STRONG CONVERGENCE OF A MODIFIED IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF Z-OPERATORS

ARIF RAFIQ

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The purpose of this note is to establish a strong convergence of a modified implicit iteration process to a common fixed point for a finite family of Z-operators.

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1. Introduction and preliminaries

We recall the following definitions in a metric space (X,d). A mapping $T : X \to X$ is called an a-contraction if

$$d(Tx,Ty) \leq ad(x,y) \quad \forall x,y \in X, \tag{1.1}$$

where $a \in (0,1)$.

The map T is called Kannan mapping [7] if there exists $b \in (0,1/2)$ such that

$$d(Tx,Ty) \leq b\big[d(x,Tx)+d(y,Ty)\big] \quad \forall x,y \in X. \tag{1.2}$$

A similar definition is due to Chatterjea [3]: there exists $c \in (0,1/2)$ such that

$$d(Tx,Ty) \leq c\big[d(x,Ty)+d(y,Tx)\big] \quad \forall x,y \in X. \tag{1.3}$$

Combining these three definitions, Zamfirescu [12] proved the following important result.

Theorem 1.1. *Let (X,d) be a complete metric space and $T : X \to X$ a mapping for which there exists the real numbers a, b, and c satisfying $a \in (0,1)$, $b,c \in (0,1/2)$ such that for each pair $x,y \in X$, at least one of the following conditions holds:*

(z_1) $d(Tx,Ty) \leq ad(x,y)$,

(z_2) $d(Tx,Ty) \leq b[d(x,Tx)+d(y,Ty)]$,

(z_3) $d(Tx,Ty) \leq c[d(x,Ty)+d(y,Tx)]$.

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Then T has a unique fixed point p and the Picard iteration $\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}, \tag{1.4}$$

converges to p for any arbitrary but fixed $x_1 \in X$.

One of the most general contraction conditions, for which the unique fixed point can be approximated by means of Picard iteration, has been obtained by Ćirić [5]: there exists $0 < h < 1$ such that

$$d(Tx, Ty) \leq h \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \quad \forall x, y \in X. \tag{QC}$$

Remark 1.2. (1) A mapping satisfying (QC) is commonly called quasicontraction. It is obvious that each of the conditions (1.1)–(1.3) and (z_1)–(z_3) implies (QC).

(2) An operator T satisfying the contractive conditions (z_1)–(z_3) in the above theorem is called Z-operator.

Let C be a nonempty closed convex subset of a normed space E.

Xu and Ori [11] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, \ldots, N\}$), with $\{\alpha_n\}$ a real sequence in $(0,1)$, and an initial point $x_0 \in C$:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1) T_1 x_1, \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2) T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N) T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_1 x_{N+1}, \\ &\vdots \end{aligned} \tag{1.5}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n \quad \forall n \geq 1, \tag{1.6}$$

where $T_n = T_{n(\operatorname{mod} N)}$ (here the mod N function takes values in I). Xu and Ori proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [13], Zhou and Chang studied the weak and strong convergences of this implicit process to a common fixed point for a finite family of nonexpansive mappings. More precisely, they proved the following result.

Theorem 1.3 [13, Theorem 3]. *Let E be a uniformly convex Banach space and let K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N semicompact nonexpansive self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$ (here $F(T_i)$ denotes the set of fixed points of T_i).*

Suppose that $x_0 \in K$ *and* $\{\alpha_n\} \subset (b,c)$ *for some* $b,c \in (0,1)$. *Then the sequence* $\{x_n\}$ *defined by the implicit iteration process (1.6) converges strongly to a common fixed point in* F.

In [4], Chidume and Shahzad studied the strong convergence of the implicit process (1.6) to a common fixed point for a finite family of nonexpansive mappings. They proved the following results.

Theorem 1.4 [4, Theorem 3.3]. *Let* E *be a uniformly convex Banach space and let* K *be a nonempty closed convex subset of* E. *Let* $\{T_i : i \in I\}$ *be* N *nonexpansive self-mappings of* K *with* $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. *Suppose that one of the mappings in* $\{T_i : i \in I\}$ *is semi-compact. Let* $\{\alpha_n\}_{n\geq 1} \subset [\delta, 1-\delta]$ *for some* $\delta \in (0,1)$. *From arbitrary* $x_0 \in K$, *define the sequence* $\{x_n\}$ *by the implicit iteration process (1.6). Then* $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_i : i \in I\}$.

Remark 1.5. It is worth mentioning here that [13, Theorem 1] by Zhou and Chang is "for convergence of modified implicit iteration process for a finite family of asymptotically nonexpansive mappings in uniformly convex Banach spaces."

Let C be a nonempty closed convex subset of a normed space E. Inspired and motivated by the above said facts, we suggest the following implicit iteration process with errors and define the sequence $\{x_n\}$ as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n + u_n \quad \forall n \geq 1, \tag{1.7}$$

where $T_n = T_{n(\text{mod}N)}$, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{u_n\}$ is a summable sequence in C.

Clearly, this iteration process contains the process (1.6) as its special case.

The purpose of this note is to study the strong convergence of implicit iteration process (1.7) to a common fixed point for a finite family of Z-operators in normed spaces.

The following lemma is proved in [2].

Lemma 1.6. *Let* $\{r_n\}$, $\{s_n\}$, *and* $\{t_n\}$ *be sequences of nonnegative numbers satisfying*

$$r_{n+1} \leq (1-s_n) r_n + s_n t_n \quad \forall n \geq 1. \tag{1.8}$$

If $\sum_{n=1}^{\infty} s_n = \infty$ *and* $\lim_{n\to\infty} t_n = 0$, *then* $\lim_{n\to\infty} r_n = 0$.

2. Main results

Theorem 2.1. *Let* C *be a nonempty closed convex subset of a normed space* E. *Let* $\{T_1, T_2, \ldots, T_N\} : C \to C$ *be* N Z-*operators with* $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. *From arbitrary* $x_0 \in C$, *define the sequence* $\{x_n\}$ *by the implicit iteration process (1.7) satisfying* $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ *and* $\|u_n\| = 0(1-\alpha_n)$. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_1, T_2, \ldots, T_N\}$.

Proof. It follows from $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$ that the operators $\{T_1, T_2, \ldots, T_N\}$ have a common fixed point in C, say w. Consider $x, y \in C$. Since each $T_i : i \in I$ is a Z-operator, at least one of the conditions (z_1), (z_2), and (z_3) is satisfied. If (z_2) holds, then

$$\begin{aligned} \|T_i x - T_i y\| &\leq b\big[\|x - T_i x\| + \|y - T_i y\|\big] \\ &\leq b\big[\|x - T_i x\| + \|y - x\| + \|x - T_i x\| + \|T_i x - T_i y\|\big] \end{aligned} \tag{2.1}$$

implies

$$(1-b)\|T_ix - T_iy\| \leq b\|x-y\| + 2b\|x - T_ix\|, \tag{2.2}$$

which yields (using the fact that $0 \leq b < 1$)

$$\|T_ix - T_iy\| \leq \frac{b}{1-b}\|x-y\| + \frac{2b}{1-b}\|x - T_ix\|. \tag{2.3}$$

If (z_3) holds, then similarly we obtain

$$\|T_ix - T_iy\| \leq \frac{c}{1-c}\|x-y\| + \frac{2c}{1-c}\|x - T_ix\|. \tag{2.4}$$

Denote

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}. \tag{2.5}$$

Then we have $0 \leq \delta < 1$ and in view of (z_1), (2.3)–(2.5) it results that the inequality

$$\|T_ix - T_iy\| \leq \delta\|x-y\| + 2\delta\|x - T_ix\| \tag{AR}$$

holds for all $x, y \in C$.

Using (1.6), we have

$$\begin{aligned}\|x_n - w\| &= \|\alpha_n x_{n-1} + (1-\alpha_n)T_nx_n + u_n - w\| \\ &= \|\alpha_n(x_{n-1} - w) + (1-\alpha_n)(T_nx_n - w) + u_n\| \\ &\leq \alpha_n\|x_{n-1} - w\| + (1-\alpha_n)\|T_nx_n - w\| + \|u_n\|.\end{aligned} \tag{2.6}$$

Now for $y = x_n$ and $x = w$, (AR) gives

$$\|Tx_n - w\| \leq \delta\|x_n - w\|, \tag{2.7}$$

and hence, by (2.6), (2.7) we obtain

$$\|x_n - w\| \leq \frac{\alpha_n}{1-\delta(1-\alpha_n)}\|x_{n-1} - w\| + \frac{1}{1-\delta(1-\alpha_n)}\|u_n\|. \tag{2.8}$$

Let

$$\begin{aligned}A_n &= \alpha_n, \\ B_n &= 1-\delta(1-\alpha_n),\end{aligned} \tag{2.9}$$

and consider

$$\begin{aligned}\beta_n &= 1 - \frac{A_n}{B_n} = 1 - \frac{\alpha_n}{1-\delta(1-\alpha_n)} \\ &= \frac{(1-\delta)(1-\alpha_n)}{1-\delta(1-\alpha_n)} \geq (1-\delta)(1-\alpha_n).\end{aligned} \tag{2.10}$$

Indeed

$$1-\delta \leq 1-\delta(1-\alpha_n) \leq 1 \tag{2.11}$$

implies

$$\frac{A_n}{B_n} \leq 1-(1-\delta)(1-\alpha_n). \tag{2.12}$$

Thus from (2.8), we get

$$||x_n - w|| \leq [1-(1-\delta)(1-\alpha_n)]||x_{n-1}-w|| + \frac{1}{1-\delta}||u_n||. \tag{2.13}$$

With the help of Lemma 1.6 and using the fact that $0 \leq \delta < 1$, $0 < \alpha_n < 1$, $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, and $\|u_n\| = 0(1-\alpha_n)$, it results that

$$\lim_{n\to\infty}||x_n - w|| = 0. \tag{2.14}$$

Consequently $x_n \to w \in F$ and this completes the proof. □

Corollary 2.2. *Let C be a nonempty closed convex subset of a normed space E_1. Let $\{T_1, T_2, \ldots, T_N\} : C \to C$ be N operators satisfying condition Z with $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. From arbitrary $x_0 \in C$, define the sequence $\{x_n\}$ by the implicit iteration process (1.6) satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.*

Remark 2.3. (1) Chatterjea's and Kannan's contractive conditions (1.3) and (1.2) are both included in the class of Zamfirescu operators.

(2) Recently the convergence problems of an implicit (or nonimplicit) iterative process to a common fixed point of finite family of nonexpansive mappings in Hilbert spaces have been considered by several authors (see, e.g., [1, 6, 8–11, 13]).

References

[1] H. H. Bauschke, *The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space*, Journal of Mathematical Analysis and Applications **202** (1996), no. 1, 150–159.

[2] S.-S. Chang, *On Chidume's open questions and approximate solutions of multivalued strongly accretive mapping equations in Banach spaces*, Journal of Mathematical Analysis and Applications **216** (1997), no. 1, 94–111.

[3] S. K. Chatterjea, *Fixed-point theorems*, Comptes Rendus de l&Académie Bulgare des Sciences **25** (1972), 727–730.

[4] C. E. Chidume and N. Shahzad, *Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings*, Nonlinear Analysis. Theory, Methods & Applications **62** (2005), no. 6, 1149–1156.

[5] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proceedings of the American Mathematical Society **45** (1974), 267–273.

[6] B. Halpern, *Fixed points of nonexpanding maps*, Bulletin of the American Mathematical Society **73** (1967), 957–961.

[7] R. Kannan, *Some results on fixed points*, Bulletin of the Calcutta Mathematical Society **60** (1968), 71–76.

[8] P.-L. Lions, *Approximation de points fixes de contractions*, Comptes Rendus des Séances de l'Académie des Sciences. Série. A-B **284** (1977), no. 21, A1357–A1359.
[9] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, Journal of Mathematical Analysis and Applications **75** (1980), no. 1, 287–292.
[10] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Archiv der Mathematik **58** (1992), no. 5, 486–491.
[11] H.-K. Xu and R. G. Ori, *An implicit iteration process for nonexpansive mappings*, Numerical Functional Analysis and Optimization **22** (2001), no. 5-6, 767–773.
[12] T. Zamfirescu, *Fix point theorems in metric spaces*, Archiv der Mathematik **23** (1972), 292–298.
[13] Y. Zhou and S.-S. Chang, *Convergence of implicit iteration process for a finite family of asymptotically nonexpansive mappings in Banach spaces*, Numerical Functional Analysis and Optimization **23** (2002), no. 7-8, 911–921.

Arif Rafiq: Department of Mathematics, COMSATS Institute of Information Technology, Plot # 30, Sector H-8/1, 44000 Islamabad, Pakistan
E-mail address: arafiq@comsats.edu.pk