DISPERSION PROPERTIES OF ERGODIC TRANSLATIONS

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We consider irrational rotations of the circle $Tx = x + \alpha \pmod 1$ and study the asymptotic behavior of sums of the type $S_n = \sum_{k=0}^{n-1} \pi_k$ with $\pi_k = \pm 1$, the sign being determined by the location of $T^k x$ with respect to a binary partition.

1. Introduction and preliminaries

Let $X = \mathbb{R}/\mathbb{Z}$ and $T : X \to X$ be the rotation of an angle $0 < \alpha < 1$, that is,

$$Tx = x + \alpha \pmod 1. \tag{1.1}$$

The Lebesgue measure μ is obviously T-invariant. The subset of X given by $\{x + k\alpha\}_{k\in\mathbb{Z}}$ is the (full) orbit of x under T. If $\alpha \in \mathbb{Q}$, say $\alpha = p/q$, then all orbits are finite (with q points), whereas for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ all orbits are dense (the system is a minimal homeomorphism of X) and μ is the unique (and thus ergodic) invariant measure.

Let us consider the partition $\zeta = (A_-, A_+)$ of X into the half-open arcs

$$A_+ = \left\{x \in X : 0 \le x < \frac{1}{2}\right\}, \qquad A_- = \left\{x \in X : \frac{1}{2} \le x < 1\right\}. \tag{1.2}$$

If the rotation angle α is irrational, then ζ is generating for T (see [4]) and the partition $\zeta^n = \zeta \vee T\zeta \vee \cdots \vee T^{n-1}\zeta$ is made out of $2n$ arcs. This can be easily realized by induction: when passing from ζ^{n-1} to ζ^n one has to add to the endpoints of the arcs belonging to ζ^{n-1} the two new points $T^n(0)$ and $T^n(1/2)$ (for rational α, say $\alpha = p/q$, the partition ζ^n has precisely $2q$ arcs for all $n \ge q$). Thus, the rotation is metrically isomorphic to the subshift given by the closure of $\pi([0,1))$ where the coding map $\pi : [0,1] \to \{-1,1\}^{\mathbb{Z}}$ can be defined by $\pi(x)_n = \vartheta(T^n(x))$ and $\vartheta : X \to \{-1,1\}$ is the function in $L^2(X,\mu)$ given by

$$\vartheta(x) = 2\chi_{A_+}(x) - 1 = \begin{cases} +1 & \text{if } x \in A_+, \\ -1 & \text{if } x \in A_-. \end{cases} \tag{1.3}$$

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Here we want to study the asymptotic behavior of the ergodic sums

$$S_n(x) = \sum_{k=0}^{n-1} \pi_k(x). \tag{1.4}$$

Since

$$\mu(S_n) = \int_X \sum_{k=0}^{n-1} \vartheta(x + k\alpha)\, \mu(dx) = n \int_X \vartheta(x)\, \mu(dx) = 0, \tag{1.5}$$

the faster S_n grows to infinity the slower the convergence in the ergodic theorem is. The reason why we are considering this problem for the particular function ϑ is that we want to study a model of random walk on $\mathbb{Z}$ with transition to the closest neighbors when the medium is defined by the rotation $T : X \to X$. In this view $|S_n(x)|$ measures how far from the origin the walker is at the nth step. Due to the above observations on the partition ζ^n there are exactly $2n$ different walks of length n that can be produced in this way (for α irrational). If we introduce the involution $\sigma : X \to X$ which acts as $\sigma(x) = (x + 1/2) (\mathrm{mod}\, 1)$, then we plainly have $S_k = -S_k \circ \sigma$, whence the paths come in pairs symmetric with respect to the origin. Several mathematical problems arise in this context (for related investigations see [1, 15]). The first is the characterization of the diffusive behavior of the walk, that is, the growth behavior of S_n. Other interesting items may arise in the attempt to understand the first passage distribution function

$$f(n) = \mu\{x \in X : \tau(x) > n\} \tag{1.6}$$

with

$$\tau(x) = \min\{n : S_n(x) = 0\}, \tag{1.7}$$

as well as the limit probability density

$$p(t) = \lim_{n \to \infty} \mu\{x \in X : \tau_n^{\pm}(x) = nt\}, \tag{1.8}$$

where $\tau_n^{\pm}$ is the time spent by the walker on the right (resp., left) side of the origin up to time n, that is,

$$\tau_n^{\pm}(x) = \frac{1}{2} \sum_{k=0}^{n-1} (1 \pm \operatorname{sgn} S_k(x)). \tag{1.9}$$

Recall that for the ordinary symmetric random walk, where the diffusive behavior is a normal one, one finds $f(n) = O(n^{-1/2})$ and $p(t) = 1/\pi\sqrt{t(1-t)}$.

In this paper only the first problem will be addressed, leaving the last two to further research. We mention in passing that this problem also arises in the attempt to understand the asymptotic behavior of the following skew product on the cylinder:

$$(x, y) \longrightarrow (x + \alpha, y + \vartheta(x)). \tag{1.10}$$

The asymptotic behavior of the y-variable is completely determined by that of S_n. This has been investigated in [5] for a class of Liouville rotation angles. On the other hand, in [7] it was shown that when the rotation angle is the golden ratio, that is, $\alpha = (\sqrt{5}-1)/2$, the arithmetic average $(1/n)\sum_{k=0}^{n-1} S_k(0)$ diverges logarithmically and this behavior is expected to persist for other quadratic irrationals. In the next two sections we will study the asymptotic behavior of S_n from two different points of view. In Section 2, using an elementary argument which combines the Ostrowski representation of a positive integer and a version of the Denjoy-Koksma inequality we obtain general upper bounds (Theorem 2.1) for $\|S_n\|_\infty$. In Section 3, after having developed some general spectral theory we obtain several lower bounds both for S_n and its Cesàro average in the L^2-norm (Theorems 3.9, 3.10 and 3.12). As a by-product we get a closed expression for the autocorrelation function of the function ϑ (Theorem 3.7) as well as estimates on some constrained sums (Proposition 3.6). The (upper and lower) bounds mentioned above turn out to depend intimately on the arithmetic properties of the rotation angle α. These, in turn, are encoded by its continued fraction expansion, some properties of which will be recalled in the following subsection.

1.1. Continued fractions and return times. In this section we collect several known facts on continued fractions as well as some results on return times for ergodic translations to be used later on. To start with, let

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \equiv [a_1, a_2, a_3, \dots] \tag{1.11}$$

be the continued fraction expansion of α. One recursively constructs a sequence p_n/q_n of rational approximants of α setting

$$\frac{p_0}{q_0} = \frac{0}{1}, \qquad \frac{p_1}{q_1} = \frac{1}{a_1}, \qquad \frac{p_{n+1}}{q_{n+1}} = \frac{a_{n+1}\,p_n + p_{n-1}}{a_{n+1}\,q_n + q_{n-1}}. \tag{1.12}$$

Then $p_n/q_n = [a_1, a_2, a_3, \dots, a_n]$ and $p_{2n}/q_{2n} < \alpha < p_{2n-1}/q_{2n-1}$ for all $n > 0$ (see [10, Chapter 1]). Given $u \in \mathbb{R}$ let $[u]$ and $\{u\}$ be its integer and fractional part, respectively. Let $G\colon [0,1] \to [0,1]$ be the *Gauss transformation* given by $G(x) = \{1/x\}$ for $x > 0$ and $G(0) = 0$. One readily sees that $\alpha = [a_1, a_2, \dots, a_n + G^n(\alpha)]$, so that $a_{n+1} = [1/(G^n(\alpha))]$ or $G^n(\alpha) = [a_{n+1}, a_{n+2}, \dots]$ (we are making the conventions that $1/0 = \infty$ and $1/\infty = 0$) and also

$$\alpha = \frac{p_{n-1} + \big(G^n(\alpha)\big)^{-1} p_n}{q_{n-1} + \big(G^n(\alpha)\big)^{-1} q_n} \tag{1.13}$$

which combined with Lagrange's formula $q_n p_{n-1} - q_{n-1} p_n = (-1)^{-n}$ yields

$$\alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n\big(q_{n-1} + \big(G^n(\alpha)\big)^{-1} q_n\big)} = \frac{(-1)^n}{q_n\big(q_{n+1} + G^{n+1}(\alpha) q_n\big)}. \tag{1.14}$$

Now assume that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this case the iterative process $G^n(\alpha)$ is endless and for all $n \geq 1$ we have $a_{n+1} < (G^n(\alpha))^{-1} < a_{n+1} + 1$. One gets the well-known inequalitites

$$\frac{1}{q_n \cdot (q_n + q_{n+1})} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n \cdot q_{n+1}}. \tag{1.15}$$

Moreover, since

$$q_n + G^n(\alpha) q_{n-1} = \left(\prod_{k=0}^{n-1} G^k(\alpha) \right)^{-1}, \tag{1.16}$$

one has the bounds

$$\frac{1}{2 \prod_{k=0}^{n-1} G^k(\alpha)} < q_n < \frac{1}{\prod_{k=0}^{n-1} G^k(\alpha)}. \tag{1.17}$$

From (1.15) one deduces the following characterization for the denominators q_n: let $|x| = \min\{|x - p| : p \in \mathbb{Z}\}$ be the distance from the nearest integer, then

$$q_n = \min \{ r > q_{n-1} : \|r\alpha\| < \|q_{n-1}\alpha\| \} \tag{1.18}$$

so that $\|r\alpha\| \geq \|q_{n-1}\alpha\|$ for all $r < q_n$ (see Figure 1.1).

If we denote by d the Euclidean metric on X, then

$$d(x, T^r x) = \min_{p \in \mathbb{Z}} |x + r\alpha - x - p| = \|r\alpha\| \tag{1.19}$$

for all $x \in X$. Set

$$f_n := d(x, T^{q_n} x) = \|q_n \alpha\| = |q_n \alpha - p_n| = (-1)^n (q_n \alpha - p_n). \tag{1.20}$$

From the above one deduces the recursion formula

$$f_{n-1} = a_{n+1} f_n + f_{n+1} \tag{1.21}$$

and also

$$f_n = \prod_{k=0}^{n} G^k(\alpha). \tag{1.22}$$

Equation (1.22) shows that for all irrational $0 < \alpha < 1$ the sequence f_n is strictly decreasing and by (1.15) satisfies the bounds

$$\frac{1}{a_{n+1} + 2} < q_n \cdot f_n < \frac{1}{a_{n+1}}. \tag{1.23}$$

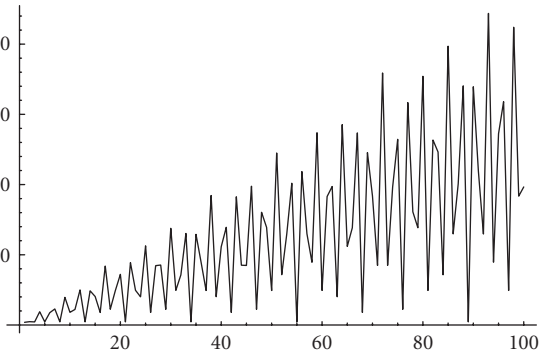


Figure 1.1. $r \cdot \|r\alpha\|$ versus r for $\alpha = (\sqrt{5}-1)/2$.

Finally, using (1.15) and (1.17) one can easily verify that the three conditions

$$\begin{aligned} r^{1+\eta} \cdot \|r\alpha\| &> C_1 \quad \forall r \geq 1, \\ q_{n+1} &< C_2 q_n^{1+\eta} \quad \forall n \geq 1, \\ \left(\prod_{k=0}^{n-1} G^k(\alpha) \right)^{\eta} &< C_3 G^n(\alpha) \quad \forall n \geq 1, \end{aligned} \tag{1.24}$$

where C_i, $i = 1,2,3$, are positive constants and $\eta \geq 0$ are equivalent to one another, and the same equivalence is true for the complementary condition where the reverse inequalities are required to hold for infinite subsequences of values of r and n.

Definition 1. The irrational number α is of type γ if

$$\gamma = \sup \left\{ s : \varliminf_{r \to \infty} r^s \cdot \|r\alpha\| = 0 \right\}. \tag{1.25}$$

By (1.18) one has that if $q_n \leq r < q_{n+1}$, then $r^s \cdot \|r\alpha\| \geq q_n^s \cdot f_n$ for $s \geq 1$ so that the lower limit in (1.25) is reached along the subsequence $r = q_n$ for such values of s. On the other hand, by (1.23) we have $q_n^{1-\epsilon} \cdot f_n < 1/q_n^{\epsilon}$ and therefore $\gamma \geq 1$. Moreover, by (1.23) it follows that $\varliminf_{r\to\infty} r \cdot \|r\alpha\| = 0$ if and only if we can find an infinite subsequence $\{n_k\}$ such that $a_{n_k} \to \infty$. As an example consider the number $\alpha = e - 2$. It is known [13] that in this case we have $a_n = 1$ except for the infinite subsequence $n_k = 3k-1$, $k = 1,2,\ldots$, with $a_{n_k} = 2k$. By (1.23) we then have $1/2(k+2) < q_{3k+1} \cdot f_{3k+1} < 1/2(k+1)$. On the other hand, (1.17) yields $q_{3k+1} \geq \prod_{h=1}^{k} 2h = 2^k k!$ and therefore $e-2$ is of type 1.

The set of type 1 numbers has measure 1 and coincides with the set of Roth-type numbers. In particular it includes those with bounded partial quotients. One has (see [13, 4.5-4.6]) $\varliminf_{r\to\infty} r \cdot \|r\alpha\| \leq 1/\sqrt{m^2+1}$ for any $\alpha = [a_1, a_2, \ldots]$ unless $a_i < m$ for all i large enough. At the opposite extreme, we say that α is *well approximable by rational numbers* if $\eta(\alpha) > 0$ where $\eta(\alpha)$ is the supremum of all η such that for all $k \geq 1$ we have $q_k \cdot \|q_k\alpha\| < Cq_k^{-\eta}$. In this case α is of type $1 + \eta(\alpha)$. If, for example, $a_k = 2^{2^k}$, then α is of type 2. More

generally, a number of type (at least) $\gamma \geq 1$ can be produced by constructing recursively its continued fraction expansion in such a way that at each step one has $a_k \geq [q_k^{\gamma-1}]$. This produces Diophantine as well as Liouville irrational numbers. A Diophantine number α of exponent $\eta \geq 0$, written $\alpha \in \mathrm{DC}(\eta)$, is such that $|\alpha - p/q| \geq Cq^{-2-\eta}$ for all $p, q \in \mathbb{Z}$. Therefore it is of type (at most) $1+\eta$ (a Liouville number is "of type ∞"). To be more precise, for $\gamma \geq 1$ the set of type γ numbers coincides with $\cap_{\eta>\gamma}\mathrm{DC}(\eta+1) \setminus \cup_{\eta<\gamma}\mathrm{DC}(\eta+1)$.

We conclude with some standard and less standard results on return times. Let $\ell \in (0, 1/2]$ and set $t_0 = 0$ and $t_i = \inf\{k > t_{i-1} : \{k\alpha\} < \ell\}$, for $i > 0$, that is, $\{t_i\}_{i\geq 0}$ is the sequence of entrance times in the half-open interval $[0, \ell)$ of the forward orbit of 0 under T. The sequence of return times $r_i = t_{i+1} - t_i$ determines a set of *gaps*, that is, the set of all possible values of r_i, $i \geq 0$. The *frequency* of a gap is defined as its frequency in the sequence $\{r_i\}_{i\geq 0}$. A classical result is the *three-gap theorem* (see [14, 6], [2, Section 4]). Let us first observe that there exists a unique expression of ℓ as

$$\ell = cf_k + f_{k+1} + g \quad \text{for some } k \geq 1,\ 0 < g \leq f_k,\ 1 \leq c \leq a_{k+1}, \tag{1.26}$$

where, by (1.21), the integer k is uniquely determined by the inequalities

$$f_k + f_{k+1} < \ell \leq f_{k-1} + f_k. \tag{1.27}$$

Indeed, the Euclidean division of ℓ by f_k gives a quotient $c_k \leq a_{k+1} + 1$ and a remainder $\ell_k \leq f_{k+1}$. Thus $\ell = c_k f_k + \ell_k = cf_k + f_{k+1} + g$ with $g = f_k - f_{k+1} + \ell_k \leq f_k$ and $c = c_k - 1$.

The three-gap theorem then says that the gaps assume at most three values, one being the sum of the other two: q_k (with frequency $(c-1)f_k + f_{k+1} + g$), $q_{k+1} - cq_k$ (with frequency g), and $q_{k+1} - (c-1)q_k$ (with frequency $f_k - g$).

By uniform distribution this result still holds when considering the gaps between the successive passages into any half-open interval I with $|I| = \ell$.

Now let

$$k_n := \min\left\{k : ||q_k\alpha|| < \frac{1}{2n}\right\} = \max\left\{k : ||q_{k-1}\alpha|| \geq \frac{1}{2n}\right\}. \tag{1.28}$$

From (1.12) one obtains the uniform lower bound $q_k > 2^{k-1}$ so that (1.15) yields $\|q_{k-1}\alpha\| < q_k^{-1} < 2^{-k+1}$. Therefore the sequence k_n cannot grow faster than $\log n$. Moreover, by (1.18) we have

$$\|r\alpha\| \geq ||q_{k_n-1}\alpha|| \geq \frac{1}{2n} \quad \forall r < q_{k_n}. \tag{1.29}$$

Therefore, by (1.19), q_{k_n} can be interpreted as the first time the orbit returns into a neighborhood of radius $1/n$ of the initial point. Note that

$$2f_{\ell+1} < \frac{1}{n} \leq 2f_\ell \Longrightarrow k_n = \ell + 1. \tag{1.30}$$

Lemma 1.1. *For $q_\ell \le n < q_{\ell+1}$ the function k_n (and thus the return time q_{k_n}) may take at most three values: $k_n \in \{\ell, \ell+1, \ell+2\}$.*

Proof. By the (inmost) inequality (1.15) we have that if $n \ge q_\ell$, then $f_{\ell-1} > 1/(2q_\ell) \ge 1/(2n)$, thus $k_n \ge \ell$. Moreover, using (1.23) we see that if $n < q_{\ell+1}$, then $f_{\ell+1} < 1/(a_{\ell+2}q_{\ell+1}) < 1/(a_{\ell+2}n)$. Therefore if $a_{\ell+2} > 1$, then $k_n \le \ell+1$. If instead $a_{\ell+2} = 1$, then $f_{\ell+2} < 1/q_{\ell+3} < 1/(2q_{\ell+1}) < 1/(2n)$. □

Remark 1.2. Due to (1.30) we see that if $1/(2f_\ell) \le q_\ell$ and $q_{\ell+1} \le 1/(2f_{\ell+1})$, then for n ranging as $q_\ell \le n < q_{\ell+1}$ the function k_n takes on but the intermediate value $\ell+1$. In turn, by (1.23), for the first inequality to be satisfied it is necessary that $a_{\ell+1} = 1$, while for the second it is sufficient that $a_{\ell+2} > 1$.

Remark 1.3. We point out that the three times dealt with in Lemma 1.1 are the three possible values of the first time the orbit comes as close to the initial point as (at least) $1/n$, when n ranges in the interval $[q_\ell, q_{\ell+1})$. A connection with the three-gap theorem is as follows: if k is chosen so as to satisfy

$$q_\ell \le n = \left(c\, f_k + f_{k+1} + g\right)^{-1} < q_{\ell+1} \tag{1.31}$$

with $1 \le c \le a_{k+2}$ and $0 < g \le f_k$, then $\{k\} \cap \{\ell, \ell+1, \ell+2\} \ne \emptyset$.

Consider, for example, the golden mean $\alpha = (\sqrt{5}-1)/2 = [1,1,1,\ldots]$. In this case $q_{k-1} = (\alpha^{-k} - (-\alpha)^k)/\sqrt{5}$ are the Fibonacci numbers whereas $f_{k-1} = \alpha^k$ by (1.22). Thus

$$\frac{1}{f_{\ell-2}} < \frac{1}{2f_{\ell-1}} < q_\ell < \frac{1}{f_{\ell-1}} < \frac{1}{2f_\ell} < q_{\ell+1} < \frac{1}{f_\ell} \tag{1.32}$$

and $k_n = \ell$ if $q_\ell \le n < 1/(2f_\ell)$ whereas $k_n = \ell+1$ if $1/(2f_\ell) \le n < q_{\ell+1}$. From this one easily sees that $k_n \asymp \log n$ (we write $a_n \asymp b_n$ whenever there are constants c_1 and c_2 such that $c_1 b_n \le a_n \le c_2 b_n$ for all n). On the other hand, the condition (1.31) becomes $q_\ell \le n = (f_{k-1} + g)^{-1} < q_{\ell+1}$. For these values of n the gaps between successive returns into an interval of size $1/n$ are q_{k-1}, q_k, and q_{k+1}. By (1.27) k is determined by the inequalities $f_{k-2}^{-1} \le n < f_{k-1}^{-1}$. Moreover $f_{k-2}^{-1} < q_k < f_{k-1}^{-1}$. Therefore the above inequalities can be satisfied choosing either $k = \ell$ or $k = \ell+1$.

We finally characterize the asymptotic behavior of the first return time q_{k_n} in terms of the type of the rotation angle α.

Lemma 1.4. *If α is of type $\gamma \ge 1$ then*

$$\lim_{n\to\infty} \frac{\log q_{k_n}}{\log n} = \frac{1}{\gamma}. \tag{1.33}$$

Proof. Under the hypothesis of the lemma we have $r^{\gamma+\epsilon} \cdot \|r\alpha\| > A_\epsilon$ for all $r \ge 1$ and some positive constant A_ϵ. We thus get

$$q_{k_n}^{\gamma+\epsilon} \ge \frac{A_\epsilon}{\|q_{k_n}\alpha\|} > 2A_\epsilon\, n \tag{1.34}$$

so that $\underline{\lim}_{n\to\infty} \log q_{k_n}/\log n > 1/(\gamma+\epsilon)$. By the arbitrariness of ϵ we thus have

$$\underline{\lim}_{n\to\infty} \frac{\log q_{k_n}}{\log n} \geq \frac{1}{\gamma}. \tag{1.35}$$

In the other direction, we have $\underline{\lim}_{r\to\infty} r^{\gamma-\epsilon} \cdot \|r\alpha\| = 0$ and therefore we can find an infinite subsequence $\{q_j\}$ so that $q_j^{\gamma-\epsilon} \cdot \|q_j\alpha\| < 1$. Choose a sequence $\{n_j\}$ such that $2n_j < q_j^{\gamma-\epsilon} \leq 4n_j$ for all j. It then follows that $\|q_j\alpha\| < 1/2n_j$ and therefore $q_{k_{n_j}} \leq q_j \leq (4n_j)^{1/(\gamma-\epsilon)}$. Reasoning as above completes the proof. □

2. The asymptotic growth of S_n

In this section we study the asymptotic growth of the sums S_n defined in (1.4) in L^∞-norm. The asymptotic behavior of S_n and its average $1/n\sum_{k=0}^{n-1} S_k(x)$ in L^2-norm will be dealt with in Section 3 using spectral techniques.

To warm up let us consider α rational, say $\alpha = p/q$. We can reason as follows: the unit circle gets partitioned by the equidistant points $0, 1/q, \ldots, q-1/q$ and for any choice of the initial point x the piece of orbit $\{x+k\alpha\}_{0\leq k<q}$ visits each subset $[i/q,(i+1)/q)$, $i = 0,\ldots,q-1$, exactly once (the order of the visits being determined by p). A short reflection shows that the discrepancy between the number of times this orbit lies in A_+ and in A_- is bounded by 2. More precisely, if q is even, then this discrepancy is zero, whereas for q odd there is i such that $i/q < 1/2 < (i+1)/q$ and the point may fall either in $[i/q, 1/2)$ or in $[1/2, (i+1)/q)$. This yields, for all $k \geq 1$,

$$\left|\left| S_{k\cdot q}\right|\right|_\infty = \begin{cases} 2k & \text{if q is odd,} \\ 0 & \text{if q is even.} \end{cases} \tag{2.1}$$

Thus, for rational α one has a nondiffusive behavior.

For α irrational we may consider its continued fraction expansion (1.11) along with the sequence of its rational approximants $p_\ell/q_\ell = [a_1, a_2, \ldots, a_\ell]$. Inequality (1.15) yields

$$\left| k\alpha - k\frac{p_\ell}{q_\ell} \right| < \frac{k}{q_\ell^2 \cdot a_{\ell+1}} < \frac{1}{q_\ell \cdot a_{\ell+1}}, \quad 1 \leq k \leq q_\ell. \tag{2.2}$$

Therefore for each $\ell \in \mathbb{N}$ the unit circle gets partitioned into the subsets $[i/q_\ell, (i+1)/q_\ell)$, $i = 0,\ldots,q_\ell - 1$, and the piece of orbit $\{x+k\alpha\}_{0\leq k<q_\ell}$ is fairly well distributed among them. More specifically, using (2.2) one realizes that the discrepancy between the number of times the orbit lies in A_+ and in A_- is bounded above by 2. We then get

$$\left|\left| S_{k\cdot q_\ell}\right|\right|_\infty \leq 2k, \quad k \geq 1. \tag{2.3}$$

In particular, the sum S_n is uniformly bounded along the subsequence $\{q_\ell\}$. But the latter has (at least) exponentially large gaps and the absence of mixing for the sequence $\{\vartheta \circ T^k\}$

may allow for a growth behavior of $\max_{q_\ell \le n < q_{\ell+1}} |S_n|$ (for sums of i.i.d. random variables it is well known that if $\max_{1\le n\le N} |S_n|$ is large, then with high probability $|S_N|$ is large as well).

THEOREM 2.1. *If α has bounded partial quotients, then for $q_\ell \le n < q_{\ell+1}$,*

$$||S_n||_\infty = O(\log n). \tag{2.4}$$

If α is of type $\gamma \ge 1$, then for all $\epsilon > 0$,

$$||S_n||_\infty = O(n^{1-(1/(\gamma+\epsilon))} \log n). \tag{2.5}$$

If moreover $\alpha \in DC(\eta)$ with $\eta \ge 0$, then

$$||S_n||_\infty = O(n^{1-(1/(\eta+1))} \log n). \tag{2.6}$$

Proof. For $q_\ell \le n < q_{\ell+1}$ we may use the Ostrowski representation for n (see [13, Chapter II.4]):

$$n = \sum_{k=0}^{\ell} c_k q_k \quad \text{with } 0 \le c_k \le a_{k+1},\ c_{k-1} = 0 \quad \text{if } c_k = a_{k+1}. \tag{2.7}$$

Using this and (2.3) we get

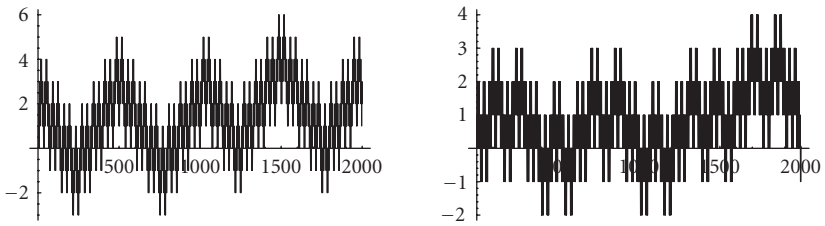
$$||S_n||_\infty \le \sum_{k=0}^{\ell} ||S_{c_k q_k}||_\infty \le 2 \sum_{k=0}^{\ell} c_k, \quad q_\ell \le n < q_{\ell+1}. \tag{2.8}$$

If the partial quotients of α are uniformly bounded (so that it is of type 1) this gives

$$||S_n||_\infty = O(\log n) \tag{2.9}$$

since $\ell = O(\log n)$. If α is of type $\gamma \ge 1$, then for each $\epsilon > 0$ we have $r^{\gamma+\epsilon} \cdot \|r\alpha\| > A_\epsilon$ for all $r \ge 1$ and some positive constant A_ϵ. Using (1.24) we see that we can find a positive constant B_ϵ so that $q_{\ell+1} < B_\epsilon q_\ell^{\gamma+\epsilon}$. Therefore $q_\ell \le n < B_\epsilon q_\ell^{\gamma+\epsilon}$. From this and (2.7) we obtain $c_\ell \le n/q_\ell < B_\epsilon^{1/(\gamma+\epsilon)} n^{1-(1/(\gamma+\epsilon))}$ and one sees inductively that the same inequality holds true for c_k with $k = 0,\dots,\ell$. Indeed, write $n = c_\ell q_\ell + n'$ with $n' = \sum_{k=0}^{\ell-1} c_k q_k$. Reasoning as above we get $c_{\ell-1} \le n'/q_{\ell-1} < B_\epsilon^{1/(\gamma+\epsilon)} (n')^{1-1/(\gamma+\epsilon)} < B_\epsilon^{1/(\gamma+\epsilon)} n^{1-1/(\gamma+\epsilon)}$, and so on. This yields the estimate (2.5). The last statement follows immediately by noting that being Diophantine of exponent $\eta \ge 0$ is equivalent to the statement $q_{\ell+1} = O(q_\ell^{\eta+1})$. □

Remark 2.2. Theorem 2.1 can be extended in various directions. For instance, one may consider irrational numbers α such that $q_{\ell+1} < e^{\lambda q_\ell^{1/\delta}}$ for some positive constants λ and δ



(a) (b)

Figure 2.1. $S_n(0)$ versus n for (A) $\alpha = e - 2 = [1,2,1,1,4,1,1,6,\ldots]$ and (B) $\alpha = (\sqrt{5}-1)/2 = [1,1,1,\ldots]$.

(such numbers may be "of type ∞"). Reasoning as in the proof of the above theorem we get in this case the estimate

$$||S_n||_\infty = O\Big(\frac{n}{\log^\delta n}\Big). \tag{2.10}$$

Furthermore, the estimate (2.5) can be improved when some additional knowledge on the partial quotients is available. Consider again the number $\alpha = e - 2$ (which is of type 1). In this case, using (2.8) along with the explicit expression for the partial quotients mentioned in the previous section, we get $\|S_n\|_\infty \le 2\sum_{i=1}^{3k+1} a_i = 2(k^2+3k+1)$ for all $q_{3k} \le n < q_{3k+1}$. On the other hand, since $q_{3k+1} \ge 2^k k!$ and $q_{3k} \ge 2q_{3k-2} \ge 2^k(k-1)!$, the condition $q_{3k} \le n < q_{3k+1}$ implies $2^k(k-1)! \le n < 2^k k!$. A simple computation then yields $k = O(\log n/\log\log n)$. Therefore, for n as above we find

$$||S_n||_\infty = O\Big(\frac{\log^2 n}{\log^2 \log n}\Big). \tag{2.11}$$

Some insight to understand the pictures in Figure 2.1 may be achieved by noting that by the three-gap theorem in the symbolic sequence $\pi(x)$ the blocks of consecutive 1's (or -1's) have at most three lengths, for all $x \in X$. As an example consider the golden ratio $\alpha = (\sqrt{5}-1)/2$. Here we find

$$f_1 = 1 - \alpha \simeq 0.38, \qquad f_2 = 2\alpha - 1 \simeq 0.24, \qquad f_3 = 2 - 3\alpha \simeq 0.14. \tag{2.12}$$

Therefore $|A_+| = 1/2 = f_1 + g$ with $g = \alpha - 1/2 \simeq 0.12$. The gaps, and thus the lengths of the blocks mentioned above, are then $q_1 = 1$, $q_2 = 2$, and $q_3 = 3$ with frequencies g, $g + f_3$, and $f_2 - g$, respectively (these are the lengths of the subintervals of A_+ on which the return time is constant and one easily checks that they sum up to 1/2).

We conclude this section with two more remarks.

Remark 2.3. The proof of the upper estimates obtained above relies on the Ostrowski representation of a positive integer along with the basic inequality (2.3). On the other

hand, the latter is a particular case of the Denjoy-Koksma inequality (see [11]): let $f : X \to \mathbb{R}$ have bounded variation and satisfy $\int_X f(x)\mu(dx) = 0$. Then for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$,

$$\left| \sum_{k=0}^{q_n-1} f(x+k\alpha) \right| \leq 2\,\mathrm{Var}(f). \tag{2.13}$$

Thus the results stated above can be straightforwardly extended to any function f with bounded variation. However, as explained in the introduction, here we are mainly interested in studying diffusion properties of the random walk on $\mathbb{Z}$ which arises using the particular function ϑ.

Remark 2.4. The behavior of the sequence $S_n(x)$ determines the analytic structure of the generating function

$$F(\alpha,x,z) := \sum_{n\geq 1} \vartheta(x+n\alpha)\, z^n. \tag{2.14}$$

In particular, the asymptotic growth of $S_n(x)$ determines the strength of the singularity of $F(\alpha,x,z)$ at $z = 1$. One moreover easily realizes that for each irrational α and all $x \in X$ the function $F(\alpha,x,z)$ has its circle of convergence (which in this case is the unit circle) as a natural boundary, a property shared (with probability one) by Taylor series whose coefficients are symmetric random variables [9]. Set indeed $z = r\,e^{2\pi i\theta}$ with $0 \leq r \leq 1$ and take $\theta = q\alpha \pmod 1$ with $q \neq 0$. By unique ergodicity we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \vartheta(x+k\alpha) e^{2\pi ikq\alpha} = e^{-2\pi iqx} \int_0^1 \vartheta(y) e^{2\pi iqy} dy = \begin{cases} \dfrac{2}{iq\pi} e^{-2\pi iqx}, & \text{q odd,} \\ 0, & \text{q even.} \end{cases} \tag{2.15}$$

Therefore, by Abel's second theorem [12, Theorem 5.2] we have

$$\lim_{r\to 1_-} (1-r) F\left(\alpha,x,re^{2\pi iq\alpha}\right) = \begin{cases} \dfrac{2}{iq\pi} e^{-2\pi iqx}, & \text{q odd,} \\ 0, & \text{q even.} \end{cases} \tag{2.16}$$

3. The spectral approach

We will now investigate the average behavior of $|S_n|$ by means of spectral techniques. Given $f : X \to \mathbb{R}$ denote by $\mu(f)$ the expectation $\int_X f(x)\mu(dx)$. As already noted $\mu(S_n) = 0$. Diffusion properties of the walk are described by the dispersion DS_n given by

$$DS_n = \mu\left(S_n^2\right) - \mu(S_n)^2 = \mu\left(S_n^2\right) \equiv ||S_n||_2^2. \tag{3.1}$$

We preliminary note that since

$$DS_n = \int_X S_n^2(x)\mu(dx) \leq ||S_n||_\infty^2, \tag{3.2}$$

an immediate consequence of Theorem 2.1 is the following estimate.

Corollary 3.1. *If α has bounded partial quotients, then*

$$DS_n = O(\log^2 n). \tag{3.3}$$

If α is of type $\gamma \geq 1$, then for each $\epsilon > 0$,

$$DS_n = O(n^{2(1-1/(\gamma+\epsilon))} \log^2 n). \tag{3.4}$$

If moreover $\alpha \in DC(\eta)$ with $\eta \geq 0$, then

$$DS_n = O(n^{2(1-1/(\eta+1))} \log^2 n). \tag{3.5}$$

In order to obtain lower bounds we will first develop some general spectral theory which will be then applied to our setting.

3.1. Some general spectral theory. Although what follows could be easily formulated for any $f : X \to \mathbb{R}$, for the sake of simplicity we will only consider the special case $f = \vartheta$. Other material on this subject can be found in [4, 8]. Let $T : X \to X$ be homeomorphism of the circle and μ an ergodic T-invariant probability measure on X. We write

$$\begin{aligned} DS_n &= \int_X \sum_{l,j=0}^{n-1} \vartheta(T^l(x))\vartheta(T^j(x))\mu(dx) \\ &= \int_X \sum_{l,j=0}^{n-1} \vartheta(x)\vartheta(T^{j-l}(x))\mu(dx) = \sum_{k=-n+1}^{n-1} (n-|k|)b_k(\vartheta), \end{aligned} \tag{3.6}$$

where we have introduced the autocorrelation function $b_k(\vartheta)$ defined by $b_0(\vartheta) = 1$ and

$$b_k(\vartheta) = \int_X \vartheta(x)\vartheta(T^k(x))\mu(dx) \equiv \mu(\vartheta \cdot \vartheta \circ T^k). \tag{3.7}$$

T being invertible, the function $b_k(\vartheta)$ satisfies $b_k(\vartheta) = b_{-k}(\vartheta)$. An interesting quantity is the *spectral type* of ϑ, that is, the probability measure σ_ϑ on $(0,1]$ such that

$$b_k(\vartheta) = \int_0^1 e^{2\pi i k\lambda} \sigma_\vartheta(d\lambda). \tag{3.8}$$

By means of this measure we can express DS_n in the following way.

Proposition 3.2.

$$DS_n = \int_0^1 \Phi_n(x)\sigma_\vartheta(dx), \tag{3.9}$$

where

$$\Phi_n(x) := \frac{\sin^2 n\pi x}{\sin^2 \pi x}. \tag{3.10}$$

Proof. The assertion follows upon combining the identity

$$\sum_{k=-n+1}^{n-1} (n-|k|)e^{ikt} = \frac{\sin^2(nt/2)}{\sin^2(t/2)} \tag{3.11}$$

with (3.6) and (3.8). □

Notice that the function $\Phi_n(x)$ is 1-periodic and satisfies

$$\int_0^1 \Phi_n(x)\,dx = n. \tag{3.12}$$

We now show that the Cesàro means

$$\langle DS_n\rangle := \frac{1}{n}\sum_{k=0}^{n-1} DS_k \tag{3.13}$$

always have a limit (finite or infinite). More specifically we have the following.

Proposition 3.3.

$$\lim_{n\to\infty}\langle DS_n\rangle = \int_0^1 \frac{\sigma_\vartheta(dx)}{2\sin^2\pi x}. \tag{3.14}$$

Proof. A straightforward computation yields for the function defined in (3.10)

$$\langle\Phi_n(x)\rangle := \frac{1}{n}\sum_{k=0}^{n-1}\Phi_k(x) = \frac{1}{\sin^2\pi x}\left[\frac{1}{2} - \frac{1}{4n}\left(1+\frac{\sin(2n-1)\pi x}{\sin\pi x}\right)\right] \tag{3.15}$$

which converges to $1/2\sin^2\pi x$ when $n\to\infty$. Upon observing that for each fixed $x\in(0,1)$ the expression between inner brackets is periodic in n (with period $1/x$) one readily shows using Lebesgue's monotone convergence theorem that $\underline{\lim}\langle\Phi_n(x)\rangle = \overline{\lim}\langle\Phi_n(x)\rangle$ and both are equal to the integral in (3.14). □

We point out that (3.9), (3.10), and (3.14) are general formulas for the dispersion, the dependence on the transformation (T,μ) being entirely contained in the spectral measure σ_ϑ. If (T,μ) is mixing, then the limit $\lim_{n\to\infty} DS_n$ is also expected to exist and to be equal to (3.14).

Example 3.4. In the case of an endomorphism $T: X\to X$ the autocorrelation sequence $b_k(\vartheta) = \mu(\vartheta\cdot\vartheta\circ T^k)$ is defined only for $k\geq 0$. On the other hand, for negative k one may put $b_k = \overline{b}_{-k}$ so that (3.6) and (3.8) remain valid (see [4]). For example, for the two-to-one map of the circle given by $T(x) = 2\cdot x \pmod 1$ we find $\vartheta(T^k(x)) = \sum_{r\ \mathrm{odd}}(2/ir\pi)\chi_{2^k r}(x)$ with $\chi_r(x) := e^{2\pi irx}$ and therefore $b_k(\vartheta) = \delta_{k,0}$ or, which is the same, the spectral measure σ_ϑ is but the Lebesgue measure. Therefore from (3.9) and (3.12) we get $DS_n = n$ for all $n\geq 1$. We point out that in this case we have $S_n = \sum_{k=1}^n \epsilon_k$ where ϵ_k is the Rademacher

sequence defined on the standard Lebesgue probability space. Whence one has the estimate (see [9, page 8]):

$$\mu\left(S_n^2 > an\right) > \frac{(1-a)^2}{3} \quad \text{for } 0 < a < 1. \tag{3.16}$$

The purely linear behavior $DS_n = n$ persists for the map $T(x) = m \cdot x \pmod 1$ provided m is an even (positive) integer. For m odd we find $b_k(\vartheta) = m^{-k}$ and a short calculation gives the spectral density

$$\begin{aligned} \rho(\lambda) &:= \frac{d\sigma_\vartheta(\lambda)}{d\lambda} = \frac{m^2-1}{m^2 - 2m\cos 2\pi\lambda + 1}, \\ DS_n &= \left(\frac{m+1}{m-1}\right)n - \frac{2m\left(1-m^{-n}\right)}{(m-1)^2} = \rho(0)n + o(n), \end{aligned} \tag{3.17}$$

to be compared to [8, Theorem 6].

We now derive general bounds which relate the asymptotic behavior of DS_n and $\langle DS_n \rangle$ to the behavior of the spectral measure σ_ϑ near the endpoints 0 and 1.

In order to estimate $\Phi_n(x)$ and $\langle \Phi_n(x) \rangle$ let us recall from basic calculus that

$$\frac{2}{\pi} \le \frac{\sin \pi x}{\pi x} \le 1 \quad \forall\, 0 \le x \le \frac{1}{2}, \tag{3.18}$$

where the upper bound holds for all $x \in \mathbb{R}_+$. This yields the following bounds:

$$\frac{4}{\pi^2} n^2 \le \Phi_n(x) \le \frac{\pi^2}{4} n^2 \quad \text{for } 0 \le x \le \frac{1}{2n}, \tag{3.19}$$

$$\Phi_n(x) \le \frac{1}{\sin^2 \pi x} \le \frac{1}{4x^2} \quad \text{for } \frac{1}{2n} \le x \le \frac{1}{2}. \tag{3.20}$$

Moreover, by (3.15) we can easily find two positive constants C_1 and C_2 so that

$$C_1 n^2 \le \langle \Phi_n(x) \rangle \le C_2 n^2 \quad \text{for } 0 \le x \le \frac{1}{2n}. \tag{3.21}$$

Using again (3.18), we find

$$\left| \frac{\sin(2n-1)\pi x}{\sin \pi x} \right| \le \frac{1}{\sin \pi x} \le n \quad \text{for } \frac{1}{2n} \le x \le \frac{1}{2}, \tag{3.22}$$

whence

$$\begin{aligned} \left| \frac{1}{2} - \frac{1}{4n}\left(1 + \frac{\sin(2n-1)\pi x}{\sin \pi x}\right) \right| &\ge \frac{1}{4}\left(1 - \frac{1}{n}\right) \ge \frac{1}{8} \quad \forall\, n > 1, \\ \left| \frac{1}{2} - \frac{1}{4n}\left(1 + \frac{\sin(2n-1)\pi x}{\sin \pi x}\right) \right| &\le \frac{3}{4}\left(1 - \frac{1}{3n}\right) \le \frac{3}{4} \quad \forall\, n \ge 1. \end{aligned} \tag{3.23}$$

Therefore we have the bounds

$$\frac{1}{8\sin^2\pi x} \le \langle \Phi_n(x)\rangle \le \frac{3}{4\sin^2\pi x} \quad \text{for } \frac{1}{2n} \le x \le \frac{1}{2}. \tag{3.24}$$

We point out that the behavior of the functions $\Phi_n(x)$ and $\langle\Phi_n(x)\rangle$ is similar in the region $0 \le x \le 1/2n$ where they are both of order n^2. Instead for $1/2n \le x \le 1/2$ their difference becomes evident in that unlike $\Phi_n(x)$, $\langle\Phi_n(x)\rangle$ never vanishes (compare (3.20) and (3.24)).

Setting $A_n = [0,1/2n] \cup [1-1/2n,1]$ and $B_n = [1/2n, 1-1/2n]$, we write

$$DS_n = \int_{A_n} \Phi_n(x)\sigma_\vartheta(dx) + \int_{B_n} \Phi_n(x)\sigma_\vartheta(dx) =: I_n + J_n, \tag{3.25}$$

$$\langle DS_n\rangle = \int_{A_n} \langle\Phi_n(x)\rangle\sigma_\vartheta(dx) + \int_{B_n} \langle\Phi_n(x)\rangle\sigma_\vartheta(dx) =: I_n' + J_n'. \tag{3.26}$$

According to the above discussion, we expect the integrals I_n and I_n' to behave similarly, the same not being true for J_n and J_n'.

To see this, let us define the spectral coefficients σ_k $(k \ge 1)$ as

$$\sigma_k := \sigma_\vartheta\left(\left[\frac{1}{2(k+1)}, \frac{1}{2k}\right] \cup \left[1-\frac{1}{2k}, 1-\frac{1}{2(k+1)}\right]\right). \tag{3.27}$$

Then (3.19) gives

$$I_n \asymp I_n' \asymp n^2 \sum_{k\ge n} \sigma_k. \tag{3.28}$$

On the other hand, (3.20) yields the upper bound

$$J_n \le \int_{B_n} \frac{\sigma_\vartheta(dx)}{4x^2} \le \sum_{k=1}^{n} (k+1)^2\sigma_k = O\left(\sum_{k=1}^{n} k^2\sigma_k\right), \tag{3.29}$$

whereas (3.24) and (3.18) give

$$J_n' \asymp \int_{B_n} \frac{\sigma_\vartheta(dx)}{x^2} \asymp \sum_{k=1}^{n} k^2\sigma_k. \tag{3.30}$$

We summarize the above as follows.

Proposition 3.5.

$$DS_n = O\left(n^2 \sum_{k\ge n} \sigma_k + \sum_{k=1}^{n} k^2\sigma_k\right), \tag{3.31}$$

and, for $n > 1$,

$$\langle DS_n\rangle \asymp n^2 \sum_{k\ge n} \sigma_k + \sum_{k=1}^{n} k^2\sigma_k. \tag{3.32}$$

By a straightforward extension of a number-theoretic lemma proved in [8, page 662] if $\{\sigma_k\}_{k\geq 1}$ is a sequence of nonnegative real numbers and $t \in [0,2)$, then for $n \to \infty$

$$\sum_{k\geq n} \sigma_k = O(n^{-t}L(n)) \Longrightarrow \sum_{k=1}^{n} k^2\sigma_k = O(n^{2-t}L(n)), \tag{3.33}$$

where $L(n)$ is a slowly varying function, that is, $L(cn) \sim L(n)$ as $n \to \infty$. When $t > 0$ the reverse implication is also valid. This entails that whenever the above conditions are satisfied the two terms in the right-hand side of (3.31) (or (3.32)) give the same upper bound for DS_n (and thus for $\langle DS_n\rangle$). Conversely, putting together (3.19) and (3.25) we obtain

$$\sum_{k\geq n} \sigma_k \equiv \sigma_\vartheta\left(\left[0,\frac{1}{2n}\right] \cup \left[1-\frac{1}{2n},1\right]\right) \leq \frac{\pi^2}{4}\frac{I_n}{n^2} \leq \frac{\pi^2}{4}\frac{DS_n}{n^2}. \tag{3.34}$$

Of course a similar estimate can be obtained for $\langle DS_n\rangle$ using (3.21).

3.2. The case of rotations. For the map $Tx = x + \alpha \pmod 1$ it is well known that the unitary operator $U_\alpha : L^2(X,\mu) \to L^2(X,\mu)$ acting as $U_\alpha f = f \circ T$ has a pure point spectrum [4], with eigenvalues λ_r and eigenfunctions χ_r, given by

$$\lambda_r = e^{2\pi i r\alpha}, \qquad \chi_r(x) = e^{2\pi i r x}, \quad r \in \mathbb{Z}. \tag{3.35}$$

Let $(f,g) = \int_{S^1} f(x)\overline{g(x)}\mu(dx)$ be the inner product of $f,g \in L^2(S^1,\mu)$. An easy computation yields

$$(\vartheta,\chi_r) = \frac{1-2e^{-\pi i r}+e^{-2\pi i r}}{2\pi i r} = \begin{cases} \dfrac{2}{ir\pi}, & \text{r odd,} \\ 0, & \text{r even.} \end{cases} \tag{3.36}$$

Therefore we have the expansions

$$\vartheta(x) = \sum_{r\,\text{odd}} \left(\frac{2}{ir\pi}\right)\chi_r(x), \qquad \vartheta(R_\alpha^k(x)) = \sum_{r\,\text{odd}} \left(\frac{2e^{2\pi i k r\alpha}}{ir\pi}\right)\chi_r(x), \tag{3.37}$$

which can be inserted into (3.7) to obtain

$$b_k(\vartheta) = \sum_{r\,\text{odd}} \left(\frac{2e^{-2\pi i k r\alpha}}{ir\pi}\right)(\vartheta,\chi_{-r}) = \sum_{r\,\text{odd}} \left(\frac{4}{r^2\pi^2}\right)e^{2\pi i k r\alpha}. \tag{3.38}$$

Notice that since $\sum_{k=0}^{\infty}(2k+1)^{-2} = \pi^2/8$, we find

$$b_0 \equiv \sigma_\vartheta(S^1) = \sum_{r\,\text{odd}} \left(\frac{4}{r^2\pi^2}\right) = 1 \tag{3.39}$$

as expected. The numbers b_k are thus the Fourier coefficients of the spectral "density" (cf. (3.8))

$$\rho(\lambda) \equiv \frac{d\sigma_\vartheta(\lambda)}{d\lambda} = \sum_{r\,\text{odd}} \left(\frac{4}{r^2\pi^2}\right)\delta(\lambda - \{r\alpha\}). \tag{3.40}$$

The coefficients (3.27) are in this case given by

$$\sigma_k = \frac{4}{\pi^2} \sum_{\substack{r\,\text{odd} \\ 1/(k+1)\le 2\|r\alpha\|\le 1/k}} \frac{1}{r^2}. \tag{3.41}$$

Estimate (3.34) along with Corollary 3.1 then yield the following result.

Proposition 3.6. *If α has bounded partial quotients, then*

$$\sum_{\substack{r\,\text{odd} \\ \|r\alpha\|\le 1/2n}} \frac{1}{r^2} = O(n^{-2}\log^2 n). \tag{3.42}$$

If α is of type $\gamma \ge 1$, then for each $\epsilon > 0$,

$$\sum_{\substack{r\,\text{odd} \\ \|r\alpha\|\le 1/2n}} \frac{1}{r^2} = O(n^{-2/\gamma+\epsilon}\log^2 n). \tag{3.43}$$

If moreover $\alpha \in DC(\eta)$ with $\eta \ge 0$, then

$$\sum_{\substack{r\,\text{odd} \\ \|r\alpha\|\le 1/2n}} \frac{1}{r^2} = O(n^{-2/\eta+1}\log^2 n). \tag{3.44}$$

We now derive a closed expression for the autocorrelation sequence b_k.

Theorem 3.7. *For all α,*

$$b_k(\vartheta) = 1 + 8\{k\alpha\}(\{k\alpha\} - 1) - 2\{2k\alpha\}(\{2k\alpha\} - 1). \tag{3.45}$$

Proof. Let $B_n(x)$ be the nth Bernoulli polynomial, defined by the generating series

$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{ze^{xz}}{e^z - 1}. \tag{3.46}$$

The first three are $B_0(x) = 1$, $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$. Let moreover $\overline{B}_n(x)$ be the function which is 1-periodic and coincides with $B_n(x)$ in the interval $[0,1)$, that is, $\overline{B}_n(x) = B_n(\{x\})$. For $n \ge 2$ the Fourier series for $\overline{B}_n(x)$ is absolutely convergent and writes (see, e.g., [3, page 14])

$$\overline{B}_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{r\ne 0} \frac{e^{2\pi irx}}{r^n}. \tag{3.47}$$

We thus have the chain of equalities:

$$b_k(\vartheta) = \frac{4}{\pi^2} \sum_{r\,\text{odd}} \frac{e^{2\pi ik\alpha r}}{r^2} = \frac{4}{\pi^2}\left[\sum_{r\ne 0} \frac{e^{2\pi ik\alpha r}}{r^2} - \frac{1}{4}\sum_{r\ne 0} \frac{e^{2\pi i2k\alpha r}}{r^2}\right] = 8\overline{B}_2(k\alpha) - 2\overline{B}_2(2k\alpha). \tag{3.48}$$

The desired formula now follows from the explicit expression of $\overline{B}_2(x)$. □

Now, putting together (3.9), (3.10), and (3.40) we get

$$DS_n = \frac{4}{\pi^2} \sum_{r\,\text{odd}} \frac{\Phi_n(\{r\alpha\})}{r^2}, \tag{3.49}$$

and, by (3.12),

$$\int_0^1 DS_n \, d\alpha = n, \tag{3.50}$$

that is, averaging over the (normalized) set of rotation angles yields a normal dispersion. Cesàro averaging over n yields instead the following asymptotic behavior.

Proposition 3.8. *For all irrational α having an infinite subsequence of odd q_k's,*

$$\lim_{n\to\infty} \langle DS_n \rangle = \infty. \tag{3.51}$$

Proof. Let us first note that the function $\Phi_n(x)$ satisfies the symmetry property $\Phi_n(x) = \Phi_n(1-x)$ for $0 \le x \le 1$, so that $\Phi_n(\{r\alpha\}) = \Phi_n(\|r\alpha\|)$ where $\|a\| = \min\{|a-p| : p \in \mathbb{Z}\}$. Formula (3.14) then reads

$$\lim_{n\to\infty} \langle DS_n \rangle = \frac{2}{\pi^2} \sum_{r\,\text{odd}} \frac{1}{r^2 \sin^2 \pi \|r\alpha\|}. \tag{3.52}$$

On the other hand, from (3.18) and $\|r\alpha\| \le 1/2$ we deduce that

$$\frac{1}{\pi^2 (r \cdot \|r\alpha\|)^2} \le \frac{1}{r^2 \sin^2 \pi \|r\alpha\|} \le \frac{1}{4(r \cdot \|r\alpha\|)^2}. \tag{3.53}$$

For α irrational we have $q_k \cdot \|q_k\alpha\| = O(q_k^{1-\gamma})$ where $\gamma \ge 1$ is the type of α. Therefore under our assumption on the q_k's the series in (3.52) is divergent. □

We conclude by showing how the different behavior of $\Phi_n(x)$ and $\langle \Phi_n(x) \rangle$ discussed in the previous subsection translates into different lower bounds for DS_n and $\langle DS_n \rangle$.

Suppose that α is chosen so that all q_k's are odd. Then using (3.19) and (1.29) we have

$$DS_n \ge c_1 \sum_{\substack{r\,\text{odd} \\ \|r\alpha\| < 1/2n}} \frac{n^2}{r^2} \ge c_2 \frac{n^2}{q_{k_n}^2}. \tag{3.54}$$

If α has bounded partial quotients, then the above estimate is not very informative in that it gives $n^2/q_{k_n}^2 \asymp \text{const}$. On the other hand, putting together (3.54) and Lemma 1.4 we get the following.

Theorem 3.9. *If α is of type $\gamma > 1$ and the q_k's are all odd, then an infinite subsequence $\{n_j\}$ can be found such that for each $\epsilon > 0$ there is a constant $C_\epsilon > 0$ so that*

$$DS_{n_j} \ge C_\epsilon n_j^{2(1-(1/(\gamma-\epsilon)))}. \tag{3.55}$$

Concerning $\langle DS_n\rangle$, from (3.24) (or else (3.41) and (3.32)) along with (1.23) we get (assuming all the q_k's odd)

$$\langle DS_n\rangle \geq c_3 \sum_{\substack{r\,\text{odd}\\ \|r\alpha\|\geq 1/2n}} \frac{1}{(r\cdot\|r\alpha\|)^2} \geq c_4 \sum_{k=0}^{k_n-1} \frac{1}{(q_k\cdot f_k)^2} \geq c_4 \sum_{k=1}^{k_n} a_k^2. \tag{3.56}$$

Using Lemma 1.1 we then obtain the following result.

Theorem 3.10. *There is a constant $C>0$ such that*

$$q_\ell \leq n < q_{\ell+1} \Longrightarrow \langle DS_n\rangle \geq C\sum_{k=1}^{\ell}{}^{*} a_k^2, \tag{3.57}$$

where the starred sum is restricted to those k for which q_{k-1} is odd.

This inequality can be used to derive lower bounds for various types of α. For example, if α is of type $\gamma>1$, the last term of the sum in (3.57) can be estimated by noting that we can find an infinite subsequence of r values so that $r^\gamma\cdot\|r\alpha\|<C_1$. By (1.24), this is equivalent to $q_{\ell+1}>C_2 q_\ell^\gamma$ and $G^\ell(\alpha)<C_3(\prod_{k=0}^{\ell-1}G^k(\alpha))^{\gamma-1}$, where, again, the inequalities are supposed to hold along a suitable infinite subsequence of ℓ values. Using (1.17) we then get for all $q_\ell\leq n\leq Aq_\ell^\gamma$,

$$a_\ell \geq \frac{1}{G^\ell(\alpha)} > c_5\left(\prod_{k=0}^{\ell-1}G^k(\alpha)\right)^{1-\gamma} > c_6\, q_\ell^{\gamma-1} > c_7 n^{1-1/\gamma}. \tag{3.58}$$

An interesting estimate can be easily obtained for a class of numbers α with bounded partial quotients. We first need the following result.

Lemma 3.11. *Given any irrational α, for every pair q_k, q_{k+1} of consecutive denominators, at least one of them is odd.*

Proof. By induction: start with $q_0=1$ and by the defining recursion $q_{k+1}=a_{k+1}q_k+q_{k-1}$ one sees that if q_{k-1} is odd and q_k is even, then q_{k+1} has to be odd. □

For example for the golden mean $\alpha=(\sqrt{5}-1)/2=[1,1,1,\ldots]$ the q_k's are the Fibonacci numbers $1,1,2,3,5,8,13,21,34,\ldots$, and the frequency of odd q_k's is 2/3. For $\alpha=\sqrt{2}-1=[2,2,2,\ldots]$ the q_k's are numbers $1,2,5,12,29,70,169,\ldots$, and the frequency of odd q_k's is exactly 1/2.

Finally, using Theorem 3.10, Lemma 3.11 and recalling that for α with bounded partial quotients the integer ℓ in (3.57) satisfies $\ell\asymp\log n$, we readily get the following inequality.

Theorem 3.12. *For all irrational α with bounded partial quotients the following holds, for n large enough:*

$$\langle DS_n\rangle \geq \frac{1}{2}\log n. \tag{3.59}$$

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