DUAL SERIES EQUATIONS INVOLVING GENERALIZED LAGUERRE POLYNOMIALS

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An exact solution is obtained for the dual series equations involving generalized Laguerre polynomials.

1. Introduction

We consider the following dual series equations:

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha)}\left[(x+b)^h\right]}{\Gamma(\alpha+n+1)} = f(x), \quad 0 < x < a, \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\sigma)}\left[(x+b)^h\right]}{\Gamma(\alpha+n+\beta)} = g(x), \quad a < x < \infty, \tag{1.2}$$

where $\alpha+\beta+1>\beta>1-m$, $\sigma+1>\alpha+\beta>0$, m is a positive integer, and $0<h<\infty$, $0\leq b<\infty$, and h and b are finite constants. $L_n^{(\alpha)}[(x+b)^h]$ is a Laguerre polynomial, A_n are unknown coefficients, and $f(x)$ and $g(x)$ are prescribed functions.

Srivastava [5, 6] has solved the following dual series equations:

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha)}(x)}{\Gamma(\alpha+n+1)} = f(x), \quad 0 < x < a, \tag{1.3}$$

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\sigma)}(x)}{\Gamma(\alpha+n+\beta)} = g(x), \quad a < x < \infty. \tag{1.4}$$

The triple series equations (1.3) and (1.4) are a special case of the dual series equations (1.1) and (1.2) when

$$h = 1, \qquad b = 0. \tag{1.5}$$

Recently, Lowndes and Srivastava [3] have solved the triple series equations involving Laguerre polynomials. References for the solutions of dual and triple series equations

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involving Laguerre polynomials are given in [3]. Connected to this work, references and solutions for dual series equations are given by Sneddon [4].

The dual series equations (1.1) and (1.2) are new in the literature and have importance due to the closed-form solution. The results of this note are shown to be in agreement with those of Srivastava [5]. The analysis is purely formal and no justification had been given for the change of the order of integrations and summation.

2. Some useful results

In this section, we will discuss some results which are useful in solving dual series equations (1.1) and (1.2). The orthogonality relation for Laguerre polynomials is given by [2, page 292, equation (2)] and [2, page 293, equation (2)], from which we have

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)dx = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)}\delta_{nm}, \quad \alpha > -1, \tag{2.1}$$

where δ_{nm} is the Kronecker delta.

We can easily find, with the help of integrals [2, page 293, equation (5)] and [2, page 405, equation (20)], that

$$\begin{aligned} &\int_0^\xi x^\alpha(\xi-x)^{\beta+m-2} L_n^{(\alpha)}(x)dx \\ &\quad = \frac{\Gamma(\alpha+n+1)\Gamma(\beta+m-1)}{\Gamma(\alpha+\beta+m+n)}\xi^{\alpha+\beta+m-1} L_n^{(\alpha+\beta+m-1)}(\xi), \quad \alpha > -1,\ \beta+m > 1, \end{aligned} \tag{2.2}$$

$$\int_\xi^\infty e^{-x}(x-\xi)^{\sigma-\alpha-\beta} L_n^{(\sigma)}(x)dx = \Gamma(\sigma-\alpha-\beta+1)e^{-\xi} L_n^{\alpha+\beta-1}(\xi), \quad \sigma+1 > \alpha+\beta > 0. \tag{2.3}$$

From [1, page 190, equation (27)], we find that

$$\frac{d^m}{dx^m}\left[x^{\alpha+m} L_n^{(\alpha+m)}(x)\right] = \frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)} x^\alpha L_n^\alpha(x). \tag{2.4}$$

3. Solution of dual series equations (1.1) and (1.2)

We assume that

$$\begin{aligned} x+b = X^{1/h}, \qquad f\left(X^{1/h}-b\right) = f_1(X), \\ g\left(X^{1/h}-b\right) = g_1(X), \qquad b^h = c, \qquad (a+b)^h = d, \end{aligned} \tag{3.1}$$

then the dual series equations (1.1) and (1.2) can be written in the following form:

$$\sum_{n=0}^\infty \frac{A_n L_n^{(\alpha)}(X)}{\Gamma(\alpha+n+1)} = f_1(X), \quad c < X < d, \tag{3.2}$$

$$\sum_{n=0}^\infty \frac{A_n L_n^{(\sigma)}(X)}{\Gamma(\alpha+\beta+n)} = g_1(X), \quad d < X < \infty. \tag{3.3}$$

We assume that

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha)}(X)}{\Gamma(\alpha+n+1)} = f_2(X), \quad 0 < X < c. \tag{3.4}$$

Combining the series equations (3.2) and (3.4), we can write the dual series equations (3.2) and (3.3) in the form

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha)}(X)}{\Gamma(\alpha+n+1)} = F(X), \quad 0 < X < d, \tag{3.5}$$

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\sigma)}(X)}{\Gamma(\alpha+\beta+n)} = g_1(X), \quad d < X < \infty, \tag{3.6}$$

where

$$F(X) = \begin{cases} f_2(X), & 0 < X < c, \\ f_1(X), & c < X < d. \end{cases} \tag{3.7}$$

Multiplying (3.5) by $X^\alpha(\xi - X)^{\beta+m-2}$, where m is a positive integer, integrating with respect to X over $(0,\xi)$, and interchanging the order of integrations, we find on using (2.2) that

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha+\beta+m-1)}(\xi)}{\Gamma(\alpha+\beta+m+n)} = \frac{\xi^{-\alpha-\beta-m+1}}{\Gamma(\beta+m-1)} \int_0^{\xi} X^\alpha(\xi - X)^{\beta+m-2} F(X)\,dX, \quad 0 < \xi < d, \tag{3.8}$$

where

$$\alpha > -1, \qquad \beta + m > 1. \tag{3.9}$$

If we now multiply (3.8) by $\xi^{\alpha+\beta+m-1}$, differentiate both sides m times with respect to ξ, and use formula (2.4), we find that

$$\sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha+\beta-1)}(\xi)}{\Gamma(\alpha+\beta+n)} = \frac{\xi^{-\alpha-\beta+1}}{\Gamma(\beta+m-1)} \frac{d^m}{d\xi^m} \int_0^{\xi} X^\alpha(\xi - X)^{\beta+m-2} F(X)\,dX, \quad 0 < \xi < d, \tag{3.10}$$

where

$$\alpha > -1, \qquad \beta + m > 1. \tag{3.11}$$

Multiplying (3.6) by $e^{-X}(X-\xi)^{\sigma-\alpha-\beta}$, integrating with respect to x over (ξ,∞), and interchanging the order of integrations, we find by using formula (2.3) that

$$\sum_{n=0}^{\infty}\frac{A_n L_n^{(\alpha+\beta-1)}(\xi)}{\Gamma(\alpha+\beta+n)}=\frac{e^{\xi}}{\Gamma(\sigma-\alpha-\beta+1)}\int_{\xi}^{\infty}e^{-X}(X-\xi)^{\sigma-\alpha-\beta}g_1(X)dX,\quad d<\xi<\infty,\quad (3.12)$$

where

$$\sigma+1>\alpha+\beta>0. \quad (3.13)$$

The left-hand sides of (3.10) and (3.12) are now identical. Making use of the orthogonality relation (2.1), we find from (3.10) and (3.12) that

$$A_n=\Gamma(n+1)\left[\int_0^d\frac{e^{-\xi}L_n^{(\alpha+\beta-1)}(\xi)F_1(\xi)d\xi}{\Gamma(\beta+m-1)}+\int_d^{\infty}\frac{\xi^{\alpha+\beta-1}L_n^{(\alpha+\beta-1)}(\xi)G(\xi)d\xi}{\Gamma(\sigma-\alpha-\beta+1)}\right], \quad (3.14)$$

where

$$F_1(\xi)=\frac{d^m}{d\xi^m}\int_0^{\xi}X^{\alpha}(\xi-X)^{\beta+m-2}F(X)dX, \quad (3.15)$$

$$G(X)=\int_{\xi}^{\infty}e^{-X}(X-\xi)^{\sigma-\alpha-\beta}g_1(X)dX, \quad (3.16)$$

provided that $\alpha+\beta+1>1-m$ and $\sigma+1>\alpha+\beta>0$.

With the help of (3.7), (3.15) can be written in the form:

$$F_1(\xi)=\frac{d^m}{d\xi^m}\left[\int_0^c X^{\alpha}(\xi-X)^{\beta+m-2}f_2(X)dX+\int_X^{\xi}X^{\alpha}(\xi-X)^{\beta+m-2}f_1(X)dX\right],\quad c<\xi. \quad (3.17)$$

When we put

$$b=0,\qquad h=1,\qquad f_2(X)=0 \quad (3.18)$$

in the solution of the dual series equations (1.1) and (1.2), we then obtain the solution of the dual series equations (1.3) and (1.4) and the results are in complete agreement with those of [5].

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