ON HORIZONTAL AND COMPLETE LIFTS FROM A MANIFOLD WITH $f_\lambda(7,1)$-STRUCTURE TO ITS COTANGENT BUNDLE

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The horizontal and complete lifts from a manifold M^n to its cotangent bundles $\overset{*}{T}(M^n)$ were studied by Yano and Ishihara, Yano and Patterson, Nivas and Gupta, Dambrowski, and many others. The purpose of this paper is to use certain methods by which $f_\lambda(7,1)$-structure in M^n can be extended to $\overset{*}{T}(M^n)$. In particular, we have studied horizontal and complete lifts of $f_\lambda(7,1)$-structure from a manifold to its cotangent bundle.

1. Introduction

Let M be a differentiable manifold of class c^∞ and of dimension n and let C_{TM} denote the cotangent bundle of M. Then C_{TM} is also a differentiable manifold of class c^∞ and dimension $2n$.

The following are notations and conventions that will be used in this paper.

(1) $\mathfrak{I}_s^r(M)$ denotes the set of tensor fields of class c^∞ and of type (r,s) on M. Similarly, $\mathfrak{I}_s^r(C_{\text{TM}})$ denotes the set of such tensor fields in C_{TM}.

(2) The map Π is the projection map of C_{TM} onto M.

(3) Vector fields in M are denoted by $X,Y,Z,\ldots$ and Lie differentiation by L_X. The Lie product of vector fields X and Y is denoted by $[X,Y]$.

(4) Suffixes $a,b,c,\ldots,h,i,j,\ldots$ take the values 1 to n and $\bar{i}=i+n$. Suffixes $A,B,C,\ldots$ take the values 1 to $2n$.

If A is a point in M, then $\Pi^{-1}(A)$ is fiber over A. Any point $p\in\Pi^{-1}(A)$ is denoted by the ordered pair (A,p_A), where p is 1-form in M and p_A is the value of p at A. Let U be a coordinate neighborhood in M such that $A\in U$. Then U induces a coordinate neighborhood $\Pi^{-1}(U)$ in C_{TM} and $p\in\Pi^{-1}(U)$.

2. Complete lift of $f_\lambda(7,1)$- structure

Let $f(\neq 0)$ be a tensor field of type $(1,1)$ and class c^∞ on M such that

$$f^7+\lambda^2 f=0, \tag{2.1}$$

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where λ is any complex number not equal to zero. We call the manifold M satisfying (2.1) as $f_\lambda(7,1)$-structure manifold. Let f_i^h be components of f at A in the coordinate neighborhood U of M. Then the complete lift f^c of f is also a tensor field of type $(1,1)$ in C_{TM} whose components $\tilde{f}_B^A$ in $\Pi^{-1}(U)$ are given by [2]

$$\tilde{f}_i^h = f_i^h; \qquad f_{\bar{i}}^h = 0, \tag{2.2}$$

$$\tilde{f}_i^{\bar{h}} = P_a\left(\frac{\partial f_h^a}{\partial x^i}\frac{\partial f_i^a}{\partial x^h}\right); \qquad \tilde{f}_{\bar{i}}^{\bar{h}} = f_h^i, \tag{2.3}$$

where $(x^1,x^2,\dots,x^n)$ are coordinates of A relative to U and p_A has a component $(p_1,p_2,\dots,p_n)$.

Thus we can write

$$f^C = \left(\tilde{f}_B^A\right) = \begin{pmatrix} f_i^h & 0 \\ p_a\left(\partial_i f_h^a - \partial_h f_i^a\right) & f_h^i \end{pmatrix}, \tag{2.4}$$

where $\partial_i = \partial/\partial x^i$.

If we put

$$\partial_i f_h^a - \partial_h f_i^a = 2\partial[i f_h^a], \tag{2.5}$$

then we can write (2.4) in the form

$$f^C = \left(f_B^A\right) = \begin{pmatrix} f_i^h & 0 \\ 2p_a\partial[i f_h^a] & f_h^i \end{pmatrix}. \tag{2.6}$$

Thus we have

$$\left(f^C\right)^2 = \begin{pmatrix} f_i^h & 0 \\ 2p_a\partial[i f_h^a] & f_h^i \end{pmatrix}\begin{pmatrix} f_j^i & 0 \\ 2p_t\partial[j f_i^t] & f_i^j \end{pmatrix}, \tag{2.7}$$

or

$$\left(f^C\right)^2 = \begin{pmatrix} f_i^h f_j^i & 0 \\ 2p_a f_j^l\partial[i f_h^a] + 2p_t f_h^i\partial[j f_i^t] & f_i^j f_h^i \end{pmatrix}. \tag{2.8}$$

If we put

$$2p_a f_j^l\partial[i f_h^a] + 2p_t f_h^i\partial[j f_i^t] = L_{hj}, \tag{2.9}$$

then (2.8) takes the form

$$\left(f^C\right)^2 = \begin{pmatrix} f_i^h f_j^i & 0 \\ L_{hj} & f_i^j f_h^i \end{pmatrix}. \tag{2.10}$$

Thus we have

$$(f^C)^4 = \begin{pmatrix} f_i^h f_j^i & 0 \\ L_{hj} & f_i^j f_h^i \end{pmatrix} \begin{pmatrix} f_k^j f_l^k & 0 \\ L_{jl} & f_k^l f_j^k \end{pmatrix}, \tag{2.11}$$

or

$$(f^C)^4 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k & 0 \\ f_k^j f_l^k L_{hj} + f_i^j f_h^i L_{jl} & f_k^l f_j^k f_i^j f_h^i \end{pmatrix}. \tag{2.12}$$

Putting again

$$f_k^j f_l^k L_{hj} + f_i^j f_h^i L_{jl} = P_{hl}, \tag{2.13}$$

then we can put (2.12) in the form

$$(f^C)^4 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k & 0 \\ P_{hl} & f_k^l f_j^k f_i^j f_h^i \end{pmatrix}. \tag{2.14}$$

Thus,

$$(f^C)^6 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k & 0 \\ P_{hl} & f_k^l f_j^k f_i^j f_h^i \end{pmatrix} \begin{pmatrix} f_m^l f_n^m & 0 \\ L_{ln} & f_m^n f_l^m \end{pmatrix}, \tag{2.15}$$

$$(f^C)^6 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k f_m^l f_n^m & 0 \\ P_{hl} f_m^l f_n^m + L_{ln} f_k^l f_j^k f_i^j f_h^i & f_m^n f_l^m f_k^l f_j^k f_i^j f_h^i \end{pmatrix}. \tag{2.16}$$

Putting again

$$P_{hl} f_m^l f_n^m + L_{ln} f_k^l f_j^k f_i^j f_h^i = Q_{hn}, \tag{2.17}$$

then (2.16) takes the form

$$(f^C)^6 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k f_m^l f_n^m & 0 \\ Q_{hn} & f_m^n f_l^m f_k^l f_j^k f_i^j f_h^i \end{pmatrix}. \tag{2.18}$$

Thus,

$$(f^C)^7 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k f_m^l f_n^m & 0 \\ Q_{hn} & f_m^n f_l^m f_k^l f_j^k f_i^j f_h^i \end{pmatrix} \begin{pmatrix} f_p^n & 0 \\ 2p_r\partial[pf_n^r] & f_n^p \end{pmatrix}, \tag{2.19}$$

$$(f^C)^7 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k f_m^l f_n^m f_p^n & 0 \\ Q_{hn} f_p^n + 2p_r\partial[pf_n^r] f_m^n f_l^m f_k^l f_j^k f_i^j f_h^i & f_n^p f_m^n f_l^m f_k^l f_j^k f_i^j f_h^i \end{pmatrix}. \tag{2.20}$$

In view of (2.1), we have

$$f_i^h f_j^i f_k^j f_l^k f_m^l f_n^m f_p^n = -\lambda^2 f_p^h, \tag{2.21}$$

and also putting

$$Q_{hn} f_p^n + 2p_r\partial[pf_n^r] f_m^n f_l^m f_k^l f_j^k f_i^j f_h^i = -\lambda^2 p_s\partial[pf_h^s], \tag{2.22}$$

then (2.20) can be given by

$$(f^C)^7 = \begin{pmatrix} -\lambda^2 f_p^n & 0 \\ -\lambda^2 p_s\partial[pf_h^s] & -\lambda^2 f_h^p \end{pmatrix}. \tag{2.23}$$

In view of (2.6) and (2.23), it follows that

$$(f^C)^7 + \lambda^2 (f^C) = 0. \tag{2.24}$$

Hence the complete lift f^C of f admits an $f_\lambda(7,1)$-structure in the cotangent bundle C_{TM}.

Thus we have the following theorem.

Theorem 2.1. *In order that the complete lift of f^C of a $(1,1)$ tensor field f admitting $f_\lambda(7,1)$-structure in M may have the similar structure in the cotangent bundle C_{TM}, it is necessary and sufficient that*

$$Q_{hn} f_p^n + 2p_r\partial[pf_n^r] f_m^n f_l^m f_k^l f_j^k f_i^j f_h^i = -\lambda^2 p_s\partial[pf_h^s]. \tag{2.25}$$

3. Horizontal lift of $f_\lambda(7,1)$-structure

Let f, g be two tensor fields of type $(1,1)$ on the manifold M. If f^H denotes the horizontal lift of f, we have

$$f^H g^H + g^H f^H = (fg + gf)^H. \tag{3.1}$$

Taking f and g identical, we get

$$(f^H)^2 = (f^2)^H. \tag{3.2}$$

Multiplying both sides by f^H and making use of the same (3.2), we get

$$(f^H)^3 = (f^3)^H \tag{3.3}$$

and so on. Thus it follows that

$$(f^H)^4 = (f^4)^H, \qquad (f^H)^5 = (f^5)^H, \tag{3.4}$$

and so on. Thus,

$$(f^H)^7 = (f^7)^H. \tag{3.5}$$

Since f gives on M the $f_\lambda(7,1)$-structure, we have

$$f^7 + \lambda^2 f = 0. \tag{3.6}$$

Taking horizontal lift, we obtain

$$(f^7)^H + \lambda^2 (f^H) = 0. \tag{3.7}$$

In view of (3.5) and (3.7), we can write

$$(f^H)^7 + \lambda^2 (f^H) = 0. \tag{3.8}$$

Thus the horizontal lift f^H of f also admits a $f_\lambda(7,1)$-structure. Hence we have the following theorem.

Theorem 3.1. *Let f be a tensor field of type $(1,1)$ admitting $f_\lambda(7,1)$-structure in M. Then the horizontal lift f^H of f also admits the similar structure in the cotangent bundle C_{TM}.*

4. Nijenhuis tensor of complete lift of f^7

The Nijenhuis tensor of a $(1,1)$ tensor field f on M is given by

$$N_{f,f}(X,Y) = [fX, fY] - f[fX,Y] - f[X,fY] + f^2[X,Y]. \tag{4.1}$$

Also for the complete lift of f^7, we have

$$\begin{aligned} N(f^7)^C,(f^7)^C(X^C,Y^C) = {} & \left[(f^7)^C X^C, (f^7)^C Y^C\right] - (f^7)^C\left[(f^7)^C X^C, Y^C\right] \\ & - (f^7)^C\left[X^C, (f^7)^C Y^C\right] + (f^7)^C (f^7)^C [X^C, Y^C]. \end{aligned} \tag{4.2}$$

In view of (2.1), the above (4.2) takes the form

$$\begin{aligned} & N(f^7)^C,(f^7)^C(X^C,Y^C) \\ & \quad = \left[(-\lambda^2 f)^C X^C, (-\lambda^2 f)^C Y^C\right] - (-\lambda^2 f)^C\left[(-\lambda^2 f)^C X^C, Y^C\right] \\ & \qquad - (-\lambda^2 f)^C\left[X^C, (-\lambda^2 f)^C Y^C\right] + (-\lambda^2 f)^C (-\lambda^2 f)^C [X^C, Y^C], \end{aligned} \tag{4.3}$$

or

$$N(f^7)^C,(f^7)^C(X^C,Y^C) = \lambda^4\left\{\begin{matrix} [(f)^CX^C,(f)^CY^C]-(f)^C[(f)^CX^C,Y^C] \\ -(f)^C[X^C,(f)^CY^C]+(f)^C(f)^C[X^C,Y^C]\end{matrix}\right\}. \tag{4.4}$$

We also know that [3]

$$(f)^CX^C = (fX)^C + v(\mathcal{L}_X f), \tag{4.5}$$

where vf has components

$$vf = \begin{pmatrix} O^a \\ P_a f_i \end{pmatrix}. \tag{4.6}$$

In view of (4.5), (4.4) takes the form

$$\begin{aligned} &N(f^7)^C,(f^7)^C X^C,Y^C \\ &\quad = \lambda^4\left\{\begin{matrix} [(fX)^C,(fY)^C]+[v(\mathcal{L}_X f),(fY)^C]+[(fX)^C,v(\mathcal{L}_Y f)] \\ +[v(\mathcal{L}_X f),v(\mathcal{L}_Y f)]-(f)^C[(fX)^C,Y^C]-(f)^C[v(\mathcal{L}_X f),Y^C] \\ -(f)^C[X^C,(fY)^C]-(f)^C\left[X^C,v(\mathcal{L}_Y f)^C\right]+(f)^C(f)^C[X^C,Y^C]\end{matrix}\right\}. \end{aligned} \tag{4.7}$$

We now suppose that

$$\mathcal{L}_X f = \mathcal{L}_Y f = 0. \tag{4.8}$$

Then from (4.7), we have

$$N(f^7)^C,(f^7)^C(X^C,Y^C) = \lambda^4\left\{\begin{matrix} [(fX)^C,(fY)^C]-(f)^C[(fX)^C,Y^C] \\ -(f)^C[X^C,(fY)^C]+(f)^C(f)^C[X^C,Y^C]\end{matrix}\right\}. \tag{4.9}$$

Further, if f acts as an identity operator on M [2], that is,

$$fX = X \quad \forall X \in \mathfrak{I}_0^1(M), \tag{4.10}$$

then we have from (4.9)

$$N(f^7)^C,(f^7)^C(X^C,Y^C) = \lambda^8\{[X^C,Y^C]-[X^C,Y^C]-[X^C,Y^C]+[X^C,Y^C]\} = 0. \tag{4.11}$$

Hence we have the following theorem.

Theorem 4.1. *The Nijenhuis tensor of the complete lift of f^7 vanishes if the Lie derivatives of the tensor field f with respect to X and Y are both zero and f acts as an identity operator on M.*

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