# ON TRANSFORMATION SEMIGROUPS WHICH ARE $\mathcal{BQ}$-SEMIGROUPS

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A semigroup whose bi-ideals and quasi-ideals coincide is called a $\mathcal{BQ}$-*semigroup*. The full transformation semigroup on a set $X$ and the semigroup of all linear transformations of a vector space $V$ over a field $F$ into itself are denoted, respectively, by $T(X)$ and $L_F(V)$. It is known that every regular semigroup is a $\mathcal{BQ}$-semigroup. Then both $T(X)$ and $L_F(V)$ are $\mathcal{BQ}$-semigroups. In 1966, Magill introduced and studied the subsemigroup $\overline{T}(X,Y)$ of $T(X)$, where $\varnothing \neq Y \subseteq X$ and $\overline{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}$. If $W$ is a subspace of $V$, the subsemigroup $\overline{L}_F(V,W)$ of $L_F(V)$ will be defined analogously. In this paper, it is shown that $\overline{T}(X,Y)$ is a $\mathcal{BQ}$-semigroup if and only if $Y = X$, $|Y| = 1$, or $|X| \leq 3$, and $\overline{L}_F(V,W)$ is a $\mathcal{BQ}$-semigroup if and only if (i) $W = V$, (ii) $W = \{0\}$, or (iii) $F = \mathbb{Z}_2$, $\dim_F V = 2$, and $\dim_F W = 1$.



## 1. Introduction

The cardinality of a set $A$ is denoted by $|A|$. The image of a map $\alpha$ at $x$ in the domain of $\alpha$ will be written by $x\alpha$.

An element $a$ of a semigroup $S$ is said to be *regular* if $a = aba$ for some $b \in S$, and $S$ is called a *regular semigroup* if every element of $S$ is regular. The set of all regular elements of $S$ is denoted by $\operatorname{Reg}(S)$.

The full transformation semigroup on a nonempty set $X$ is denoted by $T(X)$, that is, $T(X)$ is the semigroup of all mappings $\alpha : X \to X$ under composition. The semigroup $T(X)$ is known to be regular [4, page 4]. Magill [9] introduced and studied the subsemigroup

$$\overline{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\} \tag{1.1}$$

of $T(X)$, where $\varnothing \neq Y \subseteq X$. Note that $1_X$, the identity map on $X$, belongs to $\overline{T}(X,Y)$ and $\overline{T}(X,Y)$ contains $T(X,Y)$ as a subsemigroup, where $T(X,Y) = \{\alpha \in T(X) \mid \operatorname{ran}\alpha \subseteq Y\}$ and $\operatorname{ran}\alpha$ denotes the range of $\alpha$. The semigroup $T(X,Y)$ was introduced and studied by Symons [13].

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For a vector space $V$ over a field $F$, let $L_F(V)$ be the semigroup of all linear transformations $\alpha : V \to V$ under composition. It is known that $L_F(V)$ is a regular semigroup [5, page 63]. For a subspace $W$ of $V$, we define the subsemigroup $\overline{L}_F(V,W)$ of $L_F(V)$ analogously, that is,

$$\overline{L}_F(V,W) = \{\alpha \in L_F(V) \mid W\alpha \subseteq W\}. \tag{1.2}$$

Clearly, $1_V \in \overline{L}_F(V,W)$ and 0, the zero map on $V$, also belongs to $\overline{L}_F(V,W)$. In addition, $\overline{L}_F(V,W)$ contains $L_F(V,W) = \{\alpha \in L_F(V) \mid \operatorname{ran}\alpha \subseteq W\}$ as a subsemigroup.

A subsemigroup $Q$ of a semigroup $S$ is called a *quasi-ideal* of $S$ if $SQ \cap QS \subseteq Q$, and a *bi-ideal* of $S$ is a subsemigroup $B$ of $S$ such that $BSB \subseteq B$. The notions of quasi-ideal and bi-ideal for semigroups were introduced by Steinfeld [11] and Good and Hughes [3], respectively. Both quasi-ideals and bi-ideals are generalizations of one-sided ideals, and bi-ideals also generalize quasi-ideals. For a nonempty subset $A$ of $S$, let $(A)_q$ and $(A)_b$ be the quasi-ideal and the bi-ideal of $S$ generated by $A$, respectively, that is, $(A)_q[(A)_b]$ is the intersection of all quasi-ideals (bi-ideals) of $S$ containing $A$ [12, pages 10, 12]. Observe that $(A)_b \subseteq (A)_q$.

Proposition 1.1 [2, pages 84, 85]. *For a nonempty subset $A$ of a semigroup $S$,*
(i) $(A)_q = S^1A \cap AS^1$,
(ii) $(A)_b = AS^1A \cup A$.

Kapp [6] used $\mathfrak{B}\mathfrak{Q}$ to denote the class of all semigroups whose bi-ideals and quasi-ideals coincide and Mielke [10] called a semigroup in $\mathfrak{B}\mathfrak{Q}$ a *$\mathfrak{B}\mathfrak{Q}$-semigroup*. Important $\mathfrak{B}\mathfrak{Q}$-semigroups are the following ones.

Proposition 1.2 [8]. *Every regular semigroup is a $\mathfrak{B}\mathfrak{Q}$-semigroup.*

Proposition 1.3 [6]. *Every left (right) simple semigroup or every left (right) 0-simple semigroup is a $\mathfrak{B}\mathfrak{Q}$-semigroup.*

Recall that a semigroup $S$ is *left (right) simple* if $S$ has no proper left (right) ideal, and a semigroup $S$ with 0 is called *left (right) 0-simple* if $S^2 \neq \{0\}$ and $S$ has no proper nonzero left (right) ideal. Kemprasit showed in [7] that if $X$ is an infinite set, then the subsemigroup $\{\alpha \in T(X) \mid X \setminus \operatorname{ran}\alpha \text{ is infinite}\}$ of $T(X)$ is a $\mathfrak{B}\mathfrak{Q}$-semigroup but it is neither regular nor left (right) simple. In fact, $\mathfrak{B}\mathfrak{Q}$-semigroups have been characterized by Calais [1] as follows.

Proposition 1.4 [1]. *A semigroup $S$ is a $\mathfrak{B}\mathfrak{Q}$-semigroup if and only if $(x,y)_b = (x,y)_q$ for all $x,y \in S$.*

Every bi-ideal of a regular semigroup is a $\mathfrak{B}\mathfrak{Q}$-semigroup. The proof is rather simple and is as follows: let $T$ be a bi-ideal of a regular semigroup $S$ and $B$ a bi-ideal of $T$. Then $TST \subseteq T$ and $BTB \subseteq B$. Let $x \in TB \cap BT$. Since $S$ is regular, $x = xsx$ for some $s \in S$ which implies that $x = xsx \in BTsTB \subseteq BTSTB \subseteq BTB \subseteq B$. Thus $TB \cap BT \subseteq B$. Hence $B$ is a quasi-ideal of $T$, as desired. Since $T(X,Y)$ and $L_F(V,W)$ are left ideals of $T(X)$ and $L_F(V)$, respectively, it follows that $T(X,Y)$ and $L_F(V,W)$ are always $\mathfrak{B}\mathfrak{Q}$-semigroups. However, the semigroups $\overline{T}(X,Y)$ and $\overline{L}_F(V,W)$ need not be $\mathfrak{B}\mathfrak{Q}$-semigroups. Notice that if $X$ is infinite, then the semigroup $\{\alpha \in T(X) \mid X \setminus \operatorname{ran}\alpha \text{ is infinite}\}$ is a left ideal of $T(X)$. Similarly, if $V$ has infinite dimension over $F$, then the semigroup $\{\alpha \in L_F(V) \mid \dim_F(V/\operatorname{ran}\alpha) \text{ is infinite}\}$ is a left ideal of $L_F(V)$.

In Section 2, we give a necessary and sufficient condition for $\overline{T}(X,Y)$ to be a $\mathcal{BQ}$-semigroup in terms of $|X|$ and $|Y|$. In Section 3, a necessary and sufficient condition for $\overline{L}_F(V,W)$ to be a $\mathcal{BQ}$-semigroup is given in terms of $|F|$, $\dim_F V$, and $\dim_F W$.

In the remainder, let $X$ be a nonempty set, $\varnothing \neq Y \subseteq X$, $V$ a vector space over a field $F$, and $W$ a subspace of $V$.

## 2. The semigroup $\overline{T}(X,Y)$

We begin this section by characterizing regular elements of the semigroup $\overline{T}(X,Y)$. Then it is shown that $\overline{T}(X,Y)$ is a regular semigroup if and only if $Y = X$ or $Y$ contains only one element.

Proposition 2.1. *The following statements hold for the semigroup* $\overline{T}(X,Y)$.

(i) *For* $\alpha \in \overline{T}(X,Y)$, $\alpha \in \operatorname{Reg}(\overline{T}(X,Y))$ *if and only if* $\operatorname{ran}\alpha \cap Y = Y\alpha$.

(ii) *The semigroup* $\overline{T}(X,Y)$ *is regular if and only if either* $Y = X$ *or* $|Y| = 1$.

*Proof.* (i) Since $Y\alpha \subseteq Y$, we have $Y\alpha \subseteq \operatorname{ran}\alpha \cap Y$. Assume that $\alpha = \alpha\beta\alpha$ for some $\beta \in \overline{T}(X,Y)$. If $x \in \operatorname{ran}\alpha \cap Y$, then $x \in Y$ and $x = a\alpha$ for some $a \in X$ which imply that $x = a\alpha = a\alpha\beta\alpha = x\beta\alpha \in Y\beta\alpha \subseteq Y\alpha$. Hence we have $\operatorname{ran}\alpha \cap Y = Y\alpha$.

Conversely, assume that $\operatorname{ran}\alpha \cap Y = Y\alpha$. Then for each $x \in \operatorname{ran}\alpha \cap Y$, we have $x\alpha^{-1} \cap Y \neq \varnothing$. We choose an element $x' \in x\alpha^{-1} \cap Y$ for each $x \in \operatorname{ran}\alpha \cap Y$. Also, for $x \in \operatorname{ran}\alpha \setminus Y$, choose an element $\overline{x} \in x\alpha^{-1}$. Then $x'\alpha = x$ for all $x \in \operatorname{ran}\alpha \cap Y$ and $\overline{x}\alpha = x$ for all $x \in \operatorname{ran}\alpha \setminus Y$. Let $a$ be a fixed element in $Y$ and define $\beta : X \to X$ by a bracket notation as follows:

$$\beta = \begin{bmatrix} x & t & X \setminus \operatorname{ran}\alpha \\ x' & \overline{t} & a \end{bmatrix}_{\substack{x \in \operatorname{ran}\alpha \cap Y \\ t \in \operatorname{ran}\alpha \setminus Y.}} \tag{2.1}$$

Then $Y\beta \subseteq \{x' \mid x \in \operatorname{ran}\alpha \cap Y\} \cup \{a\} \subseteq Y$, and for $x \in X$,

$$x\alpha\beta\alpha = (x\alpha)\beta\alpha = \begin{cases} (x\alpha)'\alpha = x\alpha & \text{if } x\alpha \in \operatorname{ran}\alpha \cap Y, \\ (\overline{x\alpha})\alpha = x\alpha & \text{if } x\alpha \in \operatorname{ran}\alpha \setminus Y. \end{cases} \tag{2.2}$$

Hence $\beta \in \overline{T}(X,Y)$ and $\alpha = \alpha\beta\alpha$.

(ii) Suppose that $Y \subsetneq X$ and $|Y| > 1$. Let $a$ and $b$ be two distinct elements of $Y$. Define $\alpha : X \to X$ by

$$\alpha = \begin{bmatrix} Y & X \setminus Y \\ a & b \end{bmatrix}. \tag{2.3}$$

Then $\operatorname{ran}\alpha = \{a,b\} \subseteq Y$, so $\alpha \in \overline{T}(X,Y)$ and $\operatorname{ran}\alpha \cap Y = \{a,b\} \neq \{a\} = Y\alpha$. It follows from (i) that $\alpha \notin \operatorname{Reg}(\overline{T}(X,Y))$. Hence $\overline{T}(X,Y)$ is not a regular semigroup.

If $Y = X$, then $\overline{T}(X,Y) = T(X)$ which is regular. Next, assume that $Y = \{c\}$. In this case, $\overline{T}(X,Y)$ is isomorphic to the semigroup $P(X \smallsetminus Y)$ consisting of all partial transformations of $X \smallsetminus Y$, via the map $P(X \smallsetminus Y) \to \overline{T}(X,Y)$, $\alpha \mapsto \overline{\alpha}$, where

$$\overline{\alpha} = \begin{bmatrix} x & X \smallsetminus \operatorname{dom}\alpha \\ x\alpha & c \end{bmatrix}_{x \in \operatorname{dom}\alpha.} \tag{2.4}$$

It is well known that $P(X \smallsetminus Y)$ is regular [4, page 4]. Hence $\overline{T}(X,Y)$ is a regular semigroup, as required. □

To characterize when $\overline{T}(X,Y)$ is a $\mathcal{BQ}$-semigroup, Propositions 1.1, 1.2, 1.4, and 2.1 and the following three lemmas are needed.

Lemma 2.2. *Let $S$ be a semigroup. If $\varnothing \neq A \subseteq \operatorname{Reg}(S)$, then $(A)_b = (A)_q$.*

*Proof.* We know that $(A)_b \subseteq (A)_q$. Let $x \in (A)_q$. By Proposition 1.1(i), $x = sa = bt$ for some $s,t \in S^1$ and $a,b \in A$. Since $a \in \operatorname{Reg}(S)$, $a = aa'a$ for some $a' \in S$. Then

$$x = sa = saa'a = bta'a \in ASA \subseteq (A)_b \tag{2.5}$$

by Proposition 1.1(ii). Hence we have $(A)_b = (A)_q$, as desired. □

Lemma 2.3. *Let $S$ be a semigroup, let $\varnothing \neq A \subseteq S$, and let $B \subseteq \operatorname{Reg}(S)$. If $(A)_b = (A)_q$, then $(A \cup B)_b = (A \cup B)_q$.*

*Proof.* We first show that $S^1A \cap BS^1$ and $S^1B \cap AS^1$ are subsets of $(A \cup B)_b$. Let $x \in S^1A \cap BS^1$. Then $x = sa = bt$ for some $s,t \in S^1$, $a \in A$, and $b \in B$. Since $b \in \operatorname{Reg}(S)$, $b = bb'b$ for some $b' \in S$. It follows that

$$x = bt = bb'bt = bb'sa \in BSA \subseteq (A \cup B)S(A \cup B) \subseteq (A \cup B)_b. \tag{2.6}$$

This shows that $S^1A \cap BS^1 \subseteq (A \cup B)_b$. It can be shown similarly that $S^1B \cap AS^1 \subseteq (A \cup B)_b$. Consequently,

$$\begin{aligned}
(A \cup B)_q &= S^1(A \cup B) \cap (A \cup B)S^1 \\
&= (S^1A \cup S^1B) \cap (AS^1 \cup BS^1) \\
&= (S^1A \cap AS^1) \cup (S^1A \cap BS^1) \cup (S^1B \cap AS^1) \cup (S^1B \cap BS^1) \\
&= (A)_q \cup (S^1A \cap BS^1) \cup (S^1B \cap AS^1) \cup (B)_q \\
&= (A)_b \cup (S^1A \cap BS^1) \cup (S^1B \cap AS^1) \cup (B)_b, \\
&\qquad \text{from the assumption and Lemma 2.2,} \\
&\subseteq (A)_b \cup (A \cup B)_b \cup (A \cup B)_b \cup (B)_b = (A \cup B)_b.
\end{aligned} \tag{2.7}$$

But $(A \cup B)_b \subseteq (A \cup B)_q$, so $(A \cup B)_b = (A \cup B)_q$. □

Lemma 2.4. *If $|X| = 3$ and $|Y| = 2$, then for all $\alpha,\beta \in \overline{T}(X,Y)$, $(\alpha,\beta)_b = (\alpha,\beta)_q$ in $\overline{T}(X,Y)$.*

*Proof.* For convenience, let $X_a$ denote the constant map whose domain and range are $X$ and $\{a\}$, respectively.

Assume that $X = \{a,b,c\}$ and $Y = \{a,b\}$. Clearly,

$$\overline{T}(X,Y) = \left\{1_X, X_a, X_b, \begin{bmatrix} a & b & c \\ a & a & b \end{bmatrix}, \begin{bmatrix} a & b & c \\ a & a & c \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & b & a \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & b & c \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} a & b & c \\ a & b & a \end{bmatrix}, \begin{bmatrix} a & b & c \\ a & b & b \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & a & a \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & a & b \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & a & c \end{bmatrix} \right\}. \tag{2.8}$$

By Proposition 2.1(i), $\overline{T}(X,Y) \setminus \operatorname{Reg}(\overline{T}(X,Y)) = \{[\begin{smallmatrix} a & b & c \\ a & a & b \end{smallmatrix}], [\begin{smallmatrix} a & b & c \\ b & b & a \end{smallmatrix}]\}$. Let $\lambda = [\begin{smallmatrix} a & b & c \\ a & a & b \end{smallmatrix}]$ and $\eta = [\begin{smallmatrix} a & b & c \\ b & b & a \end{smallmatrix}]$. Note that $\lambda^2 = X_a = \eta\lambda$ and $\eta^2 = X_b = \lambda\eta$. To show that $(\alpha,\beta)_b = (\alpha,\beta)_q$ for all $\alpha,\beta \in \overline{T}(X,Y)$, by Lemma 2.3, it suffices to show that $(\lambda)_b = (\lambda)_q$, $(\eta)_b = (\eta)_q$, and $(\lambda, \eta)_b = (\lambda,\eta)_q$. By direct multiplication, we have

$$\begin{gathered} \overline{T}(X,Y)\lambda = \{\lambda, X_a\}, \qquad \lambda\overline{T}(X,Y) = \{\lambda, X_a, X_b, \eta\}, \qquad \lambda\overline{T}(X,Y)\lambda = \{X_a\}, \\ \overline{T}(X,Y)\eta = \{\eta, X_b\}, \qquad \eta\overline{T}(X,Y) = \{\eta, X_a, X_b, \lambda\}, \qquad \eta\overline{T}(X,Y)\eta = \{X_b\}, \\ \lambda\overline{T}(X,Y)\eta = \{X_b\}, \qquad \eta\overline{T}(X,Y)\lambda = \{X_a\}. \end{gathered} \tag{2.9}$$

Hence

$$\begin{aligned} (\lambda)_b &= \lambda\overline{T}(X,Y)\lambda \cup \{\lambda\} = \{X_a, \lambda\} = \overline{T}(X,Y)\lambda \cap \lambda\overline{T}(X,Y) = (\lambda)_q, \\ (\eta)_b &= \eta\overline{T}(X,Y)\eta \cup \{\eta\} = \{X_b, \eta\} = \overline{T}(X,Y)\eta \cap \eta\overline{T}(X,Y) = (\eta)_q, \\ (\lambda,\eta)_b &= \{\lambda,\eta\}\overline{T}(X,Y)\{\lambda,\eta\} \cup \{\lambda,\eta\} \\ &= \lambda\overline{T}(X,Y)\lambda \cup \lambda\overline{T}(X,Y)\eta \cup \eta\overline{T}(X,Y)\lambda \cup \eta\overline{T}(X,Y)\eta \cup \{\lambda,\eta\} \\ &= \{X_a, X_b, \lambda, \eta\}, \\ (\lambda,\eta)_q &= \overline{T}(X,Y)\{\lambda,\eta\} \cap \{\lambda,\eta\}\overline{T}(X,Y) \\ &= (\overline{T}(X,Y)\lambda \cup \overline{T}(X,Y)\eta) \cap (\lambda\overline{T}(X,Y) \cup \eta\overline{T}(X,Y)) \\ &= \{\lambda, X_a, \eta, X_b\} = (\lambda,\eta)_b. \end{aligned} \tag{2.10}$$

□

Theorem 2.5. *The semigroup $\overline{T}(X,Y)$ is a $\mathcal{BQ}$-semigroup if and only if one of the following statements holds.*

(i) $Y = X$.

(ii) $|Y| = 1$.

(iii) $|X| \leq 3$.

*Proof.* Assume that (i), (ii), and (iii) are false. Then $X \setminus Y \neq \varnothing$, $|Y| > 1$, and $|X| > 3$.
*Case 1* (($|Y| = 2$)). Let $Y = \{a,b\}$. Since $|X| > 3$, $|X \setminus Y| > 1$. Let $c \in X \setminus Y$. Then $X \setminus \{a,b,c\} \neq \varnothing$. Define $\alpha,\beta,\gamma \in \overline{T}(X,Y)$ by

$$\alpha = \begin{bmatrix} a & b & c & X\setminus\{a,b,c\} \\ b & b & a & c \end{bmatrix}, \qquad \beta = \begin{bmatrix} c & x \\ a & x \end{bmatrix}_{x\in X\setminus\{c\}}, \qquad \gamma = \begin{bmatrix} a & b & X\setminus\{a,b\} \\ b & b & c \end{bmatrix}. \tag{2.11}$$

Then $a\alpha\beta = b = a\gamma\alpha$, $b\alpha\beta = b = b\gamma\alpha$, $c\alpha\beta = a = c\gamma\alpha$, and $(X\setminus\{a,b,c\})\alpha\beta = \{a\} = (X \setminus \{a,b,c\})\gamma\alpha \neq (X\setminus\{a,b,c\})\alpha$, so $\alpha \neq \alpha\beta = \gamma\alpha \in (\alpha)_q$ by Proposition 1.1(i). If $\alpha\beta \in (\alpha)_b$, then by Proposition 1.1(ii), $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \overline{T}(X,Y)$. Hence we have $a = c\alpha\beta = c\alpha\eta\alpha = (a\eta)\alpha$. This implies that $a\eta = c$ which is contrary to $a \in Y$ and $c \in X \setminus Y$. Thus $(\alpha)_b \neq (\alpha)_q$, so by Proposition 1.4, $\overline{T}(X,Y)$ is not a $\mathcal{BQ}$-semigroup.
*Case 2* (($|Y| > 2$)). Let $a$, $b$, $c$ be distinct elements of $Y$. Let $\alpha,\beta,\gamma \in \overline{T}(X,Y)$ be defined by

$$\alpha = \begin{bmatrix} a & Y\setminus\{a\} & X\setminus Y \\ b & a & c \end{bmatrix}, \qquad \beta = \begin{bmatrix} a & b & x \\ b & a & x \end{bmatrix}_{x\in X\setminus\{a,b\}}, \qquad \gamma = \begin{bmatrix} a & Y\setminus\{a\} & x \\ c & a & x \end{bmatrix}_{x\in X\setminus Y.} \tag{2.12}$$

Then $a\alpha\beta = a = a\gamma\alpha \neq a\alpha$, $(Y\setminus\{a\})\alpha\beta = \{b\} = (Y\setminus\{a\})\gamma\alpha$, and $(X\setminus Y)\alpha\beta = \{c\} = (X\setminus Y)\gamma\alpha$. Thus $\alpha \neq \alpha\beta = \gamma\alpha \in (\alpha)_q$. If $\alpha\beta \in (\alpha)_b$, then $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \overline{T}(X,Y)$. Therefore we have for every $x \in X\setminus Y$, $c = x\alpha\beta = x\alpha\eta\alpha = (c\eta)\alpha$ which implies that $c\eta \in X\setminus Y$. This is a contradiction since $c \in Y$. Hence $(\alpha)_b \neq (\alpha)_q$, and so by Proposition 1.4, $\overline{T}(X,Y)$ is not a $\mathcal{BQ}$-semigroup.

If $Y = X$ or $|Y| = 1$, then $\overline{T}(X,Y)$ is regular by Proposition 2.1(ii) which implies by Proposition 1.2 that $\overline{T}(X,Y)$ is a $\mathcal{BQ}$-semigroup. If $|X| = 3$ and $|Y| = 2$, then by Lemma 2.4 and Proposition 1.4, $\overline{T}(X,Y)$ is a $\mathcal{BQ}$-semigroup.

Hence the theorem is completely proved. □

Two direct consequences of Propositions 1.2, 2.1(ii), Theorem 2.5, and the proof of Lemma 2.4 are as follows.

Corollary 2.6. *If $|X| \neq 3$, then the following statements are equivalent.*
(i) *$\overline{T}(X,Y)$ is a $\mathcal{BQ}$-semigroup.*
(ii) *$Y = X$ or $|Y| = 1$.*
(iii) *$\overline{T}(X,Y)$ is a regular semigroup.*

Corollary 2.7. *The semigroup $\overline{T}(X,Y)$ is a nonregular $\mathcal{BQ}$-semigroup if and only if $|X| = 3$ and $|Y| = 2$. Hence for each set $X$ with $|X| = 3$, there are exactly 3 semigroups $\overline{T}(X,Y)$ which are nonregular $\mathcal{BQ}$-semigroups, and each of such $\overline{T}(X,Y)$ contains 12 elements.*

*Remark 2.8.* We have mentioned that $T(X,Y)$ is a left ideal of $T(X)$. But for $\alpha \in T(X,Y)$ and $\beta \in \overline{T}(X,Y)$, $X\alpha\beta \subseteq Y\beta \subseteq Y$, so $T(X,Y)$ is an ideal of $\overline{T}(X,Y)$. We have $1_X \in \overline{T}(X,Y) \setminus T(X,Y)$ if $Y \neq X$. Hence if $Y \neq X$, then $\overline{T}(X,Y)$ is neither left nor right simple.

Therefore we deduce from Corollary 2.7 that if $|X| = 3$ and $|Y| = 2$, then $\overline{T}(X,Y)$ is an example of $\mathcal{BQ}$-semigroup which is neither regular nor left (right) simple (see Propositions 1.2 and 1.3).

## 3. The semigroup $\overline{L}_F(V,W)$

In this section, we give a necessary and sufficient condition for $\overline{L}_F(V,W)$ to be a $\mathcal{BQ}$-semigroup. We first provide the conditions of the regularity of elements of $\overline{L}_F(V,W)$ and of the semigroup $\overline{L}_F(V,W)$. The following facts about vector spaces and linear transformations will be used. If $U_1$ and $U_2$ are subspaces of $V$, $B_1$ is a basis of the subspace $U_1 \cap U_2$, $B_2 \subseteq U_1$ and $B_3 \subseteq U_2$ are such that $B_1 \cup B_2$ and $B_1 \cup B_3$ are bases of $U_1$ and $U_2$, respectively, then $B_1 \cup B_2 \cup B_3$ is a basis of the subspace $U_1 + U_2$ of $V$. If $\alpha \in L_F(V)$, $B_1$ is a basis of $\ker\alpha$, $B_2$ is a basis of $\operatorname{ran}\alpha$, and choose an element $u' \in u\alpha^{-1}$ for every $u \in B_2$, then $B_1 \cup \{u' \mid u \in B_2\}$ is a basis of $V$.

Proposition 3.1. *The following statements hold for the semigroup $\overline{L}_F(V,W)$.*
(i) *For $\alpha \in \overline{L}_F(V,W)$, $\alpha \in \operatorname{Reg}(\overline{L}_F(V,W))$ if and only if $\operatorname{ran}\alpha \cap W = W\alpha$.*
(ii) *The semigroup $\overline{L}_F(V,W)$ is regular if and only if either $W = V$ or $W = \{0\}$.*

*Proof.* (i) The proof that $\alpha \in \operatorname{Reg}(\overline{L}_F(V,W))$ implies $\operatorname{ran}\alpha \cap W = W\alpha$ is analogous to the proof of the "only if" part of Proposition 2.1(i).

Conversely, assume that $\operatorname{ran}\alpha \cap W = W\alpha$. Let $B_1$ be a basis of $\operatorname{ran}\alpha \cap W$, $B_2 \subseteq \operatorname{ran}\alpha \smallsetminus B_1$, and $B_3 \subseteq W \smallsetminus B_1$ such that $B_1 \cup B_2$ and $B_1 \cup B_3$ are bases of $\operatorname{ran}\alpha$ and $W$, respectively. Then $B_1 \cup B_2 \cup B_3$ is a basis of $\operatorname{ran}\alpha + W$. Let $B_4 \subseteq V \smallsetminus (B_1 \cup B_2 \cup B_3)$ be such that $B_1 \cup B_2 \cup B_3 \cup B_4$ is a basis of $V$. Since $B_1 \subseteq \operatorname{ran}\alpha \cap W = W\alpha$, we have $u\alpha^{-1} \cap W \neq \varnothing$ for every $u \in B_1$. For each $u \in B_1$, choose an element $u' \in u\alpha^{-1} \cap W$. Since $B_2 \subseteq \operatorname{ran}\alpha$, for each $v \in B_2$, $v\alpha^{-1} \neq \varnothing$, so choose an element $\overline{v} \in v\alpha^{-1}$. Define $\beta \in L_F(V)$ on the basis $B_1 \cup B_2 \cup B_3 \cup B_4$ by

$$\beta = \begin{bmatrix} u & v & B_3 \cup B_4 \\ u' & \overline{v} & 0 \end{bmatrix}_{\substack{u \in B_1 \\ v \in B_2.}} \tag{3.1}$$

It follows that $W\beta = \langle B_1 \cup B_3 \rangle \beta = \langle \{u' \mid u \in B_1\} \rangle \subseteq W$, so $\beta \in \overline{L}_F(V,W)$. Let $B_0$ be a basis of $\ker\alpha$. Then $B_0 \cup \{u' \mid u \in B_1\} \cup \{\overline{v} \mid v \in B_2\}$ is a basis of $V$. Since

$$\begin{gathered} B_0\alpha\beta\alpha = \{0\} = B_0\alpha, \quad u'\alpha\beta\alpha = u\beta\alpha = u'\alpha \quad \forall u \in B_1, \\ \overline{v}\alpha\beta\alpha = v\beta\alpha = \overline{v}\alpha \quad \forall v \in B_2, \end{gathered} \tag{3.2}$$

we have $\alpha = \alpha\beta\alpha$, so $\alpha$ is a regular element of $\overline{L}_F(V,W)$.

(ii) Assume that $\{0\} \neq W \subsetneq V$. Let $B_1$ be a basis of $W$ and $B$ a basis of $V$ containing $B_1$. Then $B_1 \neq \varnothing \neq B \smallsetminus B_1$. Let $w \in B_1$ and $u \in B \smallsetminus B_1$. Define $\alpha \in L_F(V)$ by

$$\alpha = \begin{bmatrix} u & B \smallsetminus \{u\} \\ w & 0 \end{bmatrix}. \tag{3.3}$$

Then $W\alpha = \langle B_1\rangle\alpha \subseteq \langle B \smallsetminus \{u\}\rangle\alpha = \{0\}$, so $\alpha \in \overline{L}_F(V,W)$. Since $\operatorname{ran}\alpha \cap W = \langle w\rangle \neq \{0\} = W\alpha$, by (i), we deduce that $\alpha$ is not a regular element of $\overline{L}_F(V,W)$. Hence $\overline{L}_F(V,W)$ is not a regular semigroup.

Since $\overline{L}_F(V,V) = L_F(V) = \overline{L}_F(V,\{0\})$, the converse holds. □

To prove the main theorem, the following lemma is also needed. Lemma 2.3 and Proposition 3.1(i) are useful to obtain this result.

Lemma 3.2. *If $F = \mathbb{Z}_2$, $\dim_F V = 2$, and $\dim_F W = 1$, then for all $\alpha,\beta \in \overline{L}_F(V,W)$, $(\alpha,\beta)_b = (\alpha,\beta)_q$ in $\overline{L}_F(V,W)$.*

*Proof.* Let $\{w\}$ be a basis of $W$ and $\{w,u\}$ a basis of $V$. Since $F = \mathbb{Z}_2$, it follows that $W = \{0,w\}$ and $V = \{0,w,u,u+w\}$. Clearly, both $\{u,u+w\}$ and $\{w,u+w\}$ are also bases of $V$. Thus $\langle w\rangle \cap \langle u\rangle = \langle w\rangle \cap \langle u+w\rangle = \langle u\rangle \cap \langle u+w\rangle = \{0\}$. All elements of $\overline{L}_F(V,W)$ defined on the basis $\{w,u\}$ of $V$ can be given as follows:

$$\overline{L}_F(V,W) = \left\{0, 1_V, \begin{bmatrix} w & u \\ 0 & w \end{bmatrix}, \begin{bmatrix} w & u \\ 0 & u \end{bmatrix}, \begin{bmatrix} w & u \\ 0 & w+u \end{bmatrix}, \begin{bmatrix} w & u \\ w & 0 \end{bmatrix}, \begin{bmatrix} w & u \\ w & w \end{bmatrix}, \begin{bmatrix} w & u \\ w & w+u \end{bmatrix}\right\}. \tag{3.4}$$

By Proposition 3.1(i), $\overline{L}_F(V,W) \smallsetminus \operatorname{Reg}(\overline{L}_F(V,W)) = \{\left[\begin{smallmatrix} w & u \\ 0 & w \end{smallmatrix}\right]\}$. Let $\lambda = \left[\begin{smallmatrix} w & u \\ 0 & w \end{smallmatrix}\right]$. Note that $\lambda^2 = 0$. To prove the lemma, by Lemma 2.3, it suffices to show that $(\lambda)_b = (\lambda)_q$. By direct multiplication, we have

$$\overline{L}_F(V,W)\lambda = \{0,\lambda\}, \qquad \lambda\overline{L}_F(V,W) = \{0,\lambda\}, \qquad \lambda\overline{L}_F(V,W)\lambda = \{0\}. \tag{3.5}$$

Consequently, $(\lambda)_b = \lambda\overline{L}_F(V,W)\lambda \cup \{\lambda\} = \{0,\lambda\} = \overline{L}_F(V,W)\lambda \cap \lambda\overline{L}_F(V,W) = (\lambda)_q$. □

Theorem 3.3. *The semigroup $\overline{L}_F(V,W)$ is a $\mathcal{BQ}$-semigroup if and only if one of the following statements holds.*

(i) $W = V$.
(ii) $W = \{0\}$.
(iii) $F = \mathbb{Z}_2$, $\dim_F V = 2$, *and* $\dim_F W = 1$.

*Proof.* Assume that (i), (ii), and (iii) are false. Then (1) $\{0\} \neq W \subsetneq V$ and (2) $F \neq \mathbb{Z}_2$, $\dim_F V > 2$, or $\dim_F W > 1$. Let $B_1$ be a basis of $W$ and $B$ a basis of $V$ containing $B_1$. Then $B_1 \neq \varnothing$ and $B \smallsetminus B_1 \neq \varnothing$.

*Case 1* (($F \neq \mathbb{Z}_2$)). Let $a \in F \smallsetminus \{0,1\}$, $w \in B_1$, and $u \in B \smallsetminus B_1$. Define $\alpha,\beta,\gamma \in \overline{L}_F(V,W)$ by

$$\alpha = \begin{bmatrix} u & B\smallsetminus\{u\} \\ w & 0 \end{bmatrix}, \qquad \beta = \begin{bmatrix} w & B\smallsetminus\{w\} \\ aw & 0 \end{bmatrix}, \qquad \gamma = \begin{bmatrix} u & B\smallsetminus\{u\} \\ au & 0 \end{bmatrix}. \tag{3.6}$$

Then we have $\alpha\beta = \left[\begin{smallmatrix} u & B\smallsetminus\{u\} \\ aw & 0 \end{smallmatrix}\right] = \gamma\alpha$. Since $a \neq 1$, we have $\alpha\beta \neq \alpha$. By Proposition 1.1(i), $\alpha\beta \in \overline{L}_F(V,W)(\alpha)_q$. Suppose that $\alpha\beta \in \overline{L}_F(V,W)(\alpha)_b$. By Proposition 1.1(ii), $\alpha\beta = \alpha\eta\alpha$

for some $\eta \in \overline{L}_F(V,W)$. Then $aw = u\alpha\beta = u\alpha\eta\alpha = (w\eta)\alpha$. But $w\eta \in W$ and $W\alpha = \langle B_1\rangle\alpha \subseteq \langle B \setminus \{u\}\rangle\alpha = \{0\}$, so $aw = 0$ which is contrary to $a \neq 0$. Thus $(\alpha)_q \neq (\alpha)_b$, so $\overline{L}_F(V,W)$ is not a $\mathcal{BQ}$-semigroup by Proposition 1.4.

*Case 2* $((\dim_F W > 1))$. Then $|B_1| > 1$. Let $w_1, w_2 \in B_1$ be such that $w_1 \neq w_2$ and $u \in B \setminus B_1$. Define $\alpha, \beta, \gamma \in \overline{L}_F(V,W)$ by

$$\alpha = \begin{bmatrix} w_1 & u & B \setminus \{w_1, u\} \\ w_2 & w_1 & 0 \end{bmatrix}, \qquad \beta = \begin{bmatrix} w_1 & B \setminus \{w_1\} \\ w_1 & 0 \end{bmatrix}, \qquad \gamma = \begin{bmatrix} u & B \setminus \{u\} \\ u & 0 \end{bmatrix}. \tag{3.7}$$

Then $\alpha\beta = \left[\begin{smallmatrix} u & B\setminus\{u\} \\ w_1 & 0 \end{smallmatrix}\right] = \gamma\alpha \neq \alpha$, so $\alpha\beta \in (\alpha)_q$. If $\alpha\beta \in (\alpha)_b$, then $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \overline{L}_F(V,W)$. Thus $w_1 = u\alpha\beta = u\alpha\eta\alpha = (w_1\eta)\alpha$. Since $w_1\eta \in W = \langle B_1\rangle$, we have $w_1\eta = aw_1 + v$ for some $a \in F$ and $v \in \langle B_1 \setminus \{w_1\}\rangle$. But $B_1 \setminus \{w_1\} \subseteq B \setminus \{w_1, u\}$, so $v\alpha = 0$. Consequently, $w_1 = (aw_1 + v)\alpha = aw_2$ which is contrary to the independence of $w_1$ and $w_2$. By Proposition 1.4, $\overline{L}_F(V,W)$ is not a $\mathcal{BQ}$-semigroup.

*Case 3* $((\dim_F V > 2$ and $\dim_F W = 1))$. Then $|B_1| = 1$ and $|B \setminus B_1| > 1$. Let $B_1 = \{w\}$ and $u_1, u_2 \in B \setminus B_1$ be such that $u_1 \neq u_2$. Let $\alpha, \beta, \gamma \in \overline{L}_F(V,W)$ be defined by

$$\alpha = \begin{bmatrix} u_1 & u_2 & B \setminus \{u_1, u_2\} \\ w & u_1 & 0 \end{bmatrix}, \qquad \beta = \begin{bmatrix} w & B \setminus \{w\} \\ w & 0 \end{bmatrix}, \qquad \gamma = \begin{bmatrix} u_1 & B \setminus \{u_1\} \\ u_1 & 0 \end{bmatrix}. \tag{3.8}$$

Then we have $\alpha\beta = \left[\begin{smallmatrix} u_1 & B\setminus\{u_1\} \\ w & 0 \end{smallmatrix}\right] = \gamma\alpha \neq \alpha$, so $\alpha\beta \in (\alpha)_q$. Suppose that $\alpha\beta \in (\alpha)_b$. It follows that $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \overline{L}_F(V,W)$. Thus $w = u_1\alpha\beta = u_1\alpha\eta\alpha = (w\eta)\alpha$. But $w\eta \in W = \langle w\rangle$ and $w\alpha = 0$, so $w = (w\eta)\alpha = 0$, a contradiction. Hence $(\alpha)_q \neq (\alpha)_b$, so $\overline{L}_F(V,W)$ is not a $\mathcal{BQ}$-semigroup, as before.

For the converse, if (i) or (ii) holds, then $\overline{L}_F(V,W) = L_F(V)$ which is a $\mathcal{BQ}$-semigroup by Proposition 1.2. If (iii) holds, then $\overline{L}_F(V,W)$ is a $\mathcal{BQ}$-semigroup by Proposition 1.4 and Lemma 3.2. □

The following corollaries follow directly from Propositions 1.2, 3.1(ii), Theorem 3.3, and the proof of Lemma 3.2.

Corollary 3.4. *If* $F \neq \mathbb{Z}_2$ *or* $\dim_F V \neq 2$, *then the following statements are equivalent.*

(i) $\overline{L}_F(V,W)$ *is a* $\mathcal{BQ}$*-semigroup.*

(ii) $W = V$ *or* $W = \{0\}$.

(iii) $\overline{L}_F(V,W)$ *is a regular semigroup.*

Corollary 3.5. *The semigroup* $\overline{L}_F(V,W)$ *is a nonregular* $\mathcal{BQ}$*-semigroup if and only if* $F = \mathbb{Z}_2$, $\dim_F V = 2$, *and* $\dim_F W = 1$. *Hence if* $F = \mathbb{Z}_2$ *and* $\dim_F V = 2$, *there are exactly 3 semigroups* $\overline{L}_F(V,W)$ *which are nonregular* $\mathcal{BQ}$*-semigroups, and each of such* $\overline{L}_F(V,W)$ *contains 8 elements.*

*Remark 3.6.* We also have that $L_F(V,W)$ is an ideal of $\overline{L}_F(V,W)$ (see Remark 2.8). Consequently, if $\{0\} \neq W \subsetneq V$, then $\overline{L}_F(V,W)$ is neither left nor right 0-simple. Hence if $F = \mathbb{Z}_2$, $\dim_F V = 2$, and $\dim_F W = 1$, then $\overline{L}_F(V,W)$ is a $\mathcal{BQ}$-semigroup which is neither regular nor left (right) 0-simple.

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