STRONG CONVERGENCE OF APPROXIMATION FIXED POINTS FOR NONEXPANSIVE NONSELF-MAPPING

RUDONG CHEN AND ZHICHUAN ZHU

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Let C be a closed convex subset of a uniformly smooth Banach space E, and $T: C \to E$ a nonexpansive nonself-mapping satisfying the weakly inwardness condition such that $F(T) \neq \emptyset$, and $f: C \to C$ a fixed contractive mapping. For $t \in (0,1)$, the implicit iterative sequence $\{x_t\}$ is defined by $x_t = P(tf(x_t) + (1-t)Tx_t)$, the explicit iterative sequence $\{x_n\}$ is given by $x_{n+1} = P(\alpha_n f(x_n) + (1-\alpha_n)Tx_n)$, where $\alpha_n \in (0,1)$ and P is a sunny nonexpansive retraction of E onto C. We prove that $\{x_t\}$ strongly converges to a fixed point of T as $t \to 0$, and $\{x_n\}$ strongly converges to a fixed point of T as α_n satisfying appropriate conditions. The results presented extend and improve the corresponding results of Hong-Kun Xu (2004) and Yisheng Song and Rudong Chen (2006).

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E, and Let $T: C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$). We use $\text{Fix}(T)$ to denote the set of fixed points of T; that is , $\text{Fix}(T) = \{x \in C : x = Tx\}$. Recall that a self-mapping $f: C \to C$ is a contraction on C if there exists a constant $\beta \in (0,1)$ such that

$$||f(x) - f(y)|| \leq \beta\|x - y\|, \quad x, y \in C. \tag{1.1}$$

Xu (see [6]) defined the following two viscosity iterations for nonexpansive mappings:

$$x_t = tf(x_t) + (1-t)Tx_t, \quad x \in C, \tag{1.2}$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \tag{1.3}$$

where α_n is a sequence in (0,1). Xu proved the strong convergence of $\{x_t\}$ defined by (1.2) as $t \to 0$ and $\{x_n\}$ defined by (1.3) in both Hilbert space and uniformly smooth Banach space.

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Recently, Song and Chen [2] proved if C is a closed subset of a real reflexive Banach space E which admits a weakly sequentially continuous duality mapping from E to E^*, and if $T : C \to E$ is a nonexpansive nonself-mapping satisfying the weakly inward condition, $F(T) \neq \phi$, $f : C \to C$ is a fixed contractive mapping, and P is a sunny nonexpansive retraction of E onto C, then the sequences $\{x_t\}$ and $\{x_n\}$ defined by

$$x_t = P(tf(x_t) + (1-t)Tx_t), \tag{1.4}$$

$$x_{n+1} = P(\alpha_n f(x_n) + (1-\alpha_n)Tx_n) \tag{1.5}$$

strongly converge to a fixed point of T.

In this paper, we establish the strong convergence of both $\{x_t\}$ defined by (1.4) and $\{x_n\}$ defined by (1.5) for a nonexpansive nonself-mapping T in a uniformly smooth Banach space. Our results extend and improve the results in [2, 6].

2. Preliminaries

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\,\|f\|, \|x\| = \|f\|\} \quad \forall x \in E, \tag{2.1}$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. In the sequence, we will denote the single-valued duality mapping by j, and $x_n \to x$ will denote strong convergence of the sequence $\{x_n\}$ to x. In Banach space E, the following result is well known [1, 3] for all $x, y \in E$, for all $j(x+y) \in J(x+y)$, for all $j(x) \in J(x)$,

$$\|x\|^2 + 2\langle y, j(x)\rangle \leq \|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle. \tag{2.2}$$

Recall that the norm of E is said to be Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.3}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Gâteaux differentiable if, for each $y \in U$, this limit is attained uniformly for $x \in U$. Finally, the norm is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit in (2.3) is attained uniformly for $(x, y) \in U \times U$. A Banach space E is said to be smooth if and only if J is single valued. It is also well known that if E is uniformly smooth, J is uniformly norm-to-norm continuous. These concepts may be found in [3].

If C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is called a retraction from C to D if $P^2 = P$. It is easily known that a mapping $P : C \to D$ is retraction, then $Px = x$, for all $x \in D$. A mapping $P : C \to D$ is called sunny if

$$P(Px + t(x - Px)) = Px \quad \forall x \in C, \tag{2.4}$$

whenever $Px + t(x - Px) \in C$ and $t > 0$. A subset D of C is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction of C onto D. For more detail, see [1, 3–5].

The following lemma is well known [3].

Lemma 2.1. *Let C be a nonempty convex subset of a smooth Banach space E, $D \in C$, $J : E \to E^*$ the (normalized) duality mapping of E, and $P : C \to D$ a retraction. Then the following are equivalent:*

(i) $\langle x - Px,\ j(y - Px)\rangle \leq 0$ *for all $x \in C$ and $y \in D$;*

(ii) *P is both sunny and nonexpansive.*

Let C be a nonempty convex subset of a Banach space E, then for $x \in C$, we define the inward set [4, 5]:

$$I_C(x) = \{y \in E : y = x + \lambda(z - x),\ z \in C \text{ and } \lambda \geq 0\}. \tag{2.5}$$

A mapping $T : C \to E$ is said to be satisfying the inward condition if $Tx \in I_C(x)$ for all $x \in C$. T is also said to be satisfying the weakly inward condition if for each $x \in C$, $Tx \in \overline{I_C(x)}$ ($\overline{I_C(x)}$ is the closure of $I_C(x)$). Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set as C is. Using above these results and definitions, we can easily show the following lemma.

Lemma 2.2 ([2], Lemma 1.2). *Let C be a nonempty closed subset of a smooth Banach space E, let $T : C \to E$ be nonexpansive nonself-mapping satisfying the weakly inward condition, and let P be a sunny nonexpansive retraction of E onto C. Then $F(T) = F(PT)$.*

Lemma 2.3 ([2], Lemma 2.1). *Let E be a Banach space and let C be a nonempty closed convex subset of E. Suppose that $T : C \to E$ is a nonexpansive mapping such that for each fixed contractive mapping $f : C \to C$, and P is a sunny nonexpansive retraction of E onto C. For each $t \in (0,1)$, $\{x_t\}$ is defined by (1.4). Suppose $u \in C$ is a fixed point of T, then*

(i) $\langle x_t - f(x_t), j(x_t - u)\rangle \leq 0$;

(ii) *$\{x_t\}$ is bounded.*

Definition 2.4. μ is called a Banach limit if μ is a continuous linear functional on l^∞ satisfying

(i) $\|\mu(e)\| = 1 = \mu(1)$, $e = (1,1,1,\ldots)$;

(ii) $\mu_n(a_n) = \mu_n(a_{n+1})$, for all $a_n \in (a_0, a_1, \ldots) \in l^\infty$;

(iii) $\liminf_{n\to\infty} a_n \leq \mu(a_n) \leq \limsup_{n\to\infty} a_n$, for all $a_n \in (a_0, a_1, \ldots) \in l^\infty$.

According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a_0, a_1, \ldots)$.

Further, we know the following result.

Lemma 2.5 ([3], Lemma 4.5.4). *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm and let $\{x_n\}$ be a bounded sequence in E. Let μ be a Banach limit and $u \in C$. Then*

$$\mu_n\|x_n - u\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2 \tag{2.6}$$

if and only if

$$\mu_n\langle x-u, J(x_n-u)\rangle \le 0 \tag{2.7}$$

for all $x\in C$.

3. Main results

Theorem 3.1. *Let E be a uniformly smooth Banach, suppose that C is a nonempty closed convex subset of E and $T: C\to E$ is a nonexpansive nonself-mapping satisfying the weakly inward condition and $F(T)\neq\emptyset$. Let $f: C\to C$ be a fixed contractive mapping, and let $\{x_t\}$ be defined by (1.4), where P is a sunny nonexpansive retraction of E onto C. Then as $t\to 0$ $\{x_t\}$ converges strongly to some fixed point q of T that q is the unique solution in $F(T)$ to the following variational inequality:*

$$\langle (I-f)q, j(q-u)\rangle \le 0 \quad \forall u\in F(T). \tag{3.1}$$

Proof. For all $u\in F(T)$ by Lemma 2.3(ii), $\{x_t\}$ is bounded, therefore the sets $\{Tx_t : t\in(0,1)\}$ and $\{f(x_t): t\in(0,1)\}$ are also bounded. From $x_t = P(tf(x_t)+(1-t)Tx_t)$, we have

$$\begin{aligned}||x_t-PTx_t|| &= ||P(tf(x_t)+(1-t)Tx_t)-PTx_t||\\ &\le ||tf(x_t)+(1-t)Tx_t-Tx_t||\\ &= t||Tx_t-f(x_t)|| \longrightarrow 0 \quad \text{as } t\longrightarrow 0.\end{aligned} \tag{3.2}$$

This implies that

$$\lim_{t\to 0}||x_t-PTx_t|| = 0. \tag{3.3}$$

Assume $t_n\to 0$, set $x_n := x_{t_n}$, and define $g: C\to\mathbb{R}$ by $g(x)=\mu_n\|x_n-x\|^2$, $x\in C$, where μ_n is a Banach limit on ℓ^∞. Let

$$K=\left\{x\in C: g(x)=\min_{y\in C}\mu_n||x_n-y||^2\right\}. \tag{3.4}$$

It is easily seen that K is a nonempty closed convex bounded subset of E, since (note $\|x_n-Tx_n\|\to 0$)

$$g(Tx)=\mu_n||x_n-Tx||^2=\mu_n||Tx_n-Tx||^2\le\mu_n||x_n-x||^2=g(x). \tag{3.5}$$

It follows that $T(K)\subset K$, that is, K is invariant under T. Since a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, T has a fixed point, say q, in K. From Lemma 2.5 we get

$$\mu_n\langle x-q, j(x_n-q)\rangle\le 0, \quad x\in C. \tag{3.6}$$

For all $q \in F(T)$, we have $tf(x_t)+(1-t)q = P[tf(x_t)+(1-t)q]$, then

$$\begin{aligned} &||x_t - [tf(x_t)+(1-t)q]|| \\ &\quad = ||P[tf(x_t)+(1-t)Tx_t] - P[tf(x_t)+(1-t)q]|| \\ &\quad \le ||(1-t)(Tx_t - q)|| \le (1-t)||x_t - q||. \end{aligned} \tag{3.7}$$

Hence from (2.2) and the above inequality we get

$$\begin{aligned} &||x_t - [tf(x_t)+(1-t)q]||^2 \\ &\quad = ||(1-t)(x_t - q) + t(x_t - f(x_t))||^2 \\ &\quad \ge (1-t)^2||x_t - q||^2 + 2t(1-t)\langle x_t - f(x_t), j(x_t - q)\rangle. \end{aligned} \tag{3.8}$$

Therefore

$$\langle x_t - f(x_t), j(x_t - q)\rangle \le 0. \tag{3.9}$$

Then

$$\begin{aligned} 0 &\ge \langle x_t - f(x_t), j(x_t - q)\rangle \\ &= ||x_t - q||^2 + \langle q - f(q), j(x_t - q)\rangle + \langle f(q) - f(x_t), j(x_t - q)\rangle \\ &\ge (1-\beta)||x_t - q||^2 + \langle q - f(q), j(x_t - q)\rangle. \end{aligned} \tag{3.10}$$

We get

$$||x_t - q||^2 \le \frac{1}{1-\beta}\langle f(q) - q, j(x_t - q)\rangle. \tag{3.11}$$

Now applying Banach limit to the above inequality, we get

$$\mu_n||x_t - q||^2 \le \mu_n\left(\frac{1}{1-\beta}\langle f(q) - q, j(x_t - q)\rangle\right). \tag{3.12}$$

Let $x = f(q)$ in (3.6), and noting (3.12), we have

$$\mu_n||x_t - q||^2 \le 0, \tag{3.13}$$

that is,

$$\mu_n||x_n - q||^2 = 0 \tag{3.14}$$

and then exists a subsequence which is still denoted by $\{x_n\}$ such that

$$x_n \longrightarrow q, \quad n \longrightarrow \infty. \tag{3.15}$$

We have proved that for any sequence $\{x_{t_n}\}$ in $\{x_t : t \in (0,1)\}$, there exists a subsequence which is still denoted by $\{x_{t_n}\}$ that converges to some point q of T. To prove that the entire net $\{x_t\}$ converges to q, suppose that there exists another sequence $\{x_{s_k}\} \subset \{x_t\}$ such that $x_{s_k} \to p$, as $s_k \to 0$, then we also have $p \in F(T)$ (using $\lim_{t\to 0}\|x_t - PTx_t\| = 0$). Next we show $p = q$ and q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I-f)q, j(q-u)\rangle \quad \forall u \in F(T). \tag{3.16}$$

Since the sets $\{x_t - u\}$ and $\{x_t - f(x_t)\}$ are bounded and the uniform smoothness of E implies that the duality map J is norm-to-norm uniformly continuous on bounded sets of E, for any $u \in F(T)$, by $x_{s_k} \to p$ $(s_k \to 0)$, we have

$$\begin{gathered} \|(I-f)x_{s_k} - (I-f)p\| \longrightarrow 0, \quad s_k \longrightarrow 0, \\ \begin{aligned} &|\langle x_{s_k} - f(x_{s_k}), j(x_{s_k}-u)\rangle - \langle (I-f)p, j(p-u)\rangle| \\ &\quad = |\langle x_{s_k} - f(x_{s_k}) - (I-f)p, j(x_{s_k}-u)\rangle - \langle (I-f)p, j(x_{s_k}-u) - j(p-u)\rangle| \\ &\quad \le \|(I-f)x_{s_k} - (I-f)p\|\|x_{s_k} - u\| \\ &\qquad + |\langle (I-f)p, j(x_{s_k}-u) - j(p-u)\rangle| \longrightarrow 0 \quad \text{as } s_k \longrightarrow 0. \end{aligned} \end{gathered} \tag{3.17}$$

Therefore, noting Lemma 2.3(i), for any $u \in F(T)$, we get

$$\langle (I-f)p, j(p-u)\rangle = \lim_{s_k\to 0}\langle x_{s_k} - f(x_{s_k}), j(x_{s_k}-u)\rangle \le 0. \tag{3.18}$$

Similarly, we also can show

$$\langle (I-f)q, j(q-u)\rangle = \langle x_{t_n} - f(x_{t_n}), j(x_{t_n}-u)\rangle \le 0. \tag{3.19}$$

Interchange q and u to obtain

$$\langle (I-f)p, j(p-q)\rangle \le 0. \tag{3.20}$$

Interchange p and u to obtain

$$\langle (I-f)q, j(q-p)\rangle \le 0. \tag{3.21}$$

This implies that

$$\langle (p-q) - (f(p) - f(q)), j(p-q)\rangle \le 0, \tag{3.22}$$

that is,

$$\|p-q\|^2 \le \beta\|p-q\|^2. \tag{3.23}$$

This is a contradiction, so we must have $q = p$.

The proof is complete. □

From Theorem 3.1 we can get the following corollary directly.

Corollary 3.2. *Let E be a uniformly smooth space, suppose C is a nonempty closed convex subset of E, $T : C \to E$ is a nonexpansive mapping satisfying the weakly inward condition, and $F(T) \neq \emptyset$. Let $f : C \to C$ be a fixed contractive mapping from C to C. $\{x_t\}$ is defined by*

$$x_t = tf(x_t) + (1-t)PTx_t, \tag{3.24}$$

where P is a sunny nonexpansive retraction of E onto C, then x_t converges strongly to some fixed point q of T as $t \to 0$ and q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I-f)q, j(q-u)\rangle \quad \forall u \in F(T). \tag{3.25}$$

Lemma 3.3 ([6], Lemma 2.1). *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n \quad \forall n \ge 0, \tag{3.26}$$

where $\{\gamma_n\} \in (0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that:
(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) *either* $\sum_{n=0}^{\infty} \delta_n < +\infty$ *or* $\limsup_{n\to\infty}(\delta_n/\gamma_n) \le 0$,
then $\lim_{n\to\infty} \alpha_n = 0$.

Theorem 3.4. *Let E be a uniformly smooth Banach space, suppose that C is a nonempty closed convex subset of E, $T : C \to E$ is a nonexpansive nonself-mapping satisfying the weakly inward condition, and $F(T) \neq \emptyset$. Let $f : C \to C$ be a fixed contractive mapping, and $\{x_n\}$ is defined by (1.5), where P is a sunny nonexpansive retraction of E onto C, and $\alpha_n \in (0,1)$ satisfies the following conditions:*
(i) $\alpha_n \to 0$, *as* $n \to \infty$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.
Then x_n converges strongly to a fixed point q of T such that q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I-f)q, j(q-u)\rangle \le 0 \quad \forall u \in F(T). \tag{3.27}$$

Proof. First we show $\{x_n\}$ is bounded. Take $u \in F(T)$, it follows that

$$\begin{aligned}
||x_{n+1} - u|| &= ||P((1-\alpha_n)Tx_n + \alpha_n f(x_n)) - Pu|| \\
&\le ||(1-\alpha_n)Tx_n + \alpha_n f(x_n) - u|| \\
&\le (1-\alpha_n)||Tx_n - u|| + \alpha_n(||f(x_n) - f(u)|| + ||f(u) - u||) \\
&\le (1-\alpha_n)||x_n - u|| + \alpha_n(\beta||x_n - u|| + ||f(u) - u||) \\
&= (1-(1-\beta)\alpha_n)||x_n - u|| + \alpha_n||f(u) - u|| \\
&\le \max\left\{||x_n - u||, \frac{1}{1-\beta}||f(u) - u||\right\}.
\end{aligned} \tag{3.28}$$

By induction,

$$||x_n - u|| \le \max\left\{||x_0 - u||, \frac{1}{1-\beta}||f(u) - u||\right\}, \quad n \ge 0, \tag{3.29}$$

and $\{x_n\}$ is bounded, so are $\{Tx_n\}$ and $\{f(x_n)\}$. We claim that

$$x_{n+1} - x_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.30}$$

Indeed we have (for some appropriate constant $M > 0$)

$$\begin{aligned}
||x_{n+1} - x_n|| &= ||P(\alpha_n f(x_n) + (1-\alpha_n)Tx_n) - P(\alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})Tx_{n-1})|| \\
&\le ||\alpha_n f(x_n) + (1-\alpha_n)Tx_n - \alpha_{n-1} f(x_{n-1}) - (1-\alpha_{n-1})Tx_{n-1}|| \\
&\le ||(1-\alpha_n)(Tx_n - Tx_{n-1}) + (\alpha_n - \alpha_{n-1})(f(x_{n-1}) - Tx_{n-1})|| \\
&\quad + \alpha_n ||f(x_n) - f(x_{n-1})|| \\
&\le (1-\alpha_n)||x_n - x_{n-1}||[3pt] + M|\alpha_n - \alpha_{n-1}| + \beta\alpha_n ||x_n - x_{n-1}|| \\
&= (1-(1-\beta)\alpha_n)||x_n - x_{n-1}||[3pt] + M|\alpha_n - \alpha_{n-1}|.
\end{aligned} \tag{3.31}$$

By Lemma 3.3 we have $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$. We now show that

$$||x_n - PTx_n|| \longrightarrow 0. \tag{3.32}$$

In fact,

$$\begin{aligned}
||x_{n+1} - PTx_n|| &= ||P(\alpha_n f(x_n) + (1-\alpha_n)Tx_n) - PTx_n|| \\
&\le \alpha_n ||f(x_n) - Tx_n||.
\end{aligned} \tag{3.33}$$

This follows from (3.30) that

$$\begin{aligned}
||x_n - PTx_n|| &\le ||x_n - x_{n+1}|| + ||x_{n+1} - PTx_n|| \\
&\le ||x_n - x_{n+1}|| + \alpha_n ||f(x_n) - Tx_n|| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.34}$$

Let $q = \lim_{t\to 0} x_t$, where $\{x_t\}$ is defined in Corollary 3.2, we get that q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I - f)q, j(q - u)\rangle \le 0 \quad \forall u \in F(T). \tag{3.35}$$

We next show that

$$\limsup_{n\to\infty} \langle f(q) - q, j(x_n - q)\rangle \le 0. \tag{3.36}$$

Form Corollary 3.2, let $x_t = tf(x_t) + (1-t)PTx_t$, indeed we can write

$$x_t - x_n = t(f(x_t) - x_n) + (1-t)(PTx_t - x_n). \tag{3.37}$$

Noting (3.32), putting

$$a_n(t) = ||x_n - PTx_n||(||x_n - PTx_n|| + 2||x_n - x_t||) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.38}$$

and using (2.2), we obtain

$$\begin{aligned} &||x_t - x_n||^2 \\ &\quad \le (1-t)^2||PTx_t - x_n||^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n)\rangle \\ &\quad \le (1-t)^2||PTx_t - PTx_n + PTx_n - x_n||^2 + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle \\ &\qquad + 2t||x_t - x_n||^2 \le (1-t)^2||x_t - x_n||^2 + (1-t)^2||x_n - PTx_n||^2 \\ &\qquad + 2(1-t)^2||PTx_n - x_n||\,||x_t - x_n|| + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle + 2t||x_t - x_n||^2 \\ &\quad \le (1+t^2)||x_t - x_n||^2 + a_n(t) + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle. \end{aligned} \tag{3.39}$$

The last inequality implies

$$\langle f(x_t) - x_t, j(x_n - x_t)\rangle \le \frac{t}{2}||x_t - x_n||^2 + \frac{1}{2t}a_n(t). \tag{3.40}$$

From $a_n(t) \to 0$ as $n \to \infty$ we get

$$\limsup_{n\to\infty} \langle f(x_t) - x_t, j(x_n - x_t)\rangle \le M \cdot \frac{t}{2}, \tag{3.41}$$

where $M > 0$ is a constant such that $M \ge \|x_t - x_n\|^2$ for all $n \ge 0$ and $t \in (0,1)$. By letting $t \to 0$ in (3.41) we have

$$\lim_{t\to 0}\limsup_{n\to\infty} \langle f(x_t) - x_t, j(x_n - x_t)\rangle \le 0. \tag{3.42}$$

On the one hand, for all $\varepsilon > 0$, $\exists\delta_1$ such that $t \in (0,\delta_1)$,

$$\limsup_{n\to\infty} \langle f(x_t) - x_t, j(x_n - x_t)\rangle \le \frac{\varepsilon}{2}. \tag{**}$$

On the other hand, $\{x_t\}$ strongly converges to q, as $t \to 0$, the set $\{x_t - x_n\}$ is bounded, and the duality map J is norm-to-norm uniformly continuous on bounded sets of uniformly smooth space E; from $x_t \to q$ $(t \to 0)$, we get

$$\|f(q) - q - (f(x_t) - x_t)\| \longrightarrow 0, \quad t \longrightarrow 0,$$

$$\begin{aligned}
&\|\langle f(q) - q, j(x_n - q)\rangle - \langle f(x_t) - x_t, j(x_n - x_t)\rangle\| \\
&\quad = \|\langle f(q) - q, j(x_n - q) - j(x_n - x_t)\rangle + \langle f(q) - q - (f(x_t) - x_t), j(x_n - x_t)\rangle\| \\
&\quad \le \|f(q) - q\| \|j(x_n - q) - j(x_n - x_t)\| \\
&\qquad + \|f(q) - q - (f(x_t) - x_t)\| \|x_n - x_t\| \longrightarrow 0, \quad t \longrightarrow 0.
\end{aligned} \tag{3.43}$$

Hence for the above $\varepsilon > 0$, $\exists \delta_2$, such that for all $t \in (0, \delta_2)$, for all n, we have

$$\|\langle f(q) - q, j(x_n - q)\rangle - \langle f(x_t) - x_t, j(x_n - x_t)\rangle\| \le \frac{\varepsilon}{2}. \tag{3.44}$$

Therefore, we have

$$\langle f(q) - q, j(x_n - q)\rangle \le \langle f(x_t) - x_t, j(x_n - x_t)\rangle + \frac{\varepsilon}{2}. \tag{3.45}$$

Noting (**) and taking $\delta = \min\{\delta_1, \delta_2\}$, for all $t \in (0, \delta)$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \langle f(q) - q, j(x_n - q)\rangle \\
&\qquad \le \limsup_{n\to\infty} \left(\langle f(x_t) - x_t, j(x_n - x_t)\rangle + \frac{\varepsilon}{2} \right) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{3.46}$$

Since ε is arbitrary, we get

$$\limsup_{n\to\infty} \langle f(q) - q, j(x_n - q)\rangle \le 0. \tag{3.47}$$

Finally we show $x_n \to q$. Indeed

$$x_{n+1} - (\alpha_n f(x_n) + (1 - \alpha_n)q) = (x_{n+1} - q) - \alpha_n (f(x_n) - q). \tag{3.48}$$

By (2.2) we have

$$\begin{aligned}
||x_{n+1}-q||^2 &= ||x_{n+1}-(\alpha_n f(x_n)+(1-\alpha_n)q)+\alpha_n(f(x_n)-q)||^2 \\
&\le ||x_{n+1}-P(\alpha_n f(x_n)+(1-\alpha_n)q)||^2+2\alpha_n\langle f(x_n)-q, j(x_{n+1}-q)\rangle \\
&\le ||P[\alpha_n f(x_n)+(1-\alpha_n)Tx_n]-P(\alpha_n f(x_n)+(1-\alpha_n)q)||^2 \\
&\quad +2\alpha_n\langle f(x_n)-q, j(x_{n+1}-q)\rangle \\
&\le (1-\alpha_n)^2||Tx_n-q||^2+2\alpha_n\langle f(x_n)-f(q), j(x_{n+1}-q)\rangle \\
&\quad +2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\
&\le (1-\alpha_n)^2||x_n-q||^2+2\alpha_n||f(q)-f(x_n)||\,||x_{n+1}-q|| \\
&\quad +2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle \\
&\le (1-\alpha_n)^2||x_n-q||^2+\alpha_n(||f(q)-f(x_n)||^2+||x_{n+1}-q||^2) \\
&\quad +2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle .
\end{aligned} \tag{3.49}$$

Therefore, we have

$$\begin{aligned}
&(1-\alpha_n)||x_{n+1}-q||^2 \\
&\quad\le (1-\alpha_n)^2||x_n-q||^2+\alpha_n\beta^2||x_n-q||^2+2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle .
\end{aligned} \tag{3.50}$$

That is,

$$\begin{aligned}
||x_{n+1}-q||^2 &\le \left(1-\frac{1-\beta^2}{1-\alpha_n}\alpha_n\right)||x_n-q||+\frac{\alpha_n^2}{1-\alpha_n}||x_n-q||^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha_n}\langle f(q)-q, j(x_{n+1}-q)\rangle \\
&\le (1-\gamma_n)||x_n-q||^2+\lambda\gamma_n\alpha_n+\frac{2}{1-\beta^2}\gamma_n\langle f(q)-q, j(x_{n+1}-q)\rangle ,
\end{aligned} \tag{3.51}$$

where $\gamma_n=((1-\beta^2)/(1-\alpha_n))\alpha_n$ and λ is a constant such that $\lambda>(1/(1-\beta^2))\|x_n-q\|^2$.

Hence,

$$||x_{n+1}-q||^2 \le (1-\gamma_n)||x_n-q||^2+\gamma_n\left(\lambda\alpha_n+\frac{2}{1-\beta^2}\langle f(q)-q, j(x_{n+1}-q)\rangle\right). \tag{3.52}$$

It is easily seen that $\gamma_n\to 0$, $\sum_{n=1}^{\infty}\gamma_n=\infty$, and (noting (3.36))

$$\limsup_{n\to\infty}\left(\lambda\alpha_n+\frac{2}{1-\beta^2}\langle f(q)-q, j(x_{n+1}-q)\rangle\right)\le 0. \tag{3.53}$$

Applying Lemma 3.3 onto (3.52), we have $x_n\to q$.

The proof is complete. □

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Rudong Chen: Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
E-mail address: chenrd@tjpu.edu.cn

Zhichuan Zhu: Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
E-mail address: zhuzcnh@yahoo.com.cn