MINIMAL CLOSED SETS AND MAXIMAL CLOSED SETS

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Some properties of minimal closed sets and maximal closed sets are obtained, which are dual concepts of maximal open sets and minimal open sets, respectively. Common properties of minimal closed sets and minimal open sets are clarified; similarly, common properties of maximal closed sets and maximal open sets are obtained. Moreover, interrelations of these four concepts are studied.

1. Introduction

Some properties of minimal open sets and maximal open sets are studied in [1, 2]. In this paper, we define dual concepts of them, namely, maximal closed set and minimal closed set. These four types of subsets appear in finite spaces, for example. More generally, minimal open sets and maximal closed sets appear in locally finite spaces such as the digital line. Minimal closed sets and maximal open sets appear in cofinite topology, for example. We have to study these four concepts to understand the relations among their properties. Some of the results in [1, 2] can be dualized using standard techniques of general topology. But we have to study the "dual results" carefully to understand the duality which we propose in this paper and in [2]. For example, considering the interrelation of these four concepts, we see that some results in [1, 2] can be generalized further.

In [1] we called a nonempty open set U of X a minimal open set if any open set which is contained in U is $\emptyset$ or U. But to consider duality, we have to consider only "*proper* nonempty open set U of X," as in the following definitions: a proper nonempty open subset U of X is said to be a *minimal open set* if any open set which is contained in U is $\emptyset$ or U. A proper nonempty open subset U of X is said to be a *maximal open set* if any open set which contains U is X or U. In this paper, we will use the following definitions.

A proper nonempty closed subset F of X is said to be a *minimal closed set* if any closed set which is contained in F is $\emptyset$ or F. A proper nonempty closed subset F of X is said to be a *maximal closed set* if any closed set which contains F is X or F.

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Let F be a subset of a topological space X. Then the following *duality principle* holds:

(1) F is a minimal closed set if and only if $X - F$ is a maximal open set;

(2) F is a maximal closed set if and only if $X - F$ is a minimal open set.

In Sections 2 and 3, we will show some results on minimal closed sets and maximal closed sets. In Section 4, we will obtain some further results on minimal open sets and maximal open sets.

The symbol $\Lambda \setminus \Gamma$ means difference of index sets, namely, $\Lambda \setminus \Gamma = \Lambda - \Gamma$, and the cardinality of a set Λ is denoted by $|\Lambda|$ in the following arguments. A subset M of a topological space X is called a *pre-open set* if $M \subset \mathrm{Int}(\mathrm{Cl}(M))$ and a subset M is called a *pre-closed set* if $X - M$ is a pre-open set.

2. Minimal closed sets

In this section, we prove some results on minimal closed sets.

The following result shows that a fundamental result of [1, Lemma 2.2] also holds for closed sets; it is the dual result of [2, Lemma 2.2].

Lemma 2.1. (1) *Let F be a minimal closed set and N a closed set. Then $F \cap N = \emptyset$ or $F \subset N$.*

(2) *Let F and S be minimal closed sets. Then $F \cap S = \emptyset$ or $F = S$.*

The proof of Lemma 2.1 is omitted, since it is obtained by an argument similar to the proof of [1, Lemma 2.2]. Now, we generalize the dual result of [2, Theorem 2.4].

Theorem 2.2. *Let F and F_λ be minimal closed sets for any element λ of Λ.*

(1) *If $F \subset \bigcup_{\lambda\in\Lambda} F_\lambda$, then there exists an element λ of Λ such that $F = F_\lambda$.*

(2) *If $F \neq F_\lambda$ for any element λ of Λ, then $(\bigcup_{\lambda\in\Lambda} F_\lambda) \cap F = \emptyset$.*

Proof. (1) Since $F \subset \bigcup_{\lambda\in\Lambda} F_\lambda$, we get $F = F \cap (\bigcup_{\lambda\in\Lambda} F_\lambda) = \bigcup_{\lambda\in\Lambda}(F \cap F_\lambda)$. If $F \neq F_\lambda$ for any element λ of Λ, then $F \cap F_\lambda = \emptyset$ for any element λ of Λ by Lemma 2.1(2), hence we have $\emptyset = \bigcup_{\lambda\in\Lambda}(F \cap F_\lambda) = F$. This contradicts our assumption that F is a minimal closed set. Thus there exists an element λ of Λ such that $F = F_\lambda$.

(2) If $(\bigcup_{\lambda\in\Lambda} F_\lambda) \cap F \neq \emptyset$, then there exists an element λ of Λ such that $F_\lambda \cap F \neq \emptyset$. By Lemma 2.1(2), we have $F_\lambda = F$, which is a contradiction. $\square$

Corollary 2.3. *Let F_λ be a minimal closed set for any element λ of Λ and $F_\lambda \neq F_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If Γ is a proper nonempty subset of Λ, then $(\bigcup_{\lambda\in\Lambda\setminus\Gamma} F_\lambda) \cap (\bigcup_{\gamma\in\Gamma} F_\gamma) = \emptyset$.*

The following result is a generalization of the dual result of [2, Theorem 2.5].

Theorem 2.4. *Let F_λ and F_γ be minimal closed sets for any element λ of Λ and γ of Γ. If there exists an element γ of Γ such that $F_\lambda \neq F_\gamma$ for any element λ of Λ, then $\bigcup_{\gamma\in\Gamma} F_\gamma \not\subset \bigcup_{\lambda\in\Lambda} F_\lambda$.*

Proof. Suppose that an element γ' of Γ satisfies $F_\lambda \neq F_{\gamma'}$ for any element λ of Λ. If $\bigcup_{\gamma\in\Gamma} F_\gamma \subset \bigcup_{\lambda\in\Lambda} F_\lambda$, then we get $F_{\gamma'} \subset \bigcup_{\lambda\in\Lambda} F_\lambda$. By Theorem 2.2(1), there exists an element λ of Λ such that $F_{\gamma'} = F_\lambda$, which is a contradiction. $\square$

The dual result of [2, Theorem 4.6] is stated as follows.

Theorem 2.5. *Let F_λ be a minimal closed set for any element λ of Λ and $F_\lambda \neq F_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If Γ is a proper nonempty subset of Λ, then $\bigcup_{\gamma\in\Gamma} F_\gamma \subsetneq \bigcup_{\lambda\in\Lambda} F_\lambda$.*

Proof. Let κ be any element of $\Lambda \setminus \Gamma$. Then $F_\kappa \cap (\bigcup_{\gamma\in\Gamma} F_\gamma) = \bigcup_{\gamma\in\Gamma}(F_\kappa \cap F_\gamma) = \varnothing$ and $F_\kappa \cap (\bigcup_{\lambda\in\Lambda} F_\lambda) = \bigcup_{\lambda\in\Lambda}(F_\kappa \cap F_\lambda) = F_\kappa$. If $\bigcup_{\gamma\in\Gamma} F_\gamma = \bigcup_{\lambda\in\Lambda} F_\lambda$, then we have $\varnothing = F_\kappa$. This contradicts our assumption that F_κ is a minimal closed set. □

The following theorem is the dual result of [2, Theorem 4.2] and is the key to the proof of Theorem 2.7, which is closely connected with [2, Theorem 4.7].

Theorem 2.6. *Assume that $|\Lambda| \geq 2$ and let F_λ be a minimal closed set for any element λ of Λ and $F_\lambda \neq F_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. Then $F_\mu \subset X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda$ and hence $\bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda \neq X$ for any element μ of Λ.*

Proof. By Corollary 2.3, we have the result. □

Theorem 2.7 (recognition principle for minimal closed sets). *Assume that $|\Lambda| \geq 2$ and let F_λ be a minimal closed set for any element λ of Λ and $F_\lambda \neq F_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. Then for any element μ of Λ,*

$$F_\mu = \left(\bigcup_{\lambda\in\Lambda} F_\lambda\right) \cap \left(X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right). \tag{2.1}$$

Proof. Let μ be an element of Λ. By Theorem 2.6, we have

$$\begin{aligned}
&\left(\bigcup_{\lambda\in\Lambda} F_\lambda\right) \cap \left(X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right)\\
&\quad= \left(\left(\bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right) \cup F_\mu\right) \cap \left(X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right)\\
&\quad= \left(\left(\bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right) \cap \left(X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right)\right) \cup \left(F_\mu \cap \left(X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right)\right)\\
&\quad= F_\mu \cap \left(X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right) = F_\mu.
\end{aligned} \tag{2.2}$$

□

As an application of Theorem 2.7, we prove the following result which is the dual result of [2, Theorem 4.8].

Theorem 2.8. *Let F_λ be a minimal closed set for any element λ of a finite set Λ and $F_\lambda \neq F_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $\bigcup_{\lambda\in\Lambda} F_\lambda$ is an open set, then F_λ is an open set for any element λ of Λ.*

Proof. Let μ be an element of Λ. By Theorem 2.7, we have $F_\mu = (\bigcup_{\lambda\in\Lambda} F_\lambda) \cap (X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda) = (\bigcup_{\lambda\in\Lambda} F_\lambda) \cap (\bigcap_{\lambda\in\Lambda\setminus\{\mu\}}(X - F_\lambda))$. By our assumption, it is seen that $\bigcap_{\lambda\in\Lambda\setminus\{\mu\}}(X - F_\lambda)$ is an open set. Hence F_μ is an open set. □

As an immediate consequence of Theorem 2.6, we have the following result which is the dual of [2, Theorem 4.9].

Theorem 2.9. *Assume that* $|\Lambda| \geq 2$ *and let* F_λ *be a minimal closed set for any element* λ *of* Λ *and* $F_\lambda \neq F_\mu$ *for any elements* λ *and* μ *of* Λ *with* $\lambda \neq \mu$. *If* $\bigcup_{\lambda\in\Lambda} F_\lambda = X$, *then* $\{F_\lambda \mid \lambda \in \Lambda\}$ *is the set of all minimal closed sets of* X.

Let $\mathcal{F} = \{F_\lambda \mid \lambda \in \Lambda\}$ be a set of minimal closed sets. We call $\cup\mathcal{F} = \bigcup_{\lambda\in\Lambda} F_\lambda$ the *coradical* of $\mathcal{F}$. The following result about coradical is obtained by [2, Theorem 4.13].

Theorem 2.10. *Let* F_λ *be a minimal closed set for any element* λ *of* Λ *and* $F_\lambda \neq F_\mu$ *for any elements* λ *and* μ *of* Λ *with* $\lambda \neq \mu$. *If* $\bigcup_{\lambda\in\Lambda} F_\lambda$ *is an open set, then* $\bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda$ *is an open set for any element* μ *of* Λ.

Theorem 2.11. *Let* F *be a minimal closed set. Then,* $\operatorname{Int}(F) = F$ *or* $\operatorname{Int}(F) = \varnothing$.

Proof. If we put $U = X - F$ in [2, Theorem 3.5], then we have the result. □

If S is a proper subset of a minimal closed set, then $\operatorname{Int}(S) = \varnothing$ and hence S is a preclosed set (cf. [2, Corollary 3.7]).

Theorem 2.12. *Let* F_λ *be a minimal closed set for any element* λ *of a finite set* Λ. *If* $\operatorname{Int}(\bigcup_{\lambda\in\Lambda} F_\lambda) \neq \varnothing$, *then there exists an element* λ *of* Λ *such that* $\operatorname{Int}(F_\lambda) = F_\lambda$.

Proof. Since $\operatorname{Int}(\bigcup_{\lambda\in\Lambda} F_\lambda) \neq \varnothing$, there exists an element x of $\operatorname{Int}(\bigcup_{\lambda\in\Lambda} F_\lambda)$. Then there exists an open set U such that $x \in U \subset \bigcup_{\lambda\in\Lambda} F_\lambda$ and hence there exists an element μ of Λ such that $x \in F_\mu$. By Theorems 2.6 and 2.7, $x \in U \cap (X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda) \subset (\bigcup_{\lambda\in\Lambda} F_\lambda) \cap (X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda) = F_\mu$. Since $U \cap (X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda)$ is an open set, we see that x is an element of $\operatorname{Int}(F_\mu)$ and hence $\operatorname{Int}(F_\mu) \neq \varnothing$. Therefore there exists an element μ of Λ such that $\operatorname{Int}(F_\mu) = F_\mu$ by Theorem 2.11. □

Remark 2.13. If Λ is an infinite set, then Theorem 2.12 does not hold. For example, see [2, Example 5.3].

Theorem 2.14 (the law of interior of coradical). *Let* F_λ *be a minimal closed set for any element* λ *of a finite set* Λ *and* $F_\lambda \neq F_\mu$ *for any elements* λ *and* μ *of* Λ *with* $\lambda \neq \mu$. *Let* Γ *be a subset of* Λ *such that*

$$\begin{aligned} \operatorname{Int}(F_\lambda) &= F_\lambda \quad \text{for any } \lambda \in \Gamma, \\ \operatorname{Int}(F_\lambda) &= \varnothing \quad \text{for any } \lambda \in \Lambda \setminus \Gamma. \end{aligned} \tag{2.3}$$

Then $\operatorname{Int}(\bigcup_{\lambda\in\Lambda} F_\lambda) = \bigcup_{\lambda\in\Gamma} F_\lambda (= \varnothing \text{ if } \Gamma = \varnothing)$.

Proof. By [2, Theorem 5.4], we have the result. □

3. Maximal closed sets

The following lemma is the dual of Lemma 2.1.

Lemma 3.1. (1) *Let* F *be a maximal closed set and* N *a closed set. Then* $F \cup N = X$ *or* $N \subset F$.

(2) *Let* F *and* S *be maximal closed sets. Then* $F \cup S = X$ *or* $F = S$.

Let F_λ be a maximal closed set for any element λ of Λ. Let $\mathcal{F} = \{F_\lambda \mid \lambda \in \Lambda\}$. We call $\bigcap \mathcal{F} = \bigcap_{\lambda\in\Lambda} F_\lambda$ the *radical* of $\mathcal{F}$.

Corollary 3.2. *Let F and F_λ be maximal closed sets for any element λ of Λ. If $F \neq F_\lambda$ for any element λ of Λ, then $(\bigcap_{\lambda\in\Lambda} F_\lambda) \cup F = X$.*

Proof. By Lemma 3.1(2), we have the result. □

Theorem 3.3. *Let F and F_λ be maximal closed sets for any element λ of Λ. If $\bigcap_{\lambda\in\Lambda} F_\lambda \subset F$, then there exists an element λ of Λ such that $F = F_\lambda$.*

Proof. Since $\bigcap_{\lambda\in\Lambda} F_\lambda \subset F$, we get $F = F \cup (\bigcap_{\lambda\in\Lambda} F_\lambda) = \bigcap_{\lambda\in\Lambda}(F \cup F_\lambda)$. If $F \cup F_\lambda = X$ for any element λ of Λ, then we have $X = \bigcap_{\lambda\in\Lambda}(F \cup F_\lambda) = F$. This contradicts our assumption that F is a maximal closed set. Then there exists an element λ of Λ such that $F \cup F_\lambda \neq X$. By Lemma 3.1(2), we have the result. □

Corollary 3.4. *Let F_λ and F_γ be maximal closed sets for any elements $\lambda \in \Lambda$ and $\gamma \in \Gamma$. If $\bigcap_{\gamma\in\Gamma} F_\gamma \subset \bigcap_{\lambda\in\Lambda} F_\lambda$, then for any element λ of Λ, there exists an element $\gamma \in \Gamma$ such that $F_\lambda = F_\gamma$.*

Corollary 3.5. *Let F_α, F_β, and F_γ be maximal closed sets which are different from each other. Then*

$$F_\alpha \cap F_\beta \not\subset F_\alpha \cap F_\gamma. \tag{3.1}$$

Theorem 3.6. *Let F_λ be a maximal closed set for any element λ of Λ and $F_\lambda \neq F_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If Γ is a proper nonempty subset of Λ, then $\bigcap_{\lambda\in\Lambda\setminus\Gamma} F_\lambda \not\subset \bigcap_{\gamma\in\Gamma} F_\gamma \not\subset \bigcap_{\lambda\in\Lambda\setminus\Gamma} F_\lambda$.*

Proof. Let γ be any element of Γ. If $\bigcap_{\lambda\in\Lambda\setminus\Gamma} F_\lambda \subset F_\gamma$, then we get $F_\lambda = F_\gamma$ for some $\lambda \in \Lambda \setminus \Gamma$ by Theorem 3.3. This contradicts our assumption. Therefore we have $\bigcap_{\lambda\in\Lambda\setminus\Gamma} F_\lambda \not\subset \bigcap_{\gamma\in\Gamma} F_\gamma$. On the other hand, since $\bigcap_{\gamma\in\Gamma} F_\gamma = \bigcap_{\gamma\in\Lambda\setminus(\Lambda\setminus\Gamma)} F_\gamma \not\subset \bigcap_{\lambda\in\Lambda\setminus\Gamma} F_\lambda$, we have $\bigcap_{\gamma\in\Gamma} F_\gamma \not\subset \bigcap_{\lambda\in\Lambda\setminus\Gamma} F_\lambda$. □

Theorem 3.7. *Let F_λ be a maximal closed set for any element λ of Λ and $F_\lambda \neq F_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If Γ is a proper nonempty subset of Λ, then $\bigcap_{\lambda\in\Lambda} F_\lambda \subsetneq \bigcap_{\gamma\in\Gamma} F_\gamma$.*

Proof. Let κ be any element of $\Lambda \setminus \Gamma$. Then $F_\kappa \cup (\bigcap_{\gamma\in\Gamma} F_\gamma) = \bigcap_{\gamma\in\Gamma}(F_\kappa \cup F_\gamma) = X$ and $F_\kappa \cup (\bigcap_{\lambda\in\Lambda} F_\lambda) = \bigcap_{\lambda\in\Lambda}(F_\kappa \cup F_\lambda) = F_\kappa$. If $\bigcap_{\gamma\in\Gamma} F_\gamma = \bigcap_{\lambda\in\Lambda} F_\lambda$, then we have $X = F_\kappa$. This contradicts our assumption that F_κ is a maximal closed set. □

Theorem 3.8. *Assume that $|\Lambda| \geq 2$ and let F_λ be a maximal closed set for any element λ of Λ and $F_\lambda \neq F_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. Then $X - \bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda \subset F_\mu$ and hence $\bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda \neq \varnothing$ for any element μ of Λ.*

Proof. Let μ be any element of Λ. By Lemma 3.1(2), we have $X - F_\mu \subset F_\lambda$ for any element λ of Λ with $\lambda \neq \mu$. Then $X - F_\mu \subset \bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda$. Therefore we have $X - \bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda \subset F_\mu$.

If $\bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda = \varnothing$, we have $X = F_\mu$. This contradicts our assumption that F_μ is a maximal closed set. Therefore we have $\bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda \neq \varnothing$. □

Theorem 3.9 (decomposition theorem for maximal closed sets). *Assume that* $|\Lambda| \geq 2$ *and let* F_λ *be a maximal closed set for any element* λ *of* Λ *and* $F_\lambda \neq F_\mu$ *for any elements* λ *and* μ *of* Λ *with* $\lambda \neq \mu$. *Then for any element* μ *of* Λ,

$$F_\mu = \left(\bigcap_{\lambda\in\Lambda} F_\lambda\right) \cup \left(X - \bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right). \tag{3.2}$$

Proof. Let μ be an element of Λ. By Theorem 3.8, we have

$$\begin{aligned}
&\left(\bigcap_{\lambda\in\Lambda} F_\lambda\right) \cup \left(X - \bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right)\\
&\quad= \left(\left(\bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right) \cap F_\mu\right) \cup \left(X - \bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right)\\
&\quad= \left(\left(\bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right) \cup \left(X - \bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right)\right) \cap \left(F_\mu \cup \left(X - \bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right)\right)\\
&\quad= F_\mu \cup \left(X - \bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda\right) = F_\mu.
\end{aligned} \tag{3.3}$$

□

Theorem 3.10. *Let* F_λ *be a maximal closed set for any element* λ *of* Λ *and* $F_\lambda \neq F_\mu$ *for any elements* λ *and* μ *of* Λ *with* $\lambda \neq \mu$. *If* $\bigcap_{\lambda\in\Lambda} F_\lambda$ *is an open set, then* F_λ *is an open set for any element* λ *of* Λ.

Proof. By Theorem 3.9, we have $F_\mu = (\bigcap_{\lambda\in\Lambda} F_\lambda) \cup (X - \bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda) = (\bigcap_{\lambda\in\Lambda} F_\lambda) \cup (\bigcup_{\lambda\in\Lambda\setminus\{\mu\}}(X - F_\lambda))$. Then $\bigcup_{\lambda\in\Lambda\setminus\{\mu\}}(X - F_\lambda)$ is an open set. Hence F_μ is an open set by our assumption. □

Theorem 3.11. *Assume that* $|\Lambda| \geq 2$ *and let* F_λ *be a maximal closed set for any element* λ *of* Λ *and* $F_\lambda \neq F_\mu$ *for any elements* λ *and* μ *of* Λ *with* $\lambda \neq \mu$. *If* $\bigcap_{\lambda\in\Lambda} F_\lambda = \emptyset$, *then* $\{F_\lambda \mid \lambda \in \Lambda\}$ *is the set of all maximal closed sets of* X.

Proof. If there exists another maximal closed set F_ν of X which is not equal to F_λ for any element λ of Λ, then $\emptyset = \bigcap_{\lambda\in\Lambda} F_\lambda = \bigcap_{\lambda\in(\Lambda\cup\{\nu\})\setminus\{\nu\}} F_\lambda$. By Theorem 3.8, we get $\bigcap_{\lambda\in(\Lambda\cup\{\nu\})\setminus\{\nu\}} F_\lambda \neq \emptyset$. This contradicts our assumption. □

Theorem 3.12. *Assume that* $|\Lambda| \geq 2$ *and let* F_λ *be a maximal closed set for any element* λ *of* Λ *and* $F_\lambda \neq F_\mu$ *for any elements* λ *and* μ *of* Λ *with* $\lambda \neq \mu$. *If* $\mathrm{Int}(\bigcap_{\lambda\in\Lambda} F_\lambda) = \emptyset$, *then* $\{F_\lambda \mid \lambda \in \Lambda\}$ *is the set of all maximal closed sets of* X.

Proof. If there exists another maximal closed set F of X which is not equal to F_λ for any element λ of Λ, then $X - F \subset \bigcap_{\lambda\in\Lambda} F_\lambda$ by Theorem 3.8. It follows that $\mathrm{Int}(\bigcap_{\lambda\in\Lambda} F_\lambda) \supset \mathrm{Int}(X - F) = X - F \neq \emptyset$. This contradicts our assumption. □

The proof of the following lemma is immediate and is omitted.

Lemma 3.13. *Let* A *and* B *be subsets of* X. *If* $A \cup B = X$, *and* $A \cap B$ *is an open set and* A *is a closed set, then* B *is an open set.*

Proposition 3.14. *Let F_λ be a closed set for any element λ of Λ and $F_\lambda \cup F_\mu = X$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $\bigcap_{\lambda\in\Lambda} F_\lambda$ is an open set, then $\bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda$ is an open set for any element μ of Λ.*

Proof. Let μ be any element of Λ. Since $F_\lambda \cup F_\mu = X$ for any element λ of Λ with $\lambda \neq \mu$, we have $F_\mu \cup (\bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda) = \bigcap_{\lambda\in\Lambda\setminus\{\mu\}}(F_\mu \cup F_\lambda) = X$. Since $F_\mu \cap (\bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda) = \bigcap_{\lambda\in\Lambda} F_\lambda$ is an open set by our assumption, $\bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda$ is an open set by Lemma 3.13. □

Theorem 3.15. *Let F_λ be a maximal closed set for any element λ of Λ and $F_\lambda \neq F_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $\bigcap_{\lambda\in\Lambda} F_\lambda$ is an open set, then $\bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda$ is an open set for any element μ of Λ.*

Proof. By Lemma 3.1(2), we have $F_\lambda \cup F_\mu = X$ for any elements λ and μ of Λ with $\lambda \neq \mu$. By Proposition 3.14, we have $\bigcap_{\lambda\in\Lambda\setminus\{\mu\}} F_\lambda$ is an open set. □

4. Minimal open sets and maximal open sets

In this section, we record some results on minimal open sets and maximal open sets, which are not proved in [1, 2]. The proofs are omitted since they are obtained by dual arguments.

Let $\mathcal{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ be a set of minimal open sets. We call $\bigcup \mathcal{U} = \bigcup_{\lambda\in\Lambda} U_\lambda$ the *coradical* of $\mathcal{U}$. By an argument similar to Theorem 2.2, we have the following result.

Theorem 4.1. *Let U and U_λ be minimal open sets for any element λ of Λ.*

(1) *If $U \subset \bigcup_{\lambda\in\Lambda} U_\lambda$, then there exists an element λ of Λ such that $U = U_\lambda$.*

(2) *If $U \neq U_\lambda$ for any element λ of Λ, then $(\bigcup_{\lambda\in\Lambda} U_\lambda) \cap U = \varnothing$.*

Corollary 4.2. *Let U_λ be a minimal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If Γ is a proper nonempty subset of Λ, then $(\bigcup_{\lambda\in\Lambda\setminus\Gamma} U_\lambda) \cap (\bigcup_{\gamma\in\Gamma} U_\gamma) = \varnothing$.*

The following results are shown by the arguments similar to the proofs of Theorems 2.4, 2.5, and 2.6, respectively.

Theorem 4.3. *Let U_λ and U_γ be minimal open sets for any elements $\lambda \in \Lambda$ and $\gamma \in \Gamma$. If there exists an element γ of Γ such that $U_\lambda \neq U_\gamma$ for any element λ of Λ, then $\bigcup_{\gamma\in\Gamma} U_\gamma \not\subset \bigcup_{\lambda\in\Lambda} U_\lambda$.*

Theorem 4.4. *Let U_λ be a minimal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If Γ is a proper nonempty subset of Λ, then $\bigcup_{\gamma\in\Gamma} U_\gamma \subsetneq \bigcup_{\lambda\in\Lambda} U_\lambda$.*

Theorem 4.5. *Assume that $|\Lambda| \geq 2$ and let U_λ be a minimal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. Then $U_\mu \subset X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} U_\lambda$ and hence $\bigcup_{\lambda\in\Lambda\setminus\{\mu\}} U_\lambda \neq X$ for any element μ of Λ.*

We obtain the following result for minimal open sets (cf. Theorems 2.7 and 3.9).

Theorem 4.6 (recognition principle for minimal open sets). *Assume that $|\Lambda| \geq 2$ and let U_λ be a minimal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ*

with $\lambda \neq \mu$. Then for any element μ of Λ,

$$U_\mu = \left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) \cap \left(X - \bigcup_{\lambda\in\Lambda\setminus\{\mu\}} U_\lambda\right). \tag{4.1}$$

Theorem 4.7. *Let U_λ be a minimal open set for any element λ of any set Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $\bigcup_{\lambda\in\Lambda} U_\lambda$ is a closed set, then U_λ is a closed set for any element λ of Λ (cf. Theorems 2.8 and 3.10).*

Theorem 4.8. *Assume that $|\Lambda| \geq 2$ and let U_λ be a minimal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $\bigcup_{\lambda\in\Lambda} U_\lambda = X$, then $\{U_\lambda \mid \lambda \in \Lambda\}$ is the set of all minimal open sets of X (cf. Theorems 2.9 and 3.11).*

Theorem 4.9. *Assume that $|\Lambda| \geq 2$ and let U_λ be a minimal open set for any element λ of Λ and $U_\lambda \neq U_\mu$ for any elements λ and μ of Λ with $\lambda \neq \mu$. If $\mathrm{Cl}(\bigcup_{\lambda\in\Lambda} U_\lambda) = X$, then $\{U_\lambda \mid \lambda \in \Lambda\}$ is the set of all minimal open sets of X (cf. Theorem 3.12).*

Example 4.10 (the digital line). The digital line is the set $\mathbb{Z}$ of the intergers equipped with the topology τ having a family of subsets $S = \{\{2k-1, 2k, 2k+1\} \mid k \in \mathbb{Z}\}$ as a subbase. We consider a set of minimal open sets $\{U_k = \{2k+1\} \mid k \in \mathbb{Z}\}$. Then $\mathrm{Cl}(\bigcup_{k\in\mathbb{Z}} U_k) = \mathrm{Cl}(\{2k+1 \mid k \in \mathbb{Z}\}) = \mathbb{Z}$ and hence $\{U_k = \{2k+1\} \mid k \in \mathbb{Z}\}$ is the set of all minimal open sets in $(\mathbb{Z}, \tau)$.

Finally we consider maximal open sets. The following results are the duals of Theorems 2.2(1) and 2.4.

Theorem 4.11. *Let U and U_λ be maximal open sets for any element λ of Λ. If $U \supset \bigcap_{\lambda\in\Lambda} U_\lambda$, then there exists an element λ of Λ such that $U = U_\lambda$.*

Theorem 4.12. *Let U_λ and U_γ be maximal open sets for any elements λ of Λ and γ of Γ. If there exists an element γ of Γ such that $U_\lambda \neq U_\gamma$ for any element λ of Λ, then $\bigcap_{\gamma\in\Gamma} U_\gamma \not\supset \bigcap_{\lambda\in\Lambda} U_\lambda$.*

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