\{x_n\}$ is eventually in P.
(3) Let $P \subset X$. P is called a sequentially open subset in X if P is a sequential neighborhood of x in X for any $x \in P$.
(4) X is called a sequential space if each sequentially open subset in X is open.

Definition 1.4 [10]. Let $\mathcal{P}$ be a collection of subsets of a space X.
(1) $\mathcal{P}$ is called a cfp-cover (i.e., compact-finite-partition cover) of compact subset K in X if there are a finite collection $\{K_\alpha : \alpha \in J\}$ of closed subsets of K and $\{P_\alpha : \alpha \in J\} \subset \mathcal{P}$ such that $K = \bigcup\{K_\alpha : \alpha \in J\}$ and each $K_\alpha \subset P_\alpha$.
(2) $\mathcal{P}$ is called a cfp-cover for X if for any compact subset K of X, there exists a finite subcollection $\mathcal{P}^\star \subset \mathcal{P}$ such that $\mathcal{P}^\star$ is a cfp-cover of K in X.
(3) $\mathcal{P}$ is called an sfp-cover (i.e., sequence-finite-partition cover) for X if for any convergent sequence (including its limit point) K in X, there exists a finite subcollection $\mathcal{P}^\star \subset \mathcal{P}$ such that $\mathcal{P}^\star$ is a cfp-cover of K in X.
(4) $\mathcal{P}$ is called a cs-cover for X, if every convergent sequence in X is eventually in some element of $\mathcal{P}$.

2. Results

Theorem 2.1. *A space X is the compact-covering π-s-image of a metric spaces if and only if X has a σ-strong network consisting of point-countable* cfp-*covers.*

Proof. To prove the only if part, suppose $f : (M,d) \to X$ is a compact-covering π-s-map, where (M,d) is a metric space. For each $n \in \mathbb{N}$, put $\mathcal{F}_n = \{f(B(z,1/n)) : z \in M\}$, where $B(z,1/n) = \{y \in M : d(z,y) < 1/n\}$. Obviously, $\mathcal{F}_{n+1}$ refines $\mathcal{F}_n$. We claim that $\bigcup\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a σ-strong network for X. In fact, for each $x \in X$, and its open neighborhood U, since f is a π-map, then there exists $n \in \mathbb{N}$ such that $d(f^{-1}(x), M \setminus f^{-1}(U)) > 1/n$.

We can pick $m \in \mathbb{N}$ such that $m \geq 2n$. If $z \in M$ with $x \in f(B(z,1/m))$, then

$$f^{-1}(x) \bigcap B(z,1/m) \neq \emptyset. \tag{2.1}$$

If $B(z,1/m) \not\subset f^{-1}(U)$, then

$$d(f^{-1}(x), M \setminus f^{1}(U)) \leq \frac{2}{m} \leq \frac{1}{n}, \tag{2.2}$$

which is a contradiction. Thus $B(z,1/m) \subset f^{-1}(U)$, so $f(B(z,1/m)) \subset U$. Hence $\operatorname{st}(x,\mathcal{F}_m) \subset U$. Therefore $\bigcup\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a σ-strong network for X.

For each $n \in \mathbb{N}$, let $\mathcal{B}_n$ be a locally finite open refinement of $\{B(z,1/n) : z \in M\}$. Since locally finite collections are closed under finite intersections, we can assume that $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$ for each $n \in \mathbb{N}$. Put $\mathcal{P}_n = f(\mathcal{B}_n)$. Obviously, $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$. Since f is an s-map, each $\mathcal{P}_n$ is point-countable in X. Because $\mathcal{P}_n$ refines $\mathcal{F}_n$ for each $n \in \mathbb{N}$, then $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is also a σ-strong network for X.

We now show that each $\mathcal{P}_n$ is a cfp-cover for X. Suppose K is compact in X, since f is compact-covering, then $f(L) = K$ for some compact subset L of M. Since $\mathcal{B}_n$ is an open cover of L in M, $\mathcal{B}_n$ have a finite subcover $\mathcal{B}_n^L$. Thus $\mathcal{B}_n^L$ can be precisely refined by some finite cover of L consisting of closed subsets of L, denoted by $\{L_\alpha : \alpha \in J_n\}$. Put $\mathcal{P}_n^K = f(\mathcal{B}_n^L)$, since $\mathcal{P}_n^K$ is precisely refined by closed cover $\{f(L_\alpha) : \alpha \in J_n\}$ of K, then $\mathcal{P}_n^K$ is a cfp-cover of K in X. Hence each $\mathcal{P}_n$ is a cfp-cover for X.

To prove the if part, suppose $\bigcup\{\mathcal{P}_i : i \in \mathbb{N}\}$ is a σ-strong network for X consisting of point-countable cfp-covers. For each $i \in \mathbb{N}$, $\mathcal{P}_i$ is a point-countable cfp-cover for X. Let $\mathcal{P}_i = \{P_\alpha : \alpha \in \Lambda_i\}$, endow Λ_i with the discrete topology, then Λ_i is a metric space. Put

$$M = \left\{\alpha = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i : \langle P_{\alpha_i} \rangle \text{ forms a local network at some point } x_\alpha \text{ in } X \right\}, \tag{2.3}$$

and endow M with the subspace topology induced from the usual product topology of the collection $\{\Lambda_i : i \in \mathbb{N}\}$ of metric spaces, then M is a metric space. Since X is Hausdroff, x_α is unique in X. For each $\alpha \in M$, we define $f : M \to X$ by $f(\alpha) = x_\alpha$. For each $x \in X$ and $i \in \mathbb{N}$, there exists $\alpha_i \in \Lambda_i$ such that $x \in P_{\alpha_i}$. Since $\bigcup\{\mathcal{P}_i : i \in \mathbb{N}\}$ is a σ-strong network for X, then $\{P_{\alpha_i} : i \in \mathbb{N}\}$ is a local network of x in X. Put $\alpha = (\alpha_i)$, then $\alpha \in M$ and $f(\alpha) = x$. Thus f is surjective. Suppose $\alpha = (\alpha_i) \in M$ and $f(\alpha) = x \in U \in \tau(X)$, then there exists $n \in \mathbb{N}$ such that $P_{\alpha_n} \subset U$. Put

$$V = \{\beta \in M : \text{the } n\text{th coordinate of } \beta \text{ is } \alpha_n\}, \tag{2.4}$$

then V is an open neighborhood of α in M, and $f(V) \subset P_{\alpha_n} \subset U$. Hence f is continuous. For each $\alpha, \beta \in M$, we define

$$d(\alpha,\beta) = \begin{cases} 0, & \alpha = \beta, \\ \max\{1/k : \pi_k(\alpha) \neq \pi_k(\beta)\}, & \alpha \neq \beta, \end{cases} \tag{2.5}$$

then d is a distance on M. Because the topology of M is the subspace topology induced from the usual product topology of the collection $\{\Lambda_i : i \in \mathbb{N}\}$ of discrete spaces, thus d

is a metric on M. For each $x \in U \in \tau(X)$, there exists $n \in \mathbb{N}$ such that $\mathrm{st}(x,\mathcal{P}_n) \subset U$. For $\alpha \in f^{-1}(x)$, $\beta \in M$, if $d(\alpha,\beta) < 1/n$, then $\pi_i(\alpha) = \pi_i(\beta)$ whenever $i \le n$. So $x \in P_{\pi_n(\alpha)} = P_{\pi_n(\beta)}$. Thus,

$$f(\beta) \in \bigcap_{i\in\mathbb{N}} P_{\pi_i(\beta)} \subset P_{\pi_n(\beta)} \subset U. \tag{2.6}$$

Hence

$$d\big(f^{-1}(x), M \setminus f^{-1}(U)\big) \ge \frac{1}{n}. \tag{2.7}$$

Therefore f is a π-map.

For each $x \in X$, it follows from the point-countable property of $\mathcal{P}_i$ that $\{\alpha \in \Lambda_i : x \in P_\alpha\}$ is countable. Put

$$L = \Big(\prod_{i\in\mathbb{N}} \{\alpha \in \Lambda_i : x \in P_\alpha\}\Big) \bigcap M, \tag{2.8}$$

then L is a hereditarily separable subspace of M, and $f^{-1}(x) \subset L$. Thus $f^{-1}(x)$ is separable in M, that is, f is an s-map.

We will prove that f is compact-covering. Suppose K is compact in X. Since each $\mathcal{P}_n$ is a cfp-cover for X, there exists finite subcollection $\mathcal{P}_n^K$ such that it is a cfp-cover of K in X. Thus there are a finite collection $\{K_\alpha : \alpha \in J_n\}$ of closed subsets of K and $\{P_\alpha : \alpha \in J_n\} \subset \mathcal{P}_n^K$ such that $K = \bigcup\{K_\alpha : \alpha \in J_n\}$ and each $K_\alpha \subset P_\alpha$. Obviously, each K_α is compact in X. Put

$$L = \Big\{(\alpha_i) : \alpha_i \in J_i, \bigcap_{i\in\mathbb{N}} K_{\alpha_i} \ne \varnothing\Big\}, \tag{2.9}$$

then

(i) L is compact in M.

In fact, for all $(\alpha_i) \notin L$, $\bigcap_{i\in\mathbb{N}} K_{\alpha_i} = \varnothing$. From $\bigcap_{i\in\mathbb{N}} K_{\alpha_i} = \varnothing$, there exists $n_0 \in \mathbb{N}$ such that $\bigcap_{i=1}^{n_0} K_{\alpha_i} = \varnothing$. Put

$$W = \{(\beta_i) : \beta_i \in J_i,\ \beta_i = \alpha_i,\ 1 \le i \le n_0\}, \tag{2.10}$$

then W is an open neighborhood of (α_i) in $\prod_{i\in\mathbb{N}} J_i$, and $W \bigcap L = \varnothing$. Thus L is closed in $\prod_{i\in\mathbb{N}} J_i$. Since $\prod_{i\in\mathbb{N}} J_i$ is compact in $\prod_{i\in\mathbb{N}} \Lambda_i$, L is compact in M.

(ii) $L \subset M$, $f(L) = K$.

In fact, for all $(\alpha_i) \in L$, $\bigcap_{i\in\mathbb{N}} K_{\alpha_i} \ne \varnothing$. Pick $x \in \bigcap_{i\in\mathbb{N}} K_{\alpha_i}$, then $\langle P_{\alpha_i}\rangle$ is a local network of x in X, so $(\alpha_i) \in M$. This implies $L \subset M$.

For all $x \in K$, for each $i \in \mathbb{N}$, pick $\alpha_i \in J_i$ such that $x \in K_{\alpha_i}$. Thus $f((\alpha_i)) = x$, so $K \subset f(L)$. Obviously, $f(L) \subset K$. Hence $f(L) = K$.

In a word, f is compact-covering. □

Corollary 2.2. *A space X is the compact-covering, quotient, and π-s-image of a metric space if and only if X has a weak-development consisting of point-countable* cfp*-covers.*

Proof. To prove the only if part, suppose X is the compact-covering, quotient, and π-s-image of a metric space M. From Theorem 2.1, X has a σ-strong network consisting of point-countable cfp-covers $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$. For each $x \in X$, $\mathrm{st}(x, \mathcal{P}_n)$ is a sequential neighborhood of x in X. Obviously, X is a sequential space. Thus $\mathrm{st}(x, \mathcal{P}_n)$ is a weak neighborhood base of x in X. Hence $\{\mathcal{P}_n\}$ is a weak-development for X.

To prove the if part, suppose X has a weak development consisting of point-countable cfp-covers. From Theorem 2.1, X is the image of a metric space under a compact-covering π-s-map f. Obviously, X is sequential. By [8, Proposition 2.1.16], f is quotient. □

Similar to the proofs of Theorem 2.1 and Corollary 2.2, we have the following theorem.

Theorem 2.3. *A space X is the pseudo-sequence-covering π-s-image of a metric space if and only if X has a σ-strong network consisting of point-countable* sfp-*covers.*

Corollary 2.4. *A space X is the pseudo-sequence-covering, quotient, and π-s-image of a metric space if and only if X has a weak-development consisting of point-countable* sfp-*covers.*

Theorem 2.5. *A space X is the sequence-covering π-s-image of a metric space if and only if X has a σ-strong network consisting of point-countable* cs-*covers.*

Proof. To prove the only if part, suppose $f : (M,d) \to X$ is a sequence-covering π-s-map, where (M,d) is a metric space. Similar to the proof of Theorem 2.1, we can show that $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-strong network consisting of point-countable covers. It suffices to show that each $\mathcal{P}_n$ is a cs-cover for X. Suppose $\{x_n\}$ converges to $x \in X$ in X. Since f is sequence-covering, then there exists a convergent sequence $\{z_i\}$ such that each $z_i \in f^{-1}(x_i)$. Suppose $\{z_i\} \to z$, then $z \in f^{-1}(x)$ and $z \in B$ for some $B \in \mathcal{B}_n$. Thus $\{z_i\}$ is eventually in B, so $\{x_i\}$ is eventually in $f(B) \in \mathcal{P}_n$. Hence each $\mathcal{P}_n$ is a cs-cover for X.

To prove the if part, suppose $\bigcup\{\mathcal{P}_i : i \in \mathbb{N}\}$ is a σ-strong network consisting of point-countable cs-covers for X. For each $i \in \mathbb{N}$, $\mathcal{P}_i$ is a point-countable cs-cover for X. Let $\mathcal{P}_i = \{P_\alpha : \alpha \in \Lambda_i\}$. Similar to the proof of Theorem 2.1, we can show that f is a π-s-map. It suffices to show that f is sequence-covering. Suppose $\{x_n\}$ converges to x in X. For each $i \in \mathbb{N}$, since $\mathcal{P}_i$ is a cs-cover for X, then there exists $P_{\alpha_i} \in \mathcal{P}_i$ such that $\{x_n\}$ is eventually in P_{α_i}. For each $n \in \mathbb{N}$, if $x_n \in P_{\alpha_i}$, let $\alpha_{in} = \alpha_i$; if $x_n \notin P_{\alpha_i}$, pick $\alpha_{in} \in \Lambda_i$ such that $x_n \in P_{\alpha_i n}$. Thus there exists $n_i \in \mathbb{N}$ such that $\alpha_{in} = \alpha_i$ for all $n > n_i$. So $\{\alpha_{in}\}$ converges to α_i. For each $n \in \mathbb{N}$, put

$$\beta_n = (\alpha_{in}) \in \prod_{i \in \mathbb{N}} \Lambda_i, \tag{2.11}$$

then $(\beta_n) \in f^{-1}(x_n)$ and $\{\beta_n\}$ converges to x. Thus f is sequence-covering. □

Similar to the proof of Corollary 2.2, we have the following corollary.

Corollary 2.6. *A space X is the sequence-covering, quotient, and π-s-image of a metric space if and only if X has a weak-development consisting of point-countable* cs-*covers.*

We give examples to illustrate the theorems of this paper.

Example 2.7. Let Z be the topological sum of the unit interval [0,1], and the collection $\{S(x) : x \in [0,1]\}$ of 2^ω convergent sequence $S(x)$. Let X be the space obtained from Z by identifying the limit point of $S(x)$ with $x \in [0,1]$, for each $x \in [0,1]$. Then, from [8, Example 2.9.27], or see [3, Example 9.8], we have the following facts.

(1) X is the compact-covering, quotient compact image of a locally compact metric space.

(2) X has no point-countable cs-network.

The above facts together with [9, Theorem 1] yield the following conclusion: compact-covering (quotient) π-s-images of metric spaces are not sequence-covering (quotient) π-s-images of metric spaces.

Example 2.8. Let X be a sequential fan S_ω (see [8, Example 1.8.7]), then X is a Fréchet and $\aleph_0$-space. So X is the sequence-covering s-image of a metric space. Because X is not g-first countable, thus X is not the pseudo-sequence-covering π-image of a metric space. Hence the following holds: sequence-covering (resp., pseudo-sequence-covering) s-images of metric spaces are not sequence-covering (resp., pseudo-sequence-covering) π-s-images of metric spaces.

Example 2.9. Let X be a Gillman-Jerison space $\psi(\mathbb{N})$ (see [8, Example 1.8.4]). Since X is developable, then X is the sequence-covering, quotient π-image of a metric space by [10, Corollary 3.1.12]. But X has no point-countable cs*-networks. Then, it follows from [8, Theorem 2.7.5] that X is not the pseudo-sequence-covering s-image of a metric space. Thus,

(1) sequence-covering (quotient) π-images of metric spaces are not sequence-covering (quotient) π-s-images of metric spaces,

(2) pseudo-sequence-covering (quotient) π-images of metric spaces are not pseudo-sequence-covering (quotient) π-s-images of metric spaces.

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