ON SEMILATTICES OF GROUPS WHOSE ARROWS ARE EPIMORPHISMS

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A q partial group is defined to be a partial group, that is, a strong semilattice of groups $S = [E(S); S_e, \varphi_{e,f}]$ such that S has an identity 1 and $\varphi_{1,e}$ is an epimorphism for all $e \in E(S)$. Every partial group S with identity contains a unique maximal q partial group $\mathbf{Q}(S)$ such that $(\mathbf{Q}(S))_1 = S_1$. This $\mathbf{Q}$ operation is proved to commute with Cartesian products and preserve normality. With $\mathbf{Q}$ extended to idempotent separating congruences on S, it is proved that $\mathbf{Q}(\rho_K) = \rho_{\mathbf{Q}(K)}$ for every normal K in S. Proper q partial groups are defined in such a way that associated to any group G, there is a proper q partial group $\mathbf{P}(G)$ with $(\mathbf{P}(G))_1 = G$. It is proved that a q partial group S is proper if and only if $S \cong \mathbf{P}(S_1)$ and hence that if S is any partial group, there exists a group M such that S is embedded in $\mathbf{P}(M)$. P epimorphisms of proper q partial groups are defined with which the category of proper q partial groups is proved to be equivalent to the category of groups and epimorphisms of groups.

1. Introduction and preliminaries

One can easily observe that Clifford semigroups have been the object of extensive study from both category and semigroup theorists. A Clifford semigroup is usually defined as a regular semigroup with central idempotents. Whence many characterizations exist including the structure theorem that characterizes them as semilattices of groups, or equivalently as strong semilattices of groups. That is if $S = [Y, S_\alpha, \varphi_{\alpha,\beta}]$ is a strong semilattice Y of groups S_α, then S is a Clifford semigroup with operation defined by

$$ab = (\varphi_{\alpha,\alpha\beta}a)(\varphi_{\beta,\alpha\beta}b) \tag{1.1}$$

for $a \in S_\alpha$, $b \in S_\beta$.

Conversely, a Clifford semigroup S is a strong semilattice $E(S)$ of groups S_f; $S = [E(S), S_f, \varphi_{f,g}]$ where $E(S)$ is the semilattice ($f \le g \Leftrightarrow fg = f$) of idempotents in S, S_f is the maximal subgroup of S with identity f, and $\varphi_{f,g}$ is the homomorphism $S_f \to S_g$, $a \mapsto ag$ if $f \ge g$. Here we observe that S may be viewed as a category with objects all S_f, $f \in E(S)$

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(are objects in the category of groups) and arrows, also called morphisms, given for two objects S_f and S_g as follows: $\mathrm{Hom}(S_f,S_g) = \{\varphi_{f,g}\}$ if $f \geq g$ and $\mathrm{Hom}(S_f,S_g) = \varnothing$ otherwise.

If $\alpha : S \to T$ is a homomorphism between Clifford semigroups, then clearly $f \geq g$ in $E(S)$ implies $\alpha(f) \geq \alpha(g)$ in $E(T)$. Also $\alpha(S_f) \subset T_{\alpha(f)}$. It follows that α is a functor between categories S and T, that sends the object S_f in S to the object $T_{\alpha(f)}$ in T and the arrow $\varphi_{f,g}$ in S to the arrow $\varphi_{\alpha(f),\alpha(g)}$ in T.

In the present work we are concerned with those Clifford semigroups S such that S is also a monoid with identity e and $\varphi_{e,f} : S_e \to S_f$ is an epimorphism for all $f \in E(S)$, that is, $S_f = S_e f$ for all $f \in E(S)$. We then call S a q partial group (or q Clifford semigroup, etc.).

A partial group is defined in [3] as a semigroup S such that every $x \in S$ has so-called a partial identity e_x and a partial inverse x^{-1} satisfying

(i) $e_x x = x e_x = x$ and if $yx = xy = x$, then $e_x y = y e_x = e_x$,

(ii) $xx^{-1} = x^{-1}x = e_x$ and $e_x x^{-1} = x^{-1} e_x = x^{-1}$,

(iii) $e_{xy} = e_x e_y$ and $(xy)^{-1} = y^{-1}x^{-1}$ for $x, y \in S$. That is the map $x \mapsto e_x$ is a homomorphism and $x \mapsto x^{-1}$ is antihomomorphism from S into S.

It turns out that a partial group S is precisely a strong semilattice of groups $[E(S), S_f, \varphi_{f,g}]$, that is, a Clifford semigroup. Also, it is proved in [3] that every partial group S is embeddable in a partial group $\overset{\triangle}{P}(S,G)$ of partial mappings from S to a suitable group G (see Theorem 1.1). Such sort of structure and representation has been the basis for developing other kinds of "partial algebras" (see [1] for partial rings and [4] for partial monoids). Here in this section we will observe that any such $\overset{\triangle}{P}(S,G)$ is not just a partial group but also a q partial group in the sense given above. Whence partial mappings (between sets and groups) may be considered as natural sources of q partial groups.

In Section 2 we give some definitions and simple observations concerning q partial groups. We introduce the $\mathbf{Q}$ operation in Section 3 and show that every partial group S with identity contains a maximal q partial group $\mathbf{Q}(S)$ which is a nontrivial if and only if S_1 is a nontrivial group, where S_1 is the maximal group in S with identity 1_S. This holds obviously for any wide (full) subpartial group T of S. We show that the $\mathbf{Q}$ operation commutes with Cartesian products in partial groups with identities, and conclude that the product of any family of q partial groups is a q partial group. In Section 4 we extend the $\mathbf{Q}$ operation to the lattice $\mathbf{C}^{\mathbf{i}}(S)$ of idempotent separating congruences ρ on a partial group S with identity and show that this operation preserves normality in S. We introduce the notion of a q congruence on S in such a way that for any $\rho \in \mathbf{C}^{\mathbf{i}}(S), \mathbf{Q}(\rho)$ is the maximal q congruence contained in ρ and that $\mathbf{Q}(\rho_N) = \rho_{\mathbf{Q}(N)}$, where $N = \ker\rho$. This allows to establish a complete modular lattice isomorphism between q normal subpartial groups of S and q congruences on S. In Section 5 we are mainly concerned with those q partial groups S for which certain conditions are satisfied, and call them proper q partial groups. We show that associated to any group G there is a proper q partial group $P(G)$, for which $(P(G))_1 = G$, and give characterizations of different kinds of q partial groups S in terms of $P(S_1)$. This allows to embed arbitrary partial groups (i.e., Clifford semigroups) in proper q partial groups of the kind $P(G)$. When morphisms in categories are restricted to epimorphisms in groups (**G**) and to a certain kind of epimorphisms (called p epimorphisms) in proper q partial groups (**PQP**), we show in Section 6 that these two categories are equivalent. We also give certain characterizations of p epimorphisms in **PQP**.

Almost all notations used throughout the paper are standard. Otherwise full explanations are accomplished. The notation 1_S (or sometimes just 1) is used either to denote the identity element of (the algebra) S or the identity mapping $S \to S$, $x \mapsto x$. Its unique meaning in a definite situation is determined by the case analysis. Some notations (symbols) from logic may be used: for all, ⇒, ⇔ (in place of for all, implies, if and only if (iff), resp.). References for different topics are as follows:

(i) semigroups, in general: [5, 8];

(ii) groups: [6, 9];

(iii) categories: [7].

Notations, definitions, and results appeared in [2, 3] and needed for our work are summarized here for the sake of reference. *A subpartial group* of a partial group S is a subsemigroup K of S such that for all $x \in K$, $e_x \in K$, $x^{-1} \in K$. A subpartial group K is *wide* (or *full*) if $E(S) \subset K$. If S,T are partial groups, then $\alpha : S \to T$ is a homomorphism of partial groups if $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x,y \in S$. Monomorphisms, epimorphisms, and so forth, and automorphisms are defined also as in semigroups. If $\alpha : S \to T$ is a homomorphism of partial groups, then $e_{\alpha(x)} = \alpha(e_x)$ and $(\alpha(x))^{-1} = \alpha(x^{-1})$ for all $x \in S$. Thus $\alpha(S_e) \subset T_{\alpha(e)}$ for all $e \in E(S)$, where S_e is the maximal group in S with identity e. Also $\operatorname{Im}\alpha$ (range α); that is, $\alpha(S)$ is a subpartial group of T.

If X is a (nonempty) set and G is a group, there is a partial group denoted $\overset{\triangle}{P}(X,G)$ whose elements are all partial mappings $f : X \rightarrowtail G$ (i.e., domain $f \subset X$) with multiplications defined as follows. For $f,g \in \overset{\triangle}{P}(X,G)$ say $f : A \to G$, $g : B \to G$ $(A,B \subset X)$, fg is the partial mapping $X \rightarrowtail G$ with domain $fg = \operatorname{dom} f \cap \operatorname{dom} g = A \cap B$ and $fg : A \cap B \to G$, $x \mapsto f(x)g(x)$ where $f(x)g(x)$ is the multiplication in G. With this operation $\overset{\triangle}{P}(X,G)$ is a strong semilattice of groups (partial group) $[Y, M(A,G), \varphi_{A,B}]$, where $Y = \{e_A : e_A : A \to G, x \mapsto 1_G,\ A \subset X\}$ is the semilattice of idempotents in $\overset{\triangle}{P}(X,G)$ [$e_A \leq e_B$ if and only if $A \subset B$], $M(A,G)$ is the maximal group in $\overset{\triangle}{P}(X,G)$ with identity e_A, that is, the set of all mappings $f : A \to G$, and for $e_A \geq e_B$, that is, $B \subset A$, $\varphi_{e_{A,e_B}}$ (or simply $\varphi_{A,B}$) is the homomorphism $\varphi_{A,B} : M(A,G) \to M(B,G)$, $f \mapsto f|_B$ where $f|_B : B \to G$ is the restriction of $f : A \to G$ on B. As we declared above, $\overset{\triangle}{P}(X,G)$ is actually a q partial group. That is for all $B \subset X$, $\varphi_{X,B} : M(X,G) \to M(B,G)$ is an epimorphism of groups. For we have that $\overset{\triangle}{P}(X,G)$ has the identity element e_X which is the identity of the maximal group $M(X,G)$, and given $f \in M(B,G)$, the mapping $\overline{f} \in M(X,G)$ defined by $\overline{f}(x) = f(x)$ if $x \in B$ and $\overline{f}(x) = 1_G$ if $x \in X - B$ satisfies $\varphi_{X,B}(\overline{f}) = f$. Thus $\varphi_{X,B}$ is an epimorphism.

Let S be any partial group (i.e., Clifford semigroup). For each $x \in S$, define $S^x = \{y \in S : e_x y = y\}$ (called the *x-ball in S* [3]). We have $x, e_x \in S_x \subset S^x$ $(\forall x \in S)$,

$S^x = S^y$ if and only if $e_x = e_y$,

$S^x \cap S^y = S^{xy}$.

Let G be the coproduct of the maximal groups S_{e_x}, $e_x \in E(S)$, that is, $G = \amalg_{e_x} S_{e_x}$. Then for each $e_x \in E(S)$, there exists an injection (monomorphism) $i_{e_x} : S_{e_x} \to G$, satisfying the desired universal property. For each $x \in S$, define

$$\beta_x : S^x \longrightarrow G \quad \text{by } a \longmapsto i_{e_a}(xe_a). \tag{1.2}$$

Thus $\beta_x \in \overset{\triangle}{P}(S,G)$ and $\beta_x\beta_y = \beta_{xy}$ for all $x, y \in S$ [3].

We have the following theorem.

Theorem 1.1 [3, the representation theorem]. *Let S, G be as above. There exists a monomorphism (embedding) of partial groups*

$$\alpha : S \longrightarrow \overset{\triangle}{P}(S,G), \tag{1.3}$$

which sends each $x \in S$ to $\beta_x \in \overset{\triangle}{P}(S,G)$.

Let S be a partial group. A subpartial group K of S is *normal* (denoted by $K \lhd S$) if K is wide (i.e., $E(S) \subset K$) and $xKx^{-1} \subset K$ for all $x \in S$. Clearly $E(S)$ is normal in S (it is the trivial normal subpartial group of S). We have the following proposition.

Proposition 1.2 [2]. *If K is a normal subpartial group of S, then K_{e_x} is a normal subgroup of S_{e_x} for all $x \in S$.*

Let $\phi : S \to T$ be a homomorphism of partial groups. The *k-kernel* of ϕ (simply $k - \ker\phi$) is

$$k - \ker\phi = \{x \in S : \phi(x) = e \text{ for some } e \in E(T)\}. \tag{1.4}$$

If S is a partial group and ρ is a congruence on S, then S/ρ is a partial group called the *quotient partial group* induced by ρ. Moreover, $e_{x\rho} = e_x\rho$ and $(x\rho)^{-1} = x^{-1}\rho$ for all $x \in S$.

If $\phi : S \to T$ is a homomorphism of partial groups, then $\ker\phi = \{(x,y) \in S \times S : \phi(x) = \phi(y)\}$ is an idempotent separating congruence on S if and only if ϕ is idempotent separating.

Let S be a partial group. Then, a congruence ρ is idempotent separating if and only if $x\rho y \Rightarrow e_x = e_y$.

Associated to every congruence ρ on S there is a unique idempotent separating congruence on S denoted by ρ^i, such that $x\rho^i = x\rho \cap S_{e_x}$ for all $x \in S$.

Theorem 1.3 [2]. *Let K be a normal subpartial group of S. Define*

$$\rho_K = \{(x,y) \in S \times S : e_x = e_y \text{ and } xy^{-1} \in K\}. \tag{1.5}$$

Then

(i) *ρ_K is an idempotent separating congruence on S and $K = \ker\rho_K = k - \ker(\rho_K^\sharp)$, where $\rho_K^\sharp : S \to S/\rho_K$ is the canonical epimorphism,*

(ii) *$x\rho_K = xK_{e_x}$ for all $x \in S$,*

(iii) *$K = (E(S))\rho_K = \cup\{e_x\rho_K : e_x \in E(S)\}$.*

Theorem 1.4 [2]. *For every idempotent separating congruence ρ on S there exists a normal subpartial group K of S with $K = \ker\rho = E(S)\rho$ and $\rho = \rho_K$.*

If K is a normal subpartial group of a partial group S, the quotient partial group S/ρ_K is denoted by S/K, where ρ_K is the unique idempotent separating congruence on S associated with K (as in Theorem 1.4). We have the following.

Proposition 1.5 [2]. *$(S/K)_{xK_{e_x}} = S_{e_x}/K_{e_x}$ for all $x \in S$.*

Proposition 1.6 [2]. *If $\phi : S \to T$ is a homomorphism of partial groups, then $\rho_{k-\ker\phi} = (\ker\phi)^i$.*

Theorem 1.7 [2]. *Let $\phi : S \to T$ be a homomorphism of partial groups.*

(a) *There exists a unique homomorphism $\alpha : S/k - \ker\phi = S/(\ker\phi)^i \to T$ such that* $\operatorname{ran}(\alpha) = \operatorname{ran}(\phi)$ *and the diagram*

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "\rho^{\neq}_{k-\ker\phi}"'] & T \\ S/(\ker\phi)^i \arrow[ur, "\alpha"'] & \end{tikzcd} \tag{1.6}$$

commutes.

(b) *α is a monomorphism if and only if ϕ is idempotent separating.*

(c) *α is an isomorphism if and only if ϕ is both an epimorphism and idempotent separating.*

Theorem 1.8 [2]. *Let $\phi : S \to T$ be a homomorphism of partial groups. If K is a normal subpartial group of S such that $K \subset k - \ker\phi$, then there exists a unique homomorphism $\alpha : S/K \to T$ such that* $\operatorname{ran}(\alpha) = \operatorname{ran}(\phi)$ *and the diagram*

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "\rho^{\neq}_{k}"'] & T \\ S/k \arrow[ur, "\alpha"'] & \end{tikzcd} \tag{1.7}$$

commutes.

Let S be a partial group. The set of all idempotent separating congruence on S is denoted by $\mathbf{C^i}(S)$. We have $\mathbf{C^i}(S) = \{\rho^i : \rho \text{ is a congruence on } S\}$, where ρ^i is the maximal idempotent separating congruence on S such that $\rho^i \subset \rho$. We have $\rho \circ \sigma = \sigma \circ \rho \in \mathbf{C^i}(S)$ for all $\rho, \sigma \in \mathbf{C^i}(S)$.

Proposition 1.9 [2]. *$(\mathbf{C^i}(S), \subset, \cap, \vee)$ is a complete modular lattice with $\rho \vee \sigma = \rho \circ \sigma$ for all $\rho, \sigma \in \mathbf{C^i}(S)$.*

The set of all normal subpartial groups of S is denoted by $\mathbf{N}(S)$.

For $M, N \in \mathbf{N}(S)$, $MN = NM$ is the minimal normal subpartial group of S containing $M \cup N$, that is, $MN = NM = \langle M \cup N \rangle$ is the (normal) subpartial group of S generated by $M \cup N$ or the join $M \vee N$ of M and N.

Proposition 1.10 [2]. *If $M, N \in \mathbf{N}(S)$, then $\rho_{M\cap N} = \rho_M \cap \rho_N$; $\rho_{MN} = \rho_M \circ \rho_N$.*

Theorem 1.11 [2]. $(\mathbf{N}(S),\subset,\cap,\vee)$ *is a complete modular lattice and the mapping*

$$\varphi:\mathbf{N}(S)\longrightarrow \mathbf{C}^{\mathbf{i}}(S),\qquad K\longmapsto \rho_K \tag{1.8}$$

is a lattice isomorphism.

2. q partial groups

Throughout this section, S stands for a partial group with identity 1. Thus S is a strong semilattice of groups $[E(S);S_f,\varphi_{f,g}]$ with $E(S)$ having upper bound 1. We call the identity 1 of S *proper* if the maximal subgroup S_1 is not the trivial group, that is, if $\{1\}$ is a proper subset of S_1. Otherwise, 1 is called *improper*. In the usual partial ordering of $E(S)$, we then have $1\geq e$ for all $e\in E(S)$, and so we have a homomorphism of groups $\varphi_{1,e}:S_1\to S_e$ for every $e\in E(S)$.

We call S a *q partial group* if $\varphi_{1,e}$ is an epimorphism for every $e\in E(S)$. This is equivalent to say that $S_1e=S_e$ for every $e\in E(S)$, that is, every $x\in S_e$ can be written as a product ye for some $y\in S_1$. Since $S_1e\subset S_e$ always holds, S is a q partial group if and only if $S_e\subset S_1e$ for all $e\in E(S)$.

If S is q partial group and $e\geq f$ in $E(S)$, we then have $\varphi_{e,f}(S_e)=S_ef=(S_1e)f=S_1(ef)=S_1f=S_f$. It follows that in a q partial group S, every homomorphism $\varphi_{e,f}$ is an epimorphism. We observe also that in a q partial group S, $S_e\cdot S_f=S_{ef}$ for all $e,f\in E(S)$. For, we have $S_e\cdot S_f\subset S_{ef}$ since S is a strong semilattice of its maximal subgroups, and if $x\in S_{ef}$, then $x=yef$ for some $y\in S_1$, which gives $x=(ye)(1f)\in S_e\cdot S_f$.

Let T be a wide subpartial group of S. Then we can call T a *q subpartial group* of S if the restriction $\varphi_{1,e}$ on T_1 is an epimorphism for every $e\in E(S)$, that is, T is a q subpartial group with the inherited operations from S. Trivially, every semilattice with upper bound is a q partial group, and hence $E(S)$ is a q subpartial group of S. If the identity 1 of S is improper, we clearly have $S=E(S)$, and so S reduces to a semilattice. The converse holds trivially. Thus for any q partial group, we have $S\neq E(S)$ if and only if 1 is proper, that is, if and only if S_1 is not the trivial group. As we observed in the introduction, the partial group $\overset{\triangle}{P}(X,G)$, for any set X and group G, is a q partial group with identity $1_X:X\to G$, $x\mapsto 1_G$, the identity 1_X is the identity of the maximal group $M(X,G)$ in $\overset{\triangle}{P}(X,G)$. Clearly $M(X,G)$ is not the trivial group if and only if X is nonempty and G is not the trivial group. Thus $\overset{\triangle}{P}(X,G)$ is a nontrivial q partial group if and only if $X\neq\varnothing$ and $G\neq 0$. We close this section by one more simple observation.

Lemma 2.1. *If S is a q partial group in which no two maximal subgroups are isomorphic, then the kernels of the epimorphisms $\varphi_{1,e}$, $e\in E(S)$, are all different.*

Proof. Let N_e denote the kernel of $\varphi_{1,e}$, $e\in E(S)$. Then $N_e=\{y\in S_1: ye=e\}$. If $e\neq f$ in $E(S)$ and $N_e=N_f$, we have by the first isomorphism theorem of groups

$$S_e\cong S_1/N_e=S_1/N_f\cong S_f \tag{2.1}$$

which contradicts the hypothesis. □

3. The Q operation

In the previous section we noticed that every partial group S with identity contains a trivial q subpartial group, namely, $E(S)$. In this section, we show that nontrivial q subpartial groups of S exist whenever S_1 is a nontrivial group. More precisely, a maximal q subpartial group of S always exists. This inherited to all wide subpartial groups T of S, and hence defines an operation $T \to \mathbf{Q}(T)$. In later work, we will show that the $\mathbf{Q}$ operation preserves normality, and commutes with the operation of taking joins. In this section, we show that it commutes with categorical products. Given a wide subpartial groups T of S, the existence of $\mathbf{Q}(T)$, the maximal q subpartial group contained in T can be verified by the axiom of choice (e.g., Zorn's lemma), but for later purpose we construct $\mathbf{Q}(T)$ explicitly. It is obtained simply by taking images of $\varphi_{1,e}$ on T_1 for all $e \in E(S)$. Formally, we have the following lemma.

Lemma 3.1. *Let S be a partial group with identity and let T be a wide subpartial group of S. There exists a q subpartial group $\mathbf{Q}(T)$ of S which is unique maximal such that $\mathbf{Q}(T) \subset T$. Moreover $\mathbf{Q}(T)$ is nontrivial (i.e., does not equal $E(S)$) if and only if T_1 is a nontrivial group.*

Proof. Since T is wide, it is a union of maximal groups indexed by $E(S)$, that is, $T = [E(S), T_e, \varphi_{e,f}]$ where T_e is a subgroup of S_e and $\varphi_{1,e} : T_1 \to T_e$ is a homomorphism ($x \mapsto xe$) for every $e \in E(S)$. Define

$$\mathbf{Q}(T) = \bigcup_{e\in E(S)} \operatorname{Im}\varphi_{1,e} = \bigcup_{e\in E(S)} T_1 e. \tag{3.1}$$

$\mathbf{Q}(T)$ is a disjoint union of groups $(\mathbf{Q}(T))_e = T_1 e$ indexed by the semilattice $E(S)$. In particular, $(\mathbf{Q}(T))_1 = \operatorname{Im}\varphi_{1,1} = T_1$ and the restriction of $\varphi_{1,e}$ on T_1 gives an epimorphism $\varphi_{1,e} : (\mathbf{Q}(T))_1 = T_1 \to (\mathbf{Q}(T))_e$ for every $e \in E(S)$.

It follows that $\mathbf{Q}(T)$ is q subpartial group of S contained in T.

For $e > f$, $\varphi_{e,f} : T_1 e \to T_1 f$ is given by $xe \mapsto xf$, $(x \in T_1)$.

If K is q subpartial group of S with $K \subset T$, then $K_1 \subset T_1 = (\mathbf{Q}(T))_1$ and for all $e \in E(S)$, $K_e = K_1 e \subset T_1 e = (\mathbf{Q}(T))_e$. This proves the unique maximality of $\mathbf{Q}(T)$.

Finally, $\mathbf{Q}(T) \neq E(S) \Rightarrow (\mathbf{Q}(T))_e \neq \{e\}$ for some $e \in E(S) \Rightarrow T_1 e \neq \{e\} \Rightarrow T_1 \neq \{1\}$, conversely, if $T_1 \neq \{1\}$, then $(\mathbf{Q}(T))_1 = T_1 \neq \{1\}$, and so $\mathbf{Q}(T) \neq E(S)$. □

Let us now consider partial groups as a part of universal algebra, that is as a variety of algebras (defined by a set of identities). This implies that, as a category, partial groups have all small limits and colimits (e.g., products, coproducts, etc.). This is also true for partial groups with identities. In the rest of this section we consider categorical products of partial groups (with identities) and show that the $\mathbf{Q}$ operation commutes with this product which implies that product of any family of q partial groups is again a q partial group. We start by characterizing products in the category of partial groups.

Lemma 3.2. *Let $\{S_i,\ i \in I\}$ be a family of partial groups and let $S = \prod_{i\in I} S_i$ be the usual Cartesian product. Then S is a partial group which is a categorical product with the usual projections $\pi_i : S \to S_i$ $(x_i) \mapsto x_i$. If each S_i has an identity 1_{s_i}, then (1_{s_i}) is the identity of S.*

Proof. Define $E(S) = \prod_{i\in I} E(S_i)$. Then $E(S)$ is a semilattice with $(e_i) \leq (f_i)$ if and only if $e_i \leq f_i$, for all $i \in I$.

We have

$$S = \prod_{i\in I} S_i = \prod_{i\in I} \bigcup_{e_i\in E(S_i)} (S_i)_{e_i} = \bigcup_{(e_i)\in E(S)} \prod_{i\in I} (S_i)_{e_i}. \tag{3.2}$$

Thus S is a disjoint union of groups $\prod_{i\in I}(S_i)_{e_i}$, $e_i \in E(S_i)$ with identities $(e_i)_{i\in I}$, $e_i \in E(S_i)$, indexed by the semilattice $E(S)$. For $(e_i) \geq (f_i)$ in $E(S)$, there is a homomorphism

$$\varphi_{(e_i),(f_i)} : \prod_{i\in I} (S_i)_{e_i} \longrightarrow \prod_{i\in I} (S_i)_{f_i}, \tag{3.3}$$

given by

$$(x_i) \longmapsto (\varphi_{e_i,f_i} x_i). \tag{3.4}$$

Now we can easily verify that S is a categorical product, and that S has identity if each S_i has a one. □

THEOREM 3.3. *Let $\{S_i,\ i \in I\}$ be a family of partial groups, with identities. Then $\mathbf{Q}(\prod_{i\in I} S_i) = \prod_{i\in I} \mathbf{Q}(S_i)$.*

Proof. By Lemma 3.2, $\prod_{i\in I} S_i$ is a partial group with identity $(1_{s_i})_{i\in I}$ which is a union of maximal subgroups $(\prod_{i\in I} S_i)_{(e_i)}$ indexed by the semilattice $E(S) = \prod_{i\in I} E(S_i)$. By Lemma 3.1, we have

$$\begin{aligned}
\mathbf{Q}\Big(\prod_{i\in I} S_i\Big) &= \bigcup_{(e_i)\in E(S)} \Big(\prod_{i\in I} S_i\Big)_{(1_{s_i})} (e_i) \\
&= \bigcup_{(e_i)\in E(S)} \Big(\prod_{i\in I} (S_i)_{1_{s_i}} e_i\Big) = \prod_{i\in I} \Big(\bigcup_{e_i\in E(S_i)} (S_i)_{1_{s_i}} e_i\Big) \\
&= \bigcup_{(e_i)\in E(S)} \Big(\prod_{i\in I} (S_i)_{1_{s_i}} e_i\Big) = \prod_{i\in I} \mathbf{Q}(S_i).
\end{aligned} \tag{3.5}$$

□

By the definition of the $\mathbf{Q}$ operation, one can show that the product of any family of q partial groups is again a q partial group. But if we notice that for any partial group S with identity, S is a q partial group if and only if $\mathbf{Q}(S) = S$, then the following is an easy consequence of Theorem 3.3.

COROLLARY 3.4. *If $\{S_i,\ i \in I\}$ is a family of q partial groups, then the product $\prod_{i\in I} S_i$ is a q partial group.*

4. Normality and q congruences

In this section we develop certain properties of the $\mathbf{Q}$ operation needed for further work. We show it preserves normality in partial groups with identities. The notation of a q normal subpartial group will play an important role in later work, so we introduce the notion of a q congruence on a partial group S with identity in such a way that ρ is a q congruence on S if and only if $\ker\rho$ is a q normal subpartial group of S. For any idempotent separating congruence ρ on S we define $\mathbf{Q}(\rho)$ and show it is the maximal q congruence contained in ρ. This extends the $\mathbf{Q}$ operation to the lattice of all idempotent separating congruences on S, with the property that if S is a q partial group, then for any normal subpartial group N of S, $\mathbf{Q}(N)$ is normal in S and $\mathbf{Q}(\rho_N) = \rho_{\mathbf{Q}(N)}$. This allows to establish a complete modular lattice isomorphism between q normal subpartial groups of S and q congruence on S, which is analogous to the classical result (see, e.g., [2]) known for idempotent separating congruences.

We begin by a technical lemma showing that the join of any family of q subpartial groups is again a q subpartial group. First, we need to recall some preliminaries.

Remark 4.1. If S is any partial group and X is any (nonempty) subset of S, the subpartial group $\langle X\rangle$ generated by X is the intersection of all subpartial groups of S containing X. Actually, $\langle X\rangle$ is the set of all finite products $x_{i_1}^{n_1}x_{i_2}^{n_2}\cdots x_{i_k}^{n_k}$ with $x_{ij}\in X$, $n_j=\pm 1$. If $X\cap S_e\neq\varnothing$ for all $e\in E(S)$, then $\langle X\rangle$ is also wide. The elements of X are the generators of $\langle X\rangle$. If $s\in\langle X\rangle$, say $s\in(\langle X\rangle)_e$ for some $e\in E(S)$ (i.e., $e_s=e$), then s may be represented by a finite product

$$s = x_{i_1}x_{i_2}\cdots x_{i_k}, \tag{4.1}$$

with $x_{ij}\in S_e$ for all $j=1,\dots,k$, (just multiply each generator in the typical expansion of s by e).

Lemma 4.2. *Let S be a partial group with identity and let $\{S_i,\ i\in I\}$ be a family of q subpartial groups of S. Then $\langle\bigcup_{i\in I}S_i\rangle$ is a q subpartial group of S.*

Proof. Clearly $\langle\bigcup_{i\in I}S_i\rangle$ is a wide subpartial group of S. Let $e\in E(S)$ and let $s\in(\langle\bigcup_{i\in I}S_i\rangle)_e$. By the above remark (and since each S_i is a subpartial group of S), we have

$$s = x_{i_1}x_{i_2}\cdots x_{i_k}, \tag{4.2}$$

with $x_{ij}\in(S_{ij})_e$, $j=1,\dots,k$.

Since S_{ij} is a q *subpartial group of* S,

$$\left(S_{ij}\right)_e = \left(S_{ij}\right)_1 e, \quad j=1,\dots,k. \tag{4.3}$$

Therefore

$$s = (s_1s_2\cdots s_k)e \in \left(\left\langle\bigcup_{i\in I}S_i\right\rangle\right)_1 e. \tag{4.4}$$

This gives

$$\left(\left\langle\bigcup_{i\in I}S_i\right\rangle\right)_e = \left(\left\langle\bigcup_{i\in I}S_i\right\rangle\right)_1 e, \tag{4.5}$$

and the result obtains. $\square$

Now we show that the $\mathbf{Q}$ operation commutes with the join operation on wide subpartial groups.

Lemma 4.3. *Let S be as in Lemma 4.2, and let $\{S_i,\ i \in I\}$ be a family of wide subpartial groups of S. Then $\mathbf{Q}(\langle \bigcup_{i\in I} S_i\rangle) = \langle \bigcup_{i\in I} \mathbf{Q}(S_i)\rangle$.*

Proof. By Lemma 4.2, $\langle \bigcup_{i\in I} \mathbf{Q}(S_i)\rangle$ is *a q subpartial group of S*, which is clearly contained in $\langle \bigcup_{i\in I} S_i\rangle$. Thus it is sufficient to show that it is maximal with respect to this property. For this, let T be *a q subpartial group of S* with $T \subset \langle \bigcup_{i\in I} S_i\rangle$, and let $t \in T$, say $t \in T_e = T_1 e$, for some $e \in E(S)$. Thus $t = se$, for some $s \in T_1$, and since $T_1 \subset (\langle \bigcup_{i\in I} S_i\rangle)_1$, we obtain

$$s = x_{i_1} x_{i_2} \cdots x_{i_k}, \tag{4.6}$$

with $x_{ij} \in (S_{ij})_1$, $j = 1,\dots,k$.

This gives

$$\begin{aligned} t &= (x_{i_1} x_{i_2} \cdots x_{i_k}) e \\ &= (x_{i_1} e) \cdots (x_{i_k} e) \in \big(\mathbf{Q}(S_{i_1})\big)_e \cdots \big(\mathbf{Q}(S_{i_k})\big)_e, \quad \text{since } (S_{ij})_1 e = \big(\mathbf{Q}(S_{i_j})\big)_e. \end{aligned} \tag{4.7}$$

Therefore

$$t \in \Big\langle \bigcup_{i\in I} \mathbf{Q}(S_i) \Big\rangle. \tag{4.8}$$

This proves maximality, and the proof is complete. □

Given a partial group S with identity, then clearly, a normal subpartial group of S need not be a q subpartial group. For a trivial example, take S such that 1_s is improper, and $S \neq E(S)$. Then S is normal in S but not a q partial group. This is also the case even if S is a q partial group, for example, take S such that 1_s is proper, $S_e \neq \{e\}$, for some $e \in E(S)$ and let $T < S$ be such that $T_1 = \{1_s\}$ and $T_e = S_e$ otherwise. On the other hand, the $\mathbf{Q}$ operation preserves normality in q partial groups. This is our next result.

Lemma 4.4. *Let S be a q partial group if N is a normal subpartial group of S, so is $\mathbf{Q}(N)$.*

Proof. Let $s \in S$, $x \in \mathbf{Q}(N)$, say $s \in S_f$ and $x \in (\mathbf{Q}(N))_g$, for some $f,g \in E(S)$.

We have

$$\big(\mathbf{Q}(N)\big)_g = \big(\mathbf{Q}(N)\big)_1 g, \tag{4.9}$$

and so $x = yg$, for some $y \in (\mathbf{Q}(N))_1$. By the definition of the $\mathbf{Q}$ operation and the normality of N, we have

$$y \in \big(\mathbf{Q}(N)\big)_1 = N_1 \lhd S_1. \tag{4.10}$$

Also $s \in S_f = S_1 f$, say, $s = s_1 f$ for some $s_1 \in S_1$.

Thus

$$s^{-1}xs = (s_1 f)^{-1}(yg)(s_1 f) = (s_1^{-1}ys)(fg) \in N_1 fg = (\mathbf{Q}(N))_{fg} \subset \mathbf{Q}(N). \tag{4.11}$$

□

Before introducing q congruences on a partial group S with identity, we consider the set of all q normal subpartial groups of S and show it is a lattice whenever S is a q partial group. We know, in the lattice $\mathbf{N}$ of all normal subpartial groups of a partial group S, the join and meet are given by $M \vee N = MN = \langle M \cup N \rangle$ and $M \wedge N = M \cap N$, respectively. For q normal subpartial groups M, N, if S has an identity, then $\langle M \cup N \rangle$ is already a q normal subpartial group of S (Lemma 4.2). However, $M \cap N$ need not be a q (normal) subpartial group of S, even if S is a q partial group, because the homomorphism $\varphi_{1,e}$ need not be one-to-one for arbitrary $e \in E(S)$. Here is a simple example.

Example 4.5. Let $X = \{a,b\}$ and $G = Z_2 = \{0,1\}$, and let $S = P(X,G)$ be the corresponding q partial group of partial mappings (see Sections 1, 2).

Then $P(X,G)$ is the union of 4 maximal groups, $\overset{\triangle}{P}(S,G) = M(X,G) \cup M(\{a\},G) \cup M(\{b\},G) \cup \{z\}$, where z represents the empty mapping $\phi \to G$, and is the zero element of $\overset{\triangle}{P}(S,G)$. Let $T_1 = \{(a,b) \mapsto (0,0),(a,b) \mapsto (1,1)\}$. Then T_1 is a subgroup of $M(X,G)$ consists of just two mappings.

Let $R_1 = \{(a,b) \mapsto (0,0),(a,b) \mapsto (1,0)\}$. Let e_a be the mapping $\{a\} \to G$, $a \mapsto 0$, and let e_b be the mapping $\{b\} \to G$, $b \mapsto 0$.

Denoting $T_1 e_a$ by T_a, and so forth, we have

$T_a = T_1 e_a = \{a \mapsto 0, a \mapsto 1\}$,
$T_b = T_1 e_b = \{b \mapsto 0, b \mapsto 1\}$,
$T_z = T_1 z = \{z\}$,
$R_a = R_1 e_a = \{a \mapsto 0, a \mapsto 1\}$,
$R_b = R_1 e_b = \{b \mapsto 0, b \mapsto 1\}$,
$R_z = R_1 z = \{z\}$.

Let $T = T_1 \cup T_a \cup T_b \cup T_z$, and $R = R_1 \cup R_a \cup R_b \cup R_z$. With the induced operation from S, both T and R are q normal subpartial groups of S.

But $T \cap R = \{(a,b) \mapsto (0,0)\} \cup \{a \mapsto 0, a \mapsto 1\} \cup \{b \mapsto 0, b \mapsto 1\} \cup \{z\}$ is a (normal) subpartial group of S, which is not a q subpartial group.

Let S be a q partial group and let $\mathbf{QN}(S)$ denote the set of all q normal subpartial groups of S. In view of Lemmas 3.1, 4.2, and 4.4, we easily obtain the following result.

Lemma 4.6. $\mathbf{QN}(S)$ *is a complete modular lattice with meet and join defined by*

$$M \wedge N = \mathbf{Q}(M \cap N), \qquad M \vee N = MN = \langle M \cup N \rangle. \tag{4.12}$$

The notion of a q normal subpartial group of a partial group S with identity is nothing but an idempotent separating congruence on S that satisfies certain condition. Here we give the definition.

Given a partial group S with identity, we call an idempotent separating congruence ρ on S a *q congruence on* S if for all $x, y \in S$, $x\rho y$ implies $x = sy$, for some $s \in (\ker\rho)_1$, that is, for some $s \in S_1$ with $s\rho 1$.

We observe that, if ρ is q congruence, then $x\rho y$ implies (also) $x = ys$, for some $s\rho 1$.

The following consequence follows at once from the definition. (Recall that ρ_K is the congruence whose kernel is K.)

Lemma 4.7. *Let S be a partial group with identity, and let ρ be an idempotent separating congruence on S. Then ρ is q congruence if and only if $K = \ker\rho$ is a q normal subpartial group of S. Equivalently, for any subpartial group K of S, K is a q normal subpartial group of S if and only if ρ_K is a q congruence.*

Let S be a partial group with identity and $\mathbf{C}^{\mathbf{i}}(S)$ the lattice of all idempotent separating congruences on S. Recall that, for $\rho,\sigma \in \mathbf{C}^{\mathbf{i}}(S)$, the join and meet are given by $\rho \circ \sigma = \sigma \circ \rho = \rho \vee \sigma$ (e.g., [2], Lemma 5.1) and $\rho \wedge \sigma = \rho \cap \sigma$, respectively.

Again, the intersection of two q congruences on S needs not be a q congruence (e.g., apply Lemma 4.7 to Example 4.5). So we define a $\mathbf{Q}$ operation on $\rho \in \mathbf{C}^{\mathbf{i}}(S)$ as follows:

$$\mathbf{Q}(\rho) = \{(x,y) : x\rho y,\ x = sy (\text{or } x = ys), \text{ for some } s \in S \text{ with } s\rho 1\}. \tag{4.13}$$

Equivalently, $\mathbf{Q}(\rho)$ is the unique maximal q congruence on S contained in ρ.

Lemma 4.8. *Let S be a q partial group and let $\rho \in \mathbf{C}^{\mathbf{i}}(S)$. Then,*

$$\mathbf{Q}(\rho) = \rho_{\mathbf{Q}(N)}, \quad \textit{where } N = \ker\rho. \tag{4.14}$$

Proof. By definition, $N = \{x \in S : x\rho e \text{ for some } e \in E(S)\}$, and $\rho = \rho_N$. By Lemma 4.4, $\mathbf{Q}(N)$ is normal in S. Now, let $(x,y) \in \mathbf{Q}(\rho) = \mathbf{Q}(\rho_N)$. Then $x\rho y$ and $x = sy$, for some $s \in S$ with $s\rho 1$, that is, for some $s \in (\ker\rho)_1 = N_1$. We have $e_x = e_y$ (since ρ is idempotent separating), and $xy^{-1} \in N_{e_x}$.

By definition of the $\mathbf{Q}$ operation

$$\big(\mathbf{Q}(N)\big)_{e_x} = N_1 e_x, \tag{4.15}$$

and so

$$xy^{-1} = (sy)y^{-1} = se_x \in N_1 e_x \subset \mathbf{Q}(N), \tag{4.16}$$

whence

$$(x,y) \in \rho_{\mathbf{Q}(N)}. \tag{4.17}$$

Thus

$$\mathbf{Q}(\rho) \subset \rho_{\mathbf{Q}(N)}. \tag{4.18}$$

Conversely, let $(x,y) \in \rho_{\mathbf{Q}(N)}$. Thus

$$e_x = e_y, \qquad xy^{-1} \in \big(\mathbf{Q}(N)\big)_{e_x} = N_1 e_x. \tag{4.19}$$

That is

$$xy^{-1} = se_x \quad \text{for some } s \in N_1. \tag{4.20}$$

Thus

$$xy^{-1} \in N, \quad x = sy \text{ for some } s \in N_1 = (\ker \rho)_1, \tag{4.21}$$

and so

$$(x, y) \in \mathbf{Q}(\rho) = \mathbf{Q}(\rho_N). \tag{4.22}$$

Therefore

$$\mathbf{Q}(\rho) = \mathbf{Q}(\rho_N) = \rho_{\mathbf{Q}(N)}. \tag{4.23}$$

□

For a q partial group S, let $\mathbf{QC^i}(S)$ denote the set of all q congruences on S. We have the following theorem.

THEOREM 4.9. $\mathbf{QC^i}(S)$ *is a complete modular lattice, with meet and join given by*

$$\begin{aligned} \rho \wedge \sigma &= \mathbf{Q}(\rho \cap \sigma), \\ \rho \circ \sigma = \sigma \circ \rho &= \rho \vee \sigma, \quad \textit{respectively.} \end{aligned} \tag{4.24}$$

Moreover, the mapping

$$\varphi : \mathbf{QN}(S) \longrightarrow \mathbf{QC^i}(S), \quad N \longmapsto \rho_N \tag{4.25}$$

is a lattice isomorphism.

Proof. It is easy to see that $\rho \circ \sigma \in \mathbf{QC^i}(S)$, if $\rho, \sigma \in \mathbf{QC^i}(S)$, and that $\mathbf{QC^i}(S)$ is a lattice. The mapping φ is well defined by Lemma 4.7, and clearly is one-to-one and onto. By Lemmas 4.6 and 4.8, we have for $N, M \in \mathbf{QN}(S)$

$$\begin{aligned} \varphi(N \wedge M) &= \varphi(\mathbf{Q}(N \cap M)) = \rho_{\mathbf{Q}(N \cap M)} = \mathbf{Q}(\rho_{N \cap M}) \\ &= \mathbf{Q}(\rho_N \cap \rho_M) = \rho_N \wedge \rho_M = \varphi(N) \wedge \varphi(M). \end{aligned} \tag{4.26}$$

Likewise,

$$\varphi(N \vee M) = \varphi(N) \vee \varphi(M). \tag{4.27}$$

□

5. Developing certain representations

Given a q partial group S, there is an isomorphism of groups $S_1/N_e \cong S_e$, for every $e \in E(S)$, where $N_e = \ker \varphi_{1,e}$. On the other hand, given a (nonzero) group G, we can (and do) generate a q partial group $P(G)$, such that $(P(G))_1 = G$ and every maximal group in $\mathrm{P}(G)$ is a quotient G/N, for some normal N in G. In this section, we develop the connection between particular types of q partial groups S, and their associated q partial groups $\mathrm{P}(S_1)$. Naturally, partial groups of partial mappings $\overset{\triangle}{P}(X, G)$ are q partial groups. This allows to embed arbitrary partial groups (i.e., Clifford semigroups) in a q partial groups of the kind $P(G)$.

Let G be a (nontrivial) group and let $\mathbf{N}$ be the set of all normal subgroups of G. We define a partial ordering $\leq$ on $\mathbf{N}$ by

$$M \leq N \Longleftrightarrow N \subset M. \tag{5.1}$$

We have a semilattice $\langle \mathbf{N}, \leq, \wedge \rangle$ with $M \wedge N = \langle M \cup N \rangle = MN$, for all $N, M \in \mathbf{N}$.

$\mathbf{N}$ has an upper bound 1_G (and lower bound $\{G\}$). If $\mathbf{N}$ is indexed by a set $\mathbf{I}$, we have a bijection

$$\mathbf{I} \longrightarrow \mathbf{N}, \quad i \longmapsto N_i. \tag{5.2}$$

This bijection is a semilattice isomorphism if we define $\leq$ on $\mathbf{I}$ by $i \leq j$ if and only if $N_i \leq N_j$; if and only if $N_j \subset N_i$, for all $i, j \in \mathbf{I}$. $i \wedge j = ij = k$ if and only if $N_i N_j = N_k$. In this case, we have the following lemma.

Lemma 5.1. *Let G be a group, and let $N = \{N_i : i \in \mathbf{I}\}$ be the semilattice of all normal subgroups of G defined as above. Let*

$$P(G) = \bigcup_{i \in \mathbf{I}} G/N_i. \tag{5.3}$$

Then $P\,(G)$ is a q partial group with multiplication

$$xN_i \cdot yN_j = xyN_iN_j = xyN_{ij}, \tag{5.4}$$

identity element $1 = 1_G$, and zero element $z = G/G$.

Proof. For $N_i \geq N_j$ (equivalently, $i \geq j$, $N_i \subset N_j$), define

$$\varphi_{i,j} : G/N_i \longrightarrow G/N_j, \quad xN_i \longmapsto xN_j. \tag{5.5}$$

We have $x_1N_i = x_2N_i \Rightarrow x_1x_2^{-1} \in N_i \subset N_j \Rightarrow x_1N_j = x_2N_j$, and hence $\varphi_{i,j}$ is well defined.

For $x_1N_i, x_2N_i \in G/N_i$,

$$\begin{aligned} \varphi_{i,j}(x_1N_i\, x_2N_i) &= \varphi_{i,j}(x_1x_2N_i) = x_1x_2N_j \\ &= x_1N_jx_2N_j = \varphi_{i,j}(x_1N_i)\varphi_{i,j}(x_2N_i). \end{aligned} \tag{5.6}$$

Thus $\varphi_{i,j}$ is a homomorphism of groups which is clearly an epimorphism. Identifying $i \in \mathbf{I}$ with the identity of G/N_i (for all $i \in \mathbf{I}$), we have for all $i \in \mathbf{I}$ an epimorphism

$$\varphi_{1,i} : G/\{1_G\} = G \longrightarrow G/N_i, \quad x \longmapsto xN_i. \tag{5.7}$$

Clearly, $\varphi_{i,i}$ is the identical automorphism, and $\varphi_{j,k} \circ \varphi_{i,j} = \varphi_{i,k,}$ for every $i \geq j \geq k$ in $\mathbf{I}$. Thus $P(G)$ is a q partial group with

$$xN_iyN_j = \varphi_{i,ij}xN_i \cdot \varphi_{j,ij}yN_j = xN_{ij}\,yN_{ij} = xyN_{ij} = xyN_iN_j. \tag{5.8}$$

The identity 1 of $P(G)$ is the identity of the maximal group $(P(G))_1 = G/\{1_G\} = G$ which is 1_G, and it has a zero $z =$ (the identity of) G/G. $\square$

Our next technicality holds for any arbitrary partial group.

Lemma 5.2. *Let S be a partial group, and let $\{T_{e'}\}$ be a family of (pairwise disjoint) groups indexed by a semilattice $\{e' : e \in E(S)\}$ isomorphic to $E(S)$, such that corresponding groups are isomorphic. Then $T = \cup T_{e'}$ is a partial group isomorphic to S.*

Proof. We have a semilattice of groups

$$S = [E(S), S_e, \varphi_{e,f}]. \tag{5.9}$$

Let $E(T)$ denote the semilattice $\{e' : e \in E(S)\}$ *isomorphic to* $E(S)$, with $e \mapsto e'$. For each $e \in E(S)$, there exists an isomorphism of groups

$$\alpha_e : S_e \xrightarrow{\cong} T_{e'}. \tag{5.10}$$

For $e' \geq f'$ in $E(T)$, $e \geq f$ in $E(S)$, and we define

$$\varphi_{e',f'} : T_{e'} \longrightarrow T_{f'}, \tag{5.11}$$

by

$$\varphi_{e',f'} = \alpha_f \varphi_{e,f} \alpha_e^{-1}. \tag{$*$}$$

It is easily seen that $\varphi_{e',f'}$ is a well-defined epimorphism of groups, that $\varphi_{e',e'}$ is the identical automorphism of $T_{e'}$, and that $\varphi_{f',g'} \cdot \varphi_{e',f'} = \varphi_{e',g'}$ for all $e' \geq f' \geq g'$ in $E(T)$. In other words, T is a semilattice $E(T)$ of groups $T_{e'}$, and we identify e' with the identity of $T_{e'}$ for each $e' \in E(T)$. Thus $E(T)$ is the semilattice of idempotents in T. We now extend the isomorphism

$$E(S) \longrightarrow E(T), \qquad e \longmapsto e' \tag{5.12}$$

to an isomorphism $S \to T$. Define $\alpha : S \to T$ as follows: for any $x \in S$, say $x = x_e \in S_e$ for some $e \in E(S)$, set

$$\alpha(x_e) = \alpha_e(x_e). \tag{5.13}$$

Thus α is a well-defined mapping (since $S_e, e \in E(S)$ are pairwise disjoint) with restriction, an isomorphism $\alpha_e : S_e \to T_{e'}$, for each $e \in E(S)$. For arbitrary $x, y \in S$, say, $x = x_e \in S_e, y = y_f \in S_f$, we have $xy \in S_{ef}$. Thus

$$\alpha(xy) = \alpha(x_e y_f) = \alpha(x_e y_f ef) = \alpha_{ef}(x_e y_f ef). \tag{5.14}$$

Using $(*)$, and since $e \mapsto e'$ is isomorphism, we also have

$$\begin{aligned}
\alpha(x)\alpha(y) &= \alpha(x_e)\alpha(y_f) = \alpha_e(x_e)\,\alpha_f(y_f) \\
&= \varphi_{e',e'f'}(\alpha_e(x_e)) \circ \varphi_{f',e'f'}(\alpha_f(y_f)) \\
&= \alpha_{ef}(\varphi_{e,ef}\,\alpha_e^{-1}(\alpha_e(x_e))) \circ \alpha_{ef}(\varphi_{f,ef}\,\alpha_f^{-1}(\alpha_f(y_f))) \\
&= \alpha_{ef}(\varphi_{e,ef}(x_e)) \circ \alpha_{ef}(\varphi_{f,ef}(y_f)) \\
&= \alpha_{ef}(\varphi_{e,ef}(x_e)\,\varphi_{f,ef}(y_f)) = \alpha_{ef}(x_e y_f ef).
\end{aligned} \tag{5.15}$$

It follows that $\alpha(xy) = \alpha(x)\alpha(y)$. Thus α is a homomorphism of partial groups which is clearly one-to-one, onto T, and the proof is complete. □

Now we discuss particular kinds of q partial groups.

Let S be a q partial group and let N_e be the kernel of $\varphi_{1,e}$, $e \in E(S)$. Let S satisfy the condition

$$e \neq f \quad \text{in } E(S) \Longrightarrow N_e \neq N_f. \tag{5.16}$$

Thus

$$E(S) \longrightarrow \mathbf{N} = \{N_e : e \in E(S)\}, \quad e \longmapsto N_e \tag{5.17}$$

is a bijection, which is an isomorphism of semilattices, where $\leq$ is defined on $\mathbf{N}$ by

$$N_e \leq N_f \Longleftrightarrow e \leq f, \tag{5.18}$$

which gives for all $e, f \in E(S)$

$$N_e \wedge N_f = N_{ef}. \tag{5.19}$$

Here we observe

$$N_e \leq N_f \Longrightarrow N_f \subset N_e, \tag{5.20}$$

equivalently, $e \leq f \Rightarrow N_f \subset N_e$, (for we have $x \in N_f \Rightarrow xf = f \Rightarrow xe = x(ef) = (xf)e = e \Rightarrow x \in N_e$).

Lemma 2.1 gives a sufficient condition for (5.16), that is a q partial group S satisfies (5.16) if the maximal groups in S are pairwise nonisomorphic.

Remark 5.3. Partial mappings provide concrete examples for q partial groups satisfying (5.16).

Let X be a nonempty set, G a nontrivial group, and let

$$S = \overset{\triangle}{P}(X, G) \tag{5.21}$$

be the corresponding q partial group of partial mappings (cf. Section 1). We have that $S_1 = M(X, G)$ is the group of all mappings from X into G, and

$$E(S) = \{e_A : A \longrightarrow G,\ x \longmapsto 1_G;\ A \subset X\} \tag{5.22}$$

is the semilattice with $e_A \leq e_B$ if and only if $A \subset B$. For all $e_A, e_B \in E(S)$, we also have

$$e_A \wedge e_B = e_{A \cap B}. \tag{5.23}$$

The identity 1 of S is the identity of S_1, that is, 1_S is the mapping $e_X : X \to G$, $x \mapsto 1_G$, (here 1_G is the identity element of the group G). For all $A \subset X$,

$$\varphi_{1,e_A} : M(X, G) \longrightarrow M(A, G) \tag{5.24}$$

is (the epimorphism) given by $f \mapsto f|_A$, (the restriction of f on A). Thus,

$$\begin{aligned} N_{e_A} &= \ker \varphi_{1,e_A} = \{f \in M(X,G) : f e_A = e_A\} \\ &= \{f \in M(X,G) : f|_A = e_A\}. \end{aligned} \tag{5.25}$$

Evidently, for all $A, B \subset X$

$$A = B \Longleftrightarrow e_A = e_B. \tag{5.26}$$

Now for all $A, B \subset X$, $N_{e_A} = N_{e_B} \Rightarrow$ for all $f \in M(X,G)$, $f|_A = e_A$ if and only if $f|_B = e_B \Leftrightarrow A = B \Leftrightarrow e_A = e_B$. It follows that

$$e_A \neq e_B \Longrightarrow N_{e_A} \neq N_{e_B}, \tag{5.27}$$

as we claimed.

Lemma 5.4. *Let S be a q partial group satisfying (5.16), and let $\mathbf{N}$ be the semilattice isomorphic to $E(S)$ given by (5.17). Then*

$$T = \bigcup_{e \in E(S)} S_1/N_e \tag{5.28}$$

is a partial group, and

$$\alpha : S \longrightarrow T, \qquad xe \longmapsto xN_e \tag{5.29}$$

is a partial group isomorphism.

Proof. For each $e \in E(S)$, there exists an isomorphism

$$\theta_e : S_1/N_e \longrightarrow S_e, \tag{5.30}$$

given by

$$\theta_e(xN_e) = \varphi_{1,e}x, \quad \forall x \in S_1. \tag{5.31}$$

That is,

$$\theta_e(xN_e) = xe. \tag{5.32}$$

Observe that $\varphi_{1,e}$ is an epimorphism, and so every element in S_e is xe, for some $x \in S_1$. Thus for every $e \in E(S)$, we have an isomorphism

$$\alpha_e : S_e \longrightarrow S_1/N_e, \tag{5.33}$$

given by

$$\alpha_e = \theta_e^{-1}, \quad \text{that is } \alpha_e(xe) = xN_e. \tag{5.34}$$

Applying Lemma 5.2 (with $N_e = e'$), it follows that T is a partial group with homomorphisms

$$\varphi_{N_e,N_f} = \alpha_f \varphi_{e,f} \alpha_e^{-1} : S_1/N_e \longrightarrow S_1/N_f, \quad \forall N_e \geq N_f \,(\text{i.e., } e \geq f \text{ in } E(S)). \tag{5.35}$$

Thus, for arbitrary $xN_e \in S_1/N_e$,

$$\varphi_{N_e,N_f}(xN_e) = \alpha_f \varphi_{e,f} \alpha_e^{-1}(xN_e) = xN_f. \tag{5.36}$$

Actually, φ_{N_e,N_f} is an epimorphism. Using (5.19), multiplication in T is given as follows: for arbitrary $xN_e, yN_f \in T(x, y \in S_1)$

$$xN_e \cdot yN_f = \varphi_{N_e,N_{ef}} xN_e \cdot \varphi_{N_f,N_{ef}} yN_f = xN_{ef} \cdot yN_{ef} = xyN_{ef}. \tag{5.37}$$

□

Lemma 5.5. *Let S, $\mathbf{N}$ be as in Lemma 5.4. Then,*

$$\forall e, f \in E(S), \quad N_e \subset N_f \Longrightarrow e \geq f \tag{5.38}$$

if and only if

$$\forall e, f \in E(S), \quad N_{ef} = \langle N_e \cup N_f \rangle = N_e N_f. \tag{5.39}$$

Proof. Suppose (5.38) holds. Then, by using (5.20), we obtain for any $e, f \in E(S)$

$$e \geq f \Longleftrightarrow N_e \geq N_f \Longleftrightarrow N_e \subset N_f. \tag{5.40}$$

Thus the partial ordering of $\mathbf{N}$ is completely given by set inclusion in a reverse manner. Since $\langle N_e \cup N_f \rangle = N_e N_f$ $(= N_f N_e)$ is the least upper bound of N_e and N_f with respect to $\subset$, it follows that, it is the greatest lower bound of them with respect to $\leq$ in $\mathbf{N}$. That is $N_e \wedge N_f = N_e N_f$. This and (5.19) give (5.39).

Conversely, suppose that (5.39) holds, and let $N_e \subset N_f (e, f \in E(S))$. Then

$$N_{ef} = N_e N_f = N_f, \tag{5.41}$$

which gives, by (5.16), $ef = f$, and so $e \geq f$. □

Let S be a q partial group satisfying (5.16) and the equivalent conditions (5.38) and (5.39). We call S *proper* if also

$$\text{for } N \subset S_1, \quad N \text{ is normal in } S_1 \text{ only if } N = N_e \text{ for some } e \in E(S), \tag{5.42}$$

(recall that $N_e = \ker \varphi_{1,e}$). By (5.16), if S is proper and $N \lhd S_1$, then $N = N_e$ for a unique $e \in E(S)$.

A typical example of a proper q partial group is the q partial group $P(G)$ associated to a (nonzero) group G (as in Lemma 5.1). Partial groups in general and (proper) q partial groups in particular can be represented by proper q partial groups of the kind $P(G)$. We devote the rest of this section for such representations.

Theorem 5.6. *Every q partial group S that satisfies (5.16) and the equivalent conditions (5.38) and (5.39) is embeddable in its associated proper q partial group $P(S_1)$.*

Proof. By Lemma 5.4, there exists an isomorphism

$$\alpha : S \longrightarrow T, \qquad xe \longmapsto xN_e, \tag{5.43}$$

where

$$T = \bigcup_{e \in E(S)} S_1/N_e, \tag{5.44}$$

and $N_e \leq N_f$ if and only if $e \leq f$. Using (5.38), we obtain

$$N_e \leq N_f \Longleftrightarrow N_f \subset N_e. \tag{5.45}$$

It follows that $E(T)$ is isomorphic to a subsemilattice of $E(P(S_1))$. Since each maximal group S_1/N_e is a maximal group in $P(S_1)$ (with identity N_e), it follows by Lemma 5.2 that T is a subpartial group of $P(S_1)$. Hence α embeds S in $P(S_1)$. □

Corollary 5.7. *Let S be an arbitrary partial group and let M be the group $M(S,G)$ of all mappings from S into G, where G is the coproduct of all maximal groups in S. Then S is embeddable in $P(M)$.*

Proof. By the representation theorem (Section 1), there exists a monomorphism

$$\beta_1 : S \longrightarrow \overset{\triangle}{P}(S,G). \tag{5.46}$$

The partial group $\overset{\triangle}{P}(S,G)$ satisfies (5.16) (see the previous remark), and it is easy to see that all these partial groups (of partial mappings) satisfy (5.38) as well. Hence by Theorem 5.6, there exists an embedding

$$\beta_2 : \overset{\triangle}{P}(S,G) \longrightarrow P\Big(\overset{\triangle}{P}(S,G)\Big)_1 = P(M(S,G)). \tag{5.47}$$

$\beta_2\beta_1$ is the required embedding. □

The proof of the following lemma is routine and omitted.

Lemma 5.8. *Let S and T be partial groups with identities, and let $\gamma : S \to T$ be a homomorphism.*

(i) *If S is a q partial group and γ maps $E(S)$ onto $E(T)$, then* $\operatorname{Im}\gamma$ *is a q subpartial group of T. In particular, if γ is an epimorphism, then T is a q partial group.*
(ii) *If γ is an isomorphism, then T is proper if (and only if) S is proper.*

Corollary 5.9. *A q partial group S is proper if and only if S is isomorphic to $P(S_1)$.*

Proof. Sufficiency follows from Lemma 5.8(ii). Necessity is a special case of Theorem 5.6 with $T = P(S_1)$. □

Nonzero groups and nontrivial commutative bands with identities are obvious examples of q partial groups which are not proper. Eventhough, it is true that q partial groups that satisfy (5.16) and (5.38) (equivalently (5.39)) need not be proper. A simple example is given by a group G that has a proper normal subgroup N. For instance, let $S = S_1 \cup S_2$ with indexed semilattice $1 > 2$, where $S_1 = Z_6$ and $S_2 = Z_6/\{0,3\}$. Then S is a q partial group satisfying (5.16) and (5.38) but does not satisfy (5.42). This can also be illustrated by partial mappings.

Example 5.10. Let X be a nonempty set and let G be as above, that is, $1_G \neq G$ is a group with a normal subgroup $1_G \neq N \neq G$, (e.g., $G = Z_6$). Let

$$T = \{f \in M(X,G) : f(X) \subset N\}, \tag{5.48}$$

clearly T is a normal subgroup of $M(X,G) = \overset{\triangle}{P}(X,G)_1$. Also $T \neq M(X,G)$, since $N \neq G$. Let $A \subset X$ be fixed but arbitrary. We have (by definition)

$$N_{e_A} = \ker \varphi_{e_X,e_A} = \{f \in M(X,G) : f|_A = e_A\}. \tag{5.49}$$

Recall that e_A is the identity of the maximal group $M(A,G)$, that is, $e_A : A \to G, x \mapsto 1_G$. Thus

$$N_{e_A} = \{f : X \to G : f(A) = \{1_G\}\}. \tag{5.50}$$

If $A = \emptyset$, then $N_{e_A} = M(X,G) \neq T$.

If $A \neq \emptyset$, choose $a \in A$, $p \in N$ and define $g : X \to G$ by

$$x \longmapsto \begin{cases} p & \text{if } x = a, \\ 1_G & \text{if } x \neq a. \end{cases} \tag{5.51}$$

Then $g \in T$, but $g \notin N_{e_A}$. Thus $T \neq N_{e_A}$. Since A is arbitrary, it follows that $T \neq \ker \varphi_{1,e_A}$, for every idempotent e_A in $\overset{\triangle}{P}(X,G)$. Hence condition (5.42) does not hold, that is $\overset{\triangle}{P}(X,G)$ is not proper.

A proper q partial group S has (up to an isomorphism) a "local" quotient group S_1/N_e at every point $e \in E(S)$. These can be integrated into a whole quotient partial group describing S. Formally, we have the following theorem.

Theorem 5.11. *Every proper q partial group S is isomorphic to some quotient partial group T/N, with $N \neq E(T)$ if 1_S is proper. Explicitly, T is a strong semilattice $E(S)$ of isomorphic copies of S_1.*

Proof. By assumption, the set of all normal subgroups of S_1 is (the semilattice)

$$\mathbf{N} = \{N_e = \ker \varphi_{1,e} : e \in E(S)\}, \tag{5.52}$$

with

$$e \geq f \Longleftrightarrow N_e \geq N_f \Longleftrightarrow N_e \subset N_f. \tag{5.53}$$

Let $\{T_e\}_{e\in E(S)}$ be disjoint isomorphic copies of S_1, with isomorphisms of groups

$$\alpha_e : S_1 \xrightarrow{\cong} T_e \quad \text{for every } e \in E(S). \tag{5.54}$$

Let $T = \bigcup_{e\in E(S)} T_e$. For all $e \geq f$ in $E(S)$,

$$\varphi'_{e,f} : T_e \longrightarrow T_f, \tag{5.55}$$

given by

$$\varphi'_{e,f} = \alpha_f \alpha_e^{-1}, \tag{5.56}$$

clearly induce a semilattice of groups $T = [E(S); T_e, \varphi'_{e,f}]$. For each $e \in E(S)$, there exists an isomorphism

$$\gamma_e : S_1/N_e \longrightarrow T_e/N'_e, \qquad xN_e \longmapsto \alpha_e(x)N'_e, \tag{5.57}$$

where N'_e is the image of N_e under α_e. Define $N'_e \geq N'_f$ if and only if $e \geq f$ in $E(S)$. Let $N = \bigcup_{e\in E(S)} N'_e$. Then N is a normal subpartial group of T and the quotient partial group T/N has maximal groups

$$(T/N)_e = T_e/N_e \cong S_1/N_e \quad (e \in E(S)). \tag{5.58}$$

By Lemma 5.2, T/N is isomorphic to $P(S_1)$, whence by Corollary 5.9, S is isomorphic to T/N. The other assertion of the theorem follows immediately. □

6. Two equivalent categories

We use previous results to show that proper q partial groups together with certain kind of epimorphisms are a category equivalent to the category of groups and epimorphisms of groups.

Lemma 6.1. *If $\alpha : S \to T$ is an epimorphism of q partial groups, then the restriction $\alpha|_{S_1}$ on S_1 is an epimorphism $S_1 \to T_1$ of groups.*

Proof. $\alpha(1_S) = e'$ for some $e' \in E(T)$. Since α is epimorphism, $\alpha(e) = 1_T$ for some $e \in E(S)$. Thus

$$1_T = \alpha(e) = \alpha(1_S e) = \alpha(1_S)\alpha(e) = \alpha(1_S)1_T = \alpha(1_S). \tag{6.1}$$

It follows that $\alpha(S_1) \subset T_1$.

If $y \in T_1$, there exists $s \in S$, say $s \in S_e$ for some $e \in E(S)$, such that $\alpha(s) = y$. Since S is a q partial group, $s = xe$ for some $x \in S_1$. Thus

$$1_T = e_y = e_{\alpha(s)} = \alpha(e_s) = \alpha(e). \tag{6.2}$$

Therefore

$$y = \alpha(s) = \alpha(x)\alpha(e) = \alpha(x)1_T = \alpha(x), \tag{6.3}$$

whence $y \in \alpha(S_1)$. The result follows. □

From now on, we use symbols to denote terms as follows: groups in general (are denoted by): $G, H, K, \ldots,$

(proper) q partial groups: $S, T, R, \ldots,$

epimorphisms of q partial groups: $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \ldots,$

restrictions of epimorphisms $\overline{\alpha} : S \to T, \overline{\beta}, \ldots,$ of proper q partial groups on maximal groups $S_1, \ldots,$ are denoted by $\alpha : S_1 \to T_1, \beta, \ldots,$ (see Lemma 6.1).

Given a group G,

$$\mathbf{N}(G) = \{N \subset G : N \lhd G\}. \tag{6.4}$$

For a q partial group S,

$$\mathbf{N}(S_1) = \{N \subset S_1 : N \lhd S_1\}. \tag{6.5}$$

In particular, if S is a proper q partial group,

$$\mathbf{N}(S_1) = \{N_e = \ker \varphi_{1,e} : e \in E(S)\}. \tag{6.6}$$

Unless stated otherwise, $\mathbf{N}(G)$ is a semilattice with $M \leq N$ if and only if $N \subset M$.

In particular, if S is a proper q partial group, $\mathbf{N}(S)$ is the semilattice for which

$$e \leq f \Longleftrightarrow N_e \leq N_f \Longleftrightarrow N_f \subset N_e \quad (\forall e, f \in E(S)) \text{ (cf. Section 5)}. \tag{6.7}$$

We call an epimorphism $\overline{\alpha} : S \to T$ (of proper q partial groups) a *p epimorphism* if

$$\alpha(N_e) = N_{\overline{\alpha}(e)} \quad \text{for every } e \in E(S). \tag{6.8}$$

In other words, $\overline{\alpha}$ is *p epimorphism* if

$$\alpha(\ker \varphi_{1,e}) = \ker \varphi_{1,\overline{\alpha}(e)} \quad \text{for every } e \in E(S). \tag{6.9}$$

The next result which gives a characterization of p epimorphisms is independent of later work. First we recall some definitions. A partially ordered set $(P, \leq)$ is said to satisfy the *maximal condition* if any nonempty subset of P contains a maximal element. This is equivalent to say that P satisfies the *ascending chain condition*: there does not exist an infinite properly ascending chain $x_1 < x_2 < \ldots,$ in P.

The *minimal condition* and the descending chain condition are defined dually. A group G satisfies the *maximal condition on normal subgroups* if $\langle \mathbf{N}(G), \subset \rangle$ satisfies the maximal condition (the partial ordering here is set inclusion). This property is identical with the *ascending chain condition* on the normal subgroups. Minimal and descending chain conditions on normal subgroups are defined dually.

As we mentioned above, where $\mathbf{N}(G)$ (or $\mathbf{N}(S)$) is written without any further mention of partial ordering, we always mean the semilattice $(\mathbf{N}(G), \leq)$, $M \leq N$ if and only if $N \subset M$. Otherwise the partial ordering is written explicitly. Thus, for instance, if S is a proper q partial group, then $E(S)$ satisfies the maximal condition if and only if $\mathbf{N}(S_1)$ satisfies the maximal condition; if and only if $(\mathbf{N}(S_1), \subset)$ satisfies the minimal condition; if and only if S_1 satisfies the minimal condition on normal subgroups; if and only if there does not exist an infinite properly descending chain $N_1 \supset N_2 \supset \cdots$ in $\mathbf{N}(S_1)$.

Lemma 6.2. *Let S, T be proper q partial groups and let $\overline{\alpha} : S \to T$ be an epimorphism.*

(i) *If $\overline{\alpha}$ is p epimorphism, then the following condition holds:*

$$\overline{\alpha}(e) = \overline{\alpha}(f) \Longleftrightarrow \alpha(N_e) = \alpha(N_f) \quad (e, f \in E(S)). \tag{6.10}$$

(ii) *If (6.10) holds and T_1 satisfies the maximal condition on normal subgroups (i.e., $E(T)$ satisfies the minimal condition), then $\overline{\alpha}$ is p epimorphism.*

Proof. (i) Trivial.

(ii) Let $e_0 \in E(S)$ be fixed but arbitrary. Clearly, $\alpha(N_{e_0}) \subset N_{\overline{\alpha}(e_0)}$ $(x \in N_{e_0} \Rightarrow xe_0 = e_0 \Rightarrow \overline{\alpha}(x)\overline{\alpha}(e_0) = \overline{\alpha}(e_0) \Rightarrow \alpha(x) \in N_{\overline{\alpha}(e_0)})$. If $\alpha(N_{e_0}) = N_{\overline{\alpha}(e_0)}$ the result obtains. Otherwise, we have $N_{\overline{\alpha}(e_0)} \lhd T_1 \Rightarrow N_{\overline{\alpha}(e_0)} = \alpha(N_{e_1}) \subset N_{\overline{\alpha}(e_1)}$ for some $e_1 \in E(S)$ (since S, T are proper and α is epimorphism). If $N_{\overline{\alpha}(e_0)} = N_{\overline{\alpha}(e_1)}$, then $\overline{\alpha}(e_0) = \overline{\alpha}(e_1)$ (by condition (5.16)) and so by (6.10), $\alpha(N_{e_0}) = \alpha(N_{e_1}) = N_{\overline{\alpha}(e_0)}$ and the result follows. Otherwise and having chosen e_k, there exists $e_{k+1} \in E(S)$ with $N_{\overline{\alpha}(e_k)} = \alpha(N_{e_{k+1}})$ and so $N_{\overline{\alpha}(e_k)} \subset N_{\overline{\alpha}(e_{k+1})}$. This would give an infinite chain $N_{\overline{\alpha}(e_0)} \subset N_{\overline{\alpha}(e_1)} \subset \cdots \subset N_{\overline{\alpha}(e_k)} \subset N_{\overline{\alpha}(e_{k+1})} \subset \cdots$.

By the maximal condition, we must have $N_{\overline{\alpha}(e_j)} = N_{\overline{\alpha}(e_{j+1})}$ for some $j \geq 1$. Thus $\overline{\alpha}(e_j) = \overline{\alpha}(e_{j+1})$, which gives by (6.10), $\alpha(N_{e_j}) = \alpha(N_{e_{j+1}}) = N_{\overline{\alpha}(e_j)} \Rightarrow N_{\overline{\alpha}(e_{j-1})} = N_{\overline{\alpha}(e_j)} \Rightarrow \overline{\alpha}(e_j) = \overline{\alpha}(e_{j-1})$. Repeating the process, we obtain $\overline{\alpha}(e_j) = \overline{\alpha}(e_{j-1}) = \cdots = \overline{\alpha}(e_1) = \overline{\alpha}(e_0)$. Thus by (6.10)

$$\alpha(N_{e_0}) = \alpha(N_{e_1}) = N_{\overline{\alpha}(e_0)}, \tag{6.11}$$

and the proof is complete. □

Lemma 6.3. *Let G, H be groups. Then every epimorphism $\alpha : G \to H$ is uniquely extended to a p epimorphism $\alpha^* : P(G) \to P(H)$ given by $xN \mapsto \alpha(x)\alpha(N)$ $(x \in G, N \lhd G)$.*

Proof. We have $P(G) = \bigcup_{N \in \mathbf{N}} G/N$ is a proper q partial group with

$$\varphi_{N,M} : G/N \longrightarrow G/M, \quad xN \longmapsto xM, \quad \text{if } N \geq M \text{ (i.e., } N \subset M, \text{ see Lemma 5.1)}, \tag{6.12}$$

since α is epimorphism, we have $\mathbf{N}(H) = \{\alpha(N) : N \lhd G\}$ and it is easy to see that $\alpha^* : P(G) \to P(H)$, $xN \mapsto \alpha(x)\alpha(N)$, $(x \in G, N \lhd G)$ is a well-defined epimorphism, and the restriction of α^* on $(P(G))_1 = G$ is $\alpha : G \to H$. For any $N \lhd G$ we have by the definition of α^*, $\alpha^* N = \alpha(N)$. Thus

$$\alpha(\ker \varphi_{1,N}) = \alpha(N) = \ker \varphi_{1,\alpha(N)} = \ker \varphi_{1,\alpha^* N}. \tag{6.13}$$

Hence α^* is p epimorphism. If $\beta^* : P(G) \to P(H)$ is a p epimorphism, then by Lemma 6.1, the restriction of β^* on $(P(G))_1 = G$ defines an epimorphism $\beta : G \to H$, $x \mapsto x'$, and for any $xN \in P(G)$, we have

$$\beta^* xN = \beta(x)\beta^* N. \tag{6.14}$$

Since β is a p epimorphism, we have

$$\beta(N) = \beta(\ker\varphi_{1,N}) = \ker\varphi_{1,\beta^* N} = \beta^* N. \tag{6.15}$$

Thus, if $\beta = \alpha$, we have for any $N \lhd G$,

$$\alpha(N) = \beta(N) = \ker\varphi_{1,\beta^* N} = \beta^* N. \tag{6.16}$$

Therefore $\beta^* = \alpha^*$. □

In view of Lemma 6.3, we may define an operation $*$ on epimorphisms of groups which sends every epimorphism $\alpha : G \to H$ to its unique extension p epimorphism $\alpha^* : P(G) \to P(H)$, $xN \mapsto \alpha(x)\alpha(N)$. Some properties of $*$ is given in the following lemma, whose proof is trivial and omitted.

Lemma 6.4. (i) *For any group G, $1_G^* = 1_G$.*
(ii) *Given a pair of epimorphisms $\alpha : G \to H$ and $\beta : H \to K$, then, $(\beta\alpha)^* = \beta^*\alpha^*$.*

Given an epimorphism $\overline{\alpha} : S \to T$ of proper q partial groups, its restriction $\alpha : S_1 \to T_1$ is an epimorphism of groups (Lemma 6.1) and so extended uniquely to a p epimorphism $\alpha^* : P(S_1) \to P(T_1)$, $xN \mapsto \alpha(x)\alpha(N)$. Recall also that, for any proper q partial group S, there is an isomorphism $\alpha_S : S \to P(S_1)$, $xe \mapsto xN_e$, $x \in S_1$, $e \in E(S)$ (see Lemma 5.4, Theorem 5.6, and Corollary 5.9).

Lemma 6.5. *Let $\overline{\alpha} : S \to T$ be an epimorphism of proper q partial groups. Then $\overline{\alpha}$ is p epimorphism if and only if $\overline{\alpha} = \alpha_T^{-1}\alpha^*\alpha_S$.*

Proof. By Lemma 6.3, α^* is p epimorphism. Thus for each $e \in E(S)$, $\alpha^* xN_e = \alpha(x)\alpha(N_e)$. Suppose $\overline{\alpha} = \alpha_T^{-1}\alpha^*\alpha_S$, let $x \in S_1$, $e \in E(S)$, and let $\alpha(N_e) = N_{e'}$ (for some $e' \in E(S)$). Then, $\alpha(x)\overline{\alpha}(e) = \overline{\alpha}(x)\overline{\alpha}(e) = \overline{\alpha}(xe) = \alpha_T^{-1}\alpha^*\alpha_S(xe) = \alpha_T^{-1}\alpha^* xN_e = \alpha_T^{-1}\alpha(x)\alpha(N_e) = \alpha_T^{-1}\alpha(x)N_{e'} = \alpha(x)e'$. Thus $\overline{\alpha}(e) = e'$ (since $\alpha(x) \in T_1$), and we have

$$\alpha(\ker\varphi_{1,e}) = \alpha(N_e) = N_{e'} = \ker\varphi_{1,e'} = \ker\varphi_{1,\overline{\alpha}(e)}. \tag{6.17}$$

Hence $\overline{\alpha}$ is p epimorphism.

Conversely, if $\overline{\alpha}$ is p epimorphism, then for all $e \in E(S)$, $\alpha(N_e) = N_{\overline{\alpha}(e)}$. Thus for $x \in S_1$, $e \in E(S)$, we have

$$\begin{aligned}\alpha_T^{-1}\alpha^*\alpha_S(xe) &= \alpha_T^{-1}\alpha^* xN_e = \alpha_T^{-1}\big(\alpha(x)\alpha(N_e)\big)\\ &= \alpha_T^{-1}\big(\alpha(x)N_{\overline{\alpha}(e)}\big) = \alpha(x)\overline{\alpha}(e) = \overline{\alpha}(xe).\end{aligned} \tag{6.18}$$

Therefore, $\alpha_T^{-1}\alpha^*\alpha_S = \overline{\alpha}$. □

We notice that given p epimorphisms $\overline{\alpha} : S \to T$ and $\overline{\beta} : T \to R$, $\overline{\beta}\overline{\alpha} : S \to R$ is a p epimorphism (by the definition or Lemma 6.5). Also $1_S : S \to S$, $x \mapsto x$ is clearly p epimorphism. Thus we can define a category:

PQP: objects are all proper q partial groups $S, T, \ldots$, and morphisms $\mathrm{Hom}(S,T) =$ (possibly empty) set of all p epimorphisms from S onto T.

Let $\mathbf{G}$ be the category of all groups with morphisms of all epimorphisms of groups. Thus for any two groups G, H, $\mathrm{Hom}(G,H)$ = (possibly empty) set of all epimorphisms from G onto H.

Recall that two categories $\mathbf{C},\mathbf{D}$ are said to be *equivalent* if there exist functors

$$F:\mathbf{C}\to\mathbf{D},\qquad G:\mathbf{D}\to\mathbf{C},\tag{6.19}$$

and natural isomorphisms

$$\eta: GF\cong I_{\mathbf{C}},\qquad \varepsilon: FG\cong I_{\mathbf{D}},\tag{6.20}$$

where $I_{\mathbf{C}}$ and $I_{\mathbf{D}}$ are the identity functors.

Theorem 6.6. $\mathbf{G}$ *and* $\mathbf{PQP}$ *are equivalent categories.*

Proof. Define $P:\mathbf{G}\to\mathbf{PQP}$ on objects by $G\mapsto P(G)$, where $P(G)$ is the proper q partial group associated to G (as in Lemma 5.1) and on morphisms by

$$\alpha: G\longrightarrow H\longmapsto \alpha^*: P(G)\longrightarrow P(H),\tag{6.21}$$

where α^* is the unique p epimorphism extension of α on $P(G)$, defined as in Lemma 6.3. By Lemma 6.4, P is a functor (actually is full and faithful). Define $R:\mathbf{PQP}\to\mathbf{G}$ on objects by $S\mapsto S_1$, and on morphisms by

$$\overline{\alpha}: S\longrightarrow T\longmapsto \alpha: S_1\longmapsto T_1,\tag{6.22}$$

where α is the restrection of $\overline{\alpha}$ on S_1 (see Lemma 6.1). Clearly R is a functor and $RP=I_{\mathbf{G}}$. In other words, there exists a natural isomorphism $\eta: RP\cong I_{\mathbf{G}}$ given by for any object G in $\mathbf{G}, RP(G)=G$ and $\eta_G: RP(G)\to G$ is the identity $1_G, x\mapsto x$. Next for each object S in $\mathbf{PQP}$, $PR(S)=P(S_1)$. Define $\varepsilon_S: PR(S)\to I(S)$, that is, $\varepsilon_S: P(S_1)\to S$ by $xN_e\mapsto xe$, that is, $\varepsilon_S=\alpha_S^{-1}$. Thus ε_S is an invertible (isomorphism) in $\mathbf{PQP}$ and for any morphism $\overline{\alpha}: S\to T$ in $\mathbf{PQP}$ we have a diagram

$$\begin{tikzcd}
PR(S)=P(S_1) \arrow[r, "\varepsilon_S=\alpha_S^{-1}"] \arrow[d, "P(\alpha)=\alpha^*"'] & I(S)=S \arrow[d, "I(\alpha^-)=\alpha^-"] \\
PR(T)=P(T_1) \arrow[r, "\varepsilon_T=\alpha_T^{-1}"] & I(T)=T
\end{tikzcd}\tag{6.23}$$

Applying Lemma 6.5, we have

$$\overline{\alpha}\varepsilon_S=\overline{\alpha}\alpha_S^{-1}=(\alpha_T^{-1}\alpha^*\alpha_S)\alpha_S^{-1}=\alpha_T^{-1}\alpha^*=\varepsilon_T\alpha^*,\tag{6.24}$$

and the above diagram is commutative. This gives a natural isomorphism $\varepsilon: PR\cong I$ whose components are $\varepsilon_S=\alpha_S^{-1}$ and the proof is complete. $\square$

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