THE DISTRIBUTION OF NONPRINCIPAL EIGENVALUES OF SINGULAR SECOND-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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We obtain the asymptotic distribution of the nonprincipal eigenvalues associated with the singular problem $x'' + \lambda q(t)x = 0$ on an infinite interval $[a, +\infty)$. Similar to the regular eigenvalue problem on compact intervals, we can prove a Weyl-type expansion of the eigenvalue counting function, and we derive the asymptotic behavior of the eigenvalues.

1. Introduction

In this work we study the second-order linear ordinary differential equation

$$x'' + \lambda q(t)x = 0, \quad t \geq a, \tag{1.1}$$

with the boundary conditions

$$x(a,\lambda) = 0, \qquad \lim_{t\to\infty}\left[x(t,\lambda) - t\right] = 0, \qquad \lim_{t\to\infty} t\left[x'(t,\lambda) - 1\right] = 0, \tag{1.2}$$

where λ is a real parameter and $q(t)$ is a positive continuous function on $[a,\infty)$ satisfying

$$\int_a^\infty t^2 q(t)\,dt < \infty. \tag{1.3}$$

A nonoscillatory solution $x_0(t,\lambda)$ of (1.1) satisfying the boundary conditions (1.2) is called a nonprincipal eigenfunction if

$$\int_a^\infty \frac{dt}{\left(x_1(t,\lambda)\right)^2} < \infty, \tag{1.4}$$

and the corresponding value of λ is called a nonprincipal eigenvalue.

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Concerning the existence and uniqueness of nonprincipal eigenvalues, the main result is due to Elbert et al. [2]. There exists a sequence of positive constants $\{\lambda_k\}_k$, $0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots \nearrow \infty$ such that, for each $\lambda = \lambda_k$, (1.1) possesses a solution $x_k(t,\lambda_k)$ satisfying the boundary condition (1.2) and having exactly k zeros in (a,∞), $k = 0,1,2,\ldots$, imposing the integrability condition (1.3) on $q(t)$.

We are interested in the distribution and asymptotic behavior of eigenvalues $\{\lambda_k\}_k$. To this end, we study the spectral counting function

$$N(\lambda) = \#\{k : \lambda_k \le \lambda\}. \tag{1.5}$$

It is well known that the eigenvalue problem in a closed interval $[a,b]$ has the asymptotic distribution (see [1]):

$$N(\lambda) \sim \frac{\lambda^{1/2}}{\pi} \int_a^b q^{1/2}(t)\,dt \tag{1.6}$$

as $\lambda \to \infty$, generalizing the Weyl formula. Here, $f \sim g$ means that $f/g \to 1$.

Our main result is the following theorem.

Theorem 1.1. *Let $\{\lambda_k\}$ be the sequence of nonprincipal eigenvalues of problem (1.1)-(1.2), and let $q(t)$ be a positive, continuous, and nonincreasing function satisfying (1.3). Then, the asymptotic expansion of $N(\lambda)$ is given by*

$$N(\lambda) = \frac{\lambda^{1/2}}{\pi} \int_a^\infty q^{1/2}(t)\,dt + o(\lambda^{1/2}) \tag{1.7}$$

as $\lambda \to \infty$. Also, the kth-eigenvalue has the following asymptotic behavior:

$$\lambda_{k-1} = \left(\frac{\pi k}{\int_a^\infty q^{1/2}(t)\,dt}\right)^2 + o(k^2) \tag{1.8}$$

as $k \to \infty$.

The paper is organized as follows. In Section 2 we prove some auxiliary results, and the proof of Theorem 1.1 is given in Section 3.

2. Sturm-Liouville bracketing of eigenvalues

Let us observe that problem (1.1)-(1.2) is not a variational one, since $x'(t) \sim 1$ as $t \to +\infty$ and $x'(t) \notin L^2(0,+\infty)$. Hence, we need the following generalization of the Dirichlet-Neumann bracketing of Courant (see [1]) in order to prove Theorem 1.1.

Theorem 2.1. *Let $N(\lambda, I)$ be the spectral counting function on $I = (a,b)$ of the problem*

$$-x'' = \lambda q(t)x, \qquad x(a) = 0 = x(b). \tag{2.1}$$

Let $c \in (a,b)$. Then,

$$N(\lambda, I) \sim N(\lambda, I_1) + N(\lambda, I_2) \tag{2.2}$$

as $\lambda \to \infty$, where $I_1 = (a,c)$ and $I_2 = (c,b)$.

Remark 2.2. For simplicity, we deal only with the Dirichlet boundary condition on a bounded interval. With minor modifications of the proof, the result is valid for different boundary conditions, including the case $b = +\infty$ and the boundary condition (1.2), since the proof is based on the Sturm-Liouville oscillation theory.

Let us sketch the proof of the Dirichlet Neumann bracketing for a second-order differential operator L with variational structure in an interval I. The eigenvalues of L are obtained minimizing a quadratic functional in a convenient subspace $H \subset H^1(I)$. We have

$$H_0^1(I_1) \oplus H_0^1(I_2) \subset H_0^1(I) \subset H \subset H^1(I) \subset H^1(I_1) \oplus H^1(I_2) \tag{2.3}$$

and we obtain the Dirichlet eigenvalues of L in I_1 and I_2 as an upper bound of the eigenvalues of L in I, and the Neumann eigenvalues of I_1 and I_2 as a lower bound.

In problem (1.1)-(1.2), the solutions and eigenvalues are obtained by a fixed point argument, instead of a minimization procedure, and we need a different argument to relate the eigenvalue of two intervals and those of the union of them. Since the eigenfunction x_k has exactly k zeros in (a,b), it is possible to obtain the asymptotic distribution of eigenvalues from the asymptotic number of zeros of solutions, an idea which goes back at least to Hartman (see [3]). For the sake of self-completeness, we prove Theorem 2.1 here.

Proof of Theorem 2.1. Let us consider the following eigenvalue problems in I_1 and I_2, with the original boundary conditions in a and b, and a Neumann boundary condition at c:

$$\begin{gathered} -u'' = \mu q(t)u, \quad t \in (a,c), \\ u(a) = 0, \quad u'(c) = 0, \end{gathered} \tag{2.4}$$

$$\begin{gathered} -v'' = \nu q(t)v, \quad t \in (c,b), \\ v'(c) = 0, \quad v(b) = 0. \end{gathered} \tag{2.5}$$

For each problem there exists a sequence of simple eigenvalues $\{\mu_k\}_k$, $\{\nu_k\}_k$ tending to infinity, and the kth eigenfunction u_k corresponding to μ_k (resp., ν_k, v_k) has exactly k zeros.

Let λ be fixed. Let λ_n be the greater eigenvalue of problem (2.1) lower or equal than λ and $x_n(t)$ the corresponding eigenfunction, which has n zeros in (a,b). Let k be the number of zeros of x_n in (a,c), and let $n-k$ be the number of zeros in (c,b).

Let μ_j be the greater eigenvalue of problem (2.4) lower or equal than λ, and let u_j be the corresponding eigenfunction. We will show that j, the number of zeros of u_j, satisfies

$$k - 1 \le j \le k + 2. \tag{2.6}$$

Let us suppose first that u_j has $k+3$ zeros. Then, the Sturmian theory gives $\mu_j > \lambda_n$. Let $x_{\mu_j}(t)$ be the unique solution of (2.1) satisfying

$$\begin{gathered} x_{\mu_j}(c) = u_j(c), \\ x'_{\mu_j}(c) = u'_j(c). \end{gathered} \tag{2.7}$$

Hence, $x_{\mu_j} \equiv u_j$ in (a,c), and $x_{\mu_j}(t)$ has at least $n-k-1$ zeros in (c,b) (let us note that one of the original zeros of $x_n(t)$ could cross the point c to the left). Thus, the solution $x_{\mu_j}(t)$ has at least $n+2$ zeros in (a,b).

However, the eigenfunction $x_{n+1}(t)$ of problem (2.1) corresponding to the eigenvalue λ_{n+1} has $n+1$ zeros and satisfy $\lambda_{n+1} < \mu_j$. Hence,

$$\lambda_{n+1} < \mu_j \leq \lambda, \tag{2.8}$$

which contradicts our assumption.

On the other hand, let us suppose that u_j has $k-2$ zeros. Clearly, $\mu_j < \lambda_n < \lambda$. Let u_{j+1} be the eigenfunction of problem (2.4) with $k-1$ zeros in (a,c), and let μ_{j+1} be the corresponding eigenvalue. By using the Sturm-Liouville theory,

$$\mu_{j+1} < \lambda_n < \lambda, \tag{2.9}$$

because $x_n(t)$ has k zeros in (a,c), which contradicts the fact that μ_j is the greater eigenvalue of problem (2.4) lower or equal than λ.

Let us consider now problem (2.5). Let ν_h be the greater eigenvalue of problem (2.5) lower or equal than λ, and let v_h be the corresponding eigenfunction. In much the same way, fixing the boundary condition at $t=b$, we can show that h, the number of zeros of v_h, satisfy

$$n-k-2 \leq h \leq n-k+1. \tag{2.10}$$

Then, from inequalities (2.6) and (2.10),

$$N(\lambda,I_1)+N(\lambda,I_2)-3 \leq N(\lambda,I) \leq N(\lambda,I_1)+N(\lambda,I_2)+3 \tag{2.11}$$

and the proof is finished. □

3. Asymptotic of nonprincipal eigenvalues

In this section we prove Theorem 1.1. First, we need the following lemma.

Lemma 3.1. *Let $q(t)$ be a positive continuous function satisfying*

$$\int_a^\infty t^2 q(t)\,dt < \infty. \tag{3.1}$$

Then,

$$\int_a^\infty q^{1/2}(t)\,dt < \infty. \tag{3.2}$$

Proof. It follows from Holder's inequality:

$$\int_a^\infty q^{1/2}(t)\,dt < \left(\int_a^\infty t^2 q(t)\,dt\right)^{1/2}\left(\int_a^\infty t^{-2}\,dt\right)^{1/2} < \infty. \tag{3.3}$$

□

We divide the proof of Theorem 1.1 in three parts. We obtain an optimal lower bound for $N(\lambda)$; then we obtain an upper bound for $N(\lambda)$; and finally, we improve the upper bound.

PROPOSITION 3.2. *Let $N(\lambda)$ be the eigenvalue counting function of Theorem 1.1. The following inequality holds:*

$$\frac{\lambda^{1/2}}{\pi}\int_a^{+\infty} q^{1/2}(t)dt + o(\lambda^{1/2}) \le N(\lambda). \tag{3.4}$$

Proof. Let $\varepsilon > 0$ be fixed, there exist T_ε such that

$$\frac{1}{\pi}\int_{T_\varepsilon}^{\infty} q^{1/2}(t)dt \le \frac{\varepsilon}{2}. \tag{3.5}$$

Let us consider the Dirichlet eigenvalue problem on $[a, T_\varepsilon]$:

$$-y''(t) = \mu q(t)y(t), \tag{3.6}$$

$$y(a) = 0 = y(T_\varepsilon). \tag{3.7}$$

It is well known that there exists a sequence of eigenvalues $\{\mu_k\}_{k\ge 0}$, with associated eigenfunctions $\{y_k\}_{k\ge 0}$. Each eigenvalue is isolated and y_k has exactly k zeros in the open interval (a, T_ε).

The spectral counting function $N_D(\lambda, [a, T_\varepsilon])$ of problem (3.6) has the following asymptotic expansion:

$$N_D(\lambda, [a, T_\varepsilon]) = \frac{\lambda^{1/2}}{\pi}\int_a^{T_\varepsilon} q^{1/2}(t)dt + o(\lambda^{1/2}). \tag{3.8}$$

Therefore, for the same $\varepsilon > 0$, there exists $\lambda(\varepsilon)$ such that

$$\left| \frac{N_D(\lambda, [a, T_\varepsilon])}{\lambda^{1/2}} - \frac{1}{\pi}\int_a^{T_\varepsilon} q^{1/2}(t)dt \right| \le \frac{\varepsilon}{2} \tag{3.9}$$

for every $\lambda \ge \lambda(\varepsilon)$.

By the Sturmian comparison theorem, we have the inequality $\lambda_k \le \mu_k$, which gives the lower bound for $N(\lambda)$:

$$N_D(\lambda, [a, T_\varepsilon]) \le N(\lambda). \tag{3.10}$$

Hence,

$$\frac{N(\lambda)}{\lambda^{1/2}} \ge \frac{N_D(\lambda, [a, T_\varepsilon])}{\lambda^{1/2}} \ge \frac{1}{\pi}\int_a^{T_\varepsilon} q^{1/2}(t)dt - \frac{\varepsilon}{2} \ge \frac{1}{\pi}\int_a^{\infty} q^{1/2}(t)dt - \varepsilon \tag{3.11}$$

for every $\lambda \ge \lambda(\varepsilon)$, and the proof is finished. □

Remark 3.3. Let us note that Proposition 3.2 is valid whenever $\int_a^\infty q^{1/2}(t)dt < +\infty$, which is guaranteed by Lemma 3.1, without any monotonicity assumption.

Proposition 3.4. *Let $N(\lambda)$ be the eigenvalue counting function of Theorem 1.1. The following inequality holds:*

$$\frac{4\lambda^{1/2}}{\pi}\int_a^{+\infty} q^{1/2}(t)dt + o(\lambda^{1/2}) \geq N(\lambda). \tag{3.12}$$

Proof. We need a lower bound for eigenvalues due to Nehari [4]. Let $q(t)$ be a monotonic function, and μ_k the kth Dirichlet eigenvalue of (1.1) in (a,b). Then,

$$\mu_k\left(\int_a^b q^{1/2}(t)dt\right)^2 \geq \frac{\pi^2k^2}{4}. \tag{3.13}$$

Let $\{\lambda_k\}_{k\geq 0}$ be the nonprincipal eigenvalues of problem (1.1)-(1.2), and let t_k be the kth zero of the associated eigenfunction $x_k(t)$. Clearly, λ_k coincides with the kth Dirichlet eigenvalue in (a,t_k).

Hence,

$$\lambda_k \geq \frac{\pi^2k^2}{4\left(\int_a^{t_k} q^{1/2}(t)dt\right)^2} \geq \frac{\pi^2k^2}{4\left(\int_a^{\infty} q^{1/2}(t)dt\right)^2}. \tag{3.14}$$

We obtain

$$\begin{aligned} N(\lambda) &= \#\{k : \lambda_k \leq \lambda\} \\ &\leq \#\left\{k : \frac{\pi^2k^2}{4\left(\int_a^{\infty} q^{1/2}(t)dt\right)^2} \leq \lambda\right\} \\ &= \#\left\{k : k \leq \frac{2\lambda^{1/2}}{\pi}\int_a^{\infty} q^{1/2}(t)dt\right\} \\ &\leq \frac{2\lambda^{1/2}}{\pi}\int_a^{\infty} q^{1/2}(t)dt + O(1), \end{aligned} \tag{3.15}$$

and the proof is finished. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Let be T_ε such that

$$\int_{T_\varepsilon}^{+\infty} q^{1/2}(t)dt < \varepsilon. \tag{3.16}$$

Applying Theorem 2.1 we obtain

$$N(\lambda) \sim N(\lambda,(a,T_\varepsilon)) + N(\lambda,(T_\varepsilon,\infty)). \tag{3.17}$$

The asymptotic behavior of $N(\lambda,(a,T_\varepsilon))$ is obtained from the classical theory,

$$N(\lambda,(a,T_\varepsilon)) \sim \frac{\lambda^{1/2}}{\pi}\int_a^{T_\varepsilon} q^{1/2}(t)dt. \tag{3.18}$$

Hence, for $\lambda \geq \lambda(\varepsilon)$, we have

$$N(\lambda,(a,T_\varepsilon)) \leq \frac{\lambda^{1/2}}{\pi}\int_a^{T_\varepsilon} q^{1/2}(t)dt + \varepsilon\lambda^{1/2} \leq \frac{\lambda^{1/2}}{\pi}\int_a^{+\infty} q^{1/2}(t)dt + \varepsilon\lambda^{1/2}. \tag{3.19}$$

Now, $N(\lambda,(T_\varepsilon,\infty))$ can be bounded by using Proposition 3.4:

$$N(\lambda,(T_\varepsilon,\infty)) \leq \frac{2\lambda^{1/2}}{\pi}\int_{T_\varepsilon}^{+\infty} q^{1/2}(t)dt \leq \varepsilon\frac{2\lambda^{1/2}}{\pi}. \tag{3.20}$$

Hence,

$$N(\lambda) \leq \frac{\lambda^{1/2}}{\pi}\int_a^{+\infty} q^{1/2}(t)dt + \varepsilon\lambda^{1/2} + \varepsilon\frac{2\lambda^{1/2}}{\pi}. \tag{3.21}$$

Since ε is arbitrarily small, and by using Proposition 3.2, we have the asymptotic expansion

$$N(\lambda) \sim \frac{\lambda^{1/2}}{\pi}\int_a^{+\infty} q^{1/2}(t)dt. \tag{3.22}$$

Finally, from (3.22), we have

$$k = N(\lambda_{k-1}) \sim \frac{\lambda_k^{1/2}}{\pi}\int_a^{\infty} q^{1/2}(t)dt, \tag{3.23}$$

which gives the asymptotic behavior of the kth-eigenvalue,

$$\lambda_k = \left(\frac{\pi k}{\int_a^\infty q^{1/2}(t)dt}\right)^2 + o(k^2). \tag{3.24}$$

This completes the proof. □

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