ON P.P.-RINGS WHICH ARE REDUCED

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Denote the 2×2 upper triangular matrix rings over $\mathbb{Z}$ and $\mathbb{Z}_p$ by $\mathrm{UTM}_2(\mathbb{Z})$ and $\mathrm{UTM}_2(\mathbb{Z}_p)$, respectively. We prove that if a ring R is a p.p.-ring, then R is reduced if and only if R does not contain any subrings isomorphic to $\mathrm{UTM}_2(\mathbb{Z})$ or $\mathrm{UTM}_2(\mathbb{Z}_p)$. Other conditions for a p.p.-ring to be reduced are also given. Our results strengthen and extend the results of Fraser and Nicholson on r.p.p.-rings.

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1. Introduction

Throughout the paper, all rings are associative rings with identity 1. The set of all idempotents of a ring R is denoted by $E(R)$. Also, for a subset $X \subseteq R$, we denote the right [resp., left] annihilator of X by $r(X)$ [resp., $\ell(X)$].

We call a ring R a *left* p.p.-*ring* [3], in brevity, an l.p.p.-ring, if every principal left ideal of R, regarded as a left R-module, is projective. Dually, we may define the *right* p.p.-*rings* (r.p.p.-*rings*). We call a ring R a p.p.-*ring* if R is both an l.p.p.- and r.p.p.-ring. It can be easily observed that the class of p.p.-rings contains the classes of regular (von Neumann) rings, hereditary rings, Baer rings, and semihereditary rings as its proper subclasses. In the literature, p.p.-rings have been extensively studied by many authors and many interesting results have been obtained (see [1–7]). It is noteworthy that the definition of p.p.-rings can also be extended to semigroups.

We now call a ring R *reduced* if it contains no nonzero nilpotent elements. Obviously, the left annihilator $\ell(X)$ of X in a reduced ring R is always a two-sided ideal of R. Moreover, if R is a reduced ring, then $ef = 0$ if and only if $fe = 0$ for any nonzero idempotents $e, f \in R$. Reduced rings with the maximum condition on annihilator were first studied by Cornish and Stewart [2]. By using the concept of annihilator and reduced ring, Fraser and Nicholson [3] showed that a ring R is a reduced p.p.-ring if and only if R is a (left, right) p.p.-ring in which every idempotent is central.

In this paper, we will prove that a p.p.-ring R is reduced if and only if R contains no subrings which are isomorphic to the matrix rings $\mathrm{UTM}_2(\mathbb{Z})$ or $\mathrm{UTM}_2(\mathbb{Z}_p)$. Thus, our

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results strengthen and extend the results obtained by Fraser and Nicholson in [3]. Also, some of our results can be applied to r.p.p.-monoids with zero.

2. Definitions and basic results

The following crucial lemma of p.p.-rings was given by Fraser and Nicholson in [3].

Lemma 2.1 [3]. *Let R be a ring and $a \in R$. Then R is an l.p.p.-ring if and only if $\ell(a) = Re$ for some idempotent $e \in E(R)$.*

By using Lemma 2.1, we can give some properties of a p.p.-ring which is reduced.

Theorem 2.2. *Let R be a p.p.-ring and $E(R)$ the set of all idempotents of R. Then the following statements are equivalent:*

(i) *R is reduced;*
(ii) *$ef = fe$ for all $e, f \in E(R)$;*
(iii) *$E(R)$ is a subsemigroup of the semigroup $(R, \cdot)$;*
(iv) *$ef = 0$ if and only if $fe = 0$ for all $e, f \in E(R)$;*
(v) *$eR = Re$ for all $e \in E(R)$.*

Proof. (i)⇒(ii)⇒(iii) are trivial.

(iii)⇒(iv). Let $e, f \in E(R)$. Suppose that $ef = 0$. Then by (iii), we have $fe \in E(R)$ and so $fe = (fe)^2 = f(ef)e = 0$. Similarly, we can show that if $fe = 0$, then $ef = 0$. This proves (iv).

(iv)⇒(v). Let $x \in r(e)$. Then $ex = 0$ and so $e \in \ell(x)$. Since R is a p.p.-ring, by Lemma 2.1, we have $\ell(x) = Rf$, for some $f \in E(R)$. Now, by Pierce decomposition, we have $R = R(1-f) \oplus Rf$ and hence $\ell(1-f) = Rf$. Consequently $e \in \ell(1-f) = \ell(x)$ and thereby $e(1-f) = 0$ since $ex = 0$. Because $(1-f) \in E(R)$, by (iv), we have $(1-f)e = 0$. It is now easy to check that $e + xe \in E(R)$. Since $(e+xe)(1-f) = 0$, we have, by (iv), $0 = (1-f)(e+xe) = (1-f)xe$. However, by $\ell(x) = Rf$ and $1 \in R$, we have $fx = 0$ so that $fxe = 0$. This leads to $xe = (1-f)xe + fxe = 0$, and thereby $x \in \ell(e)$. Thus $r(e) \subseteq \ell(e)$. Dually, we can show that $\ell(e) \subseteq r(e)$. Therefore $r(e) = \ell(e)$. Thus, for all $e \in R$, $r(1-e) = \ell(1-e)$, that is, $eR = Re$. This proves (v).

(v)⇒(i). Since (v) easily yields that the idempotents of R are central, so (v)⇒(i) by [3]. □

The following example illustrates that there exists a p.p.-ring which is not reduced.

Example 2.3. Let $\mathrm{UTM}_2(\mathbb{R})$ be the subring of the matrix ring $M_2(\mathbb{R})$ consisting of all 2×2 upper triangular matrices over the field $\mathbb{R}$. We claim that $\mathrm{UTM}_2(\mathbb{R})$ is a p.p.-ring. In order to establish our claim, let

$$A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, \qquad B = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \tag{2.1}$$

be elements of $\mathrm{UTM}_2(\mathbb{R})$. Then we see immediately that $AB = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ if and only if $ax = 0$, $by = 0$ and $az + cy = 0$. The following cases now arise.

(i) $x \neq 0$ and $y \neq 0$. In this case, we have $AB = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ if and only if $a = b = c = 0$. Hence, we have

$$\ell(B) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} = \mathrm{UTM}_2(\mathbb{R}) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.2}$$

(ii) $x \neq 0$ and $y = 0$. In this case, we have $AB = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ if and only if $a = 0$. This leads to

$$\ell(B) = \left\{ \begin{pmatrix} 0 & c \\ 0 & b \end{pmatrix} : b, c \in \mathbb{R} \right\} = \mathrm{UTM}_2(\mathbb{R}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.3}$$

(iii) $x = 0$ and $y \neq 0$. In this case, we have $AB = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ if and only if $b = 0$ and $c = azy^{-1}$. This leads to

$$\ell(B) = \left\{ \begin{pmatrix} a & azy^{-1} \\ 0 & 0 \end{pmatrix} : a \in R \right\} = \mathrm{UTM}_2(\mathbb{R}) \begin{pmatrix} 1 & zy^{-1} \\ 0 & 0 \end{pmatrix}. \tag{2.4}$$

Summing up the above cases, we can easily see that $\ell(B)$ of $\mathrm{UTM}_2(\mathbb{R})$ is generated by an idempotent. Clearly, $\mathrm{UTM}_2(\mathbb{R})$ is not reduced.

3. Main theorem

In proving the main theorem of this paper, we first denote by $o(r)$ the (additive) order of $r \in R$, that is, the smallest positive integer n such that $nr = 0$. If r is of infinite order, then we simply write $o(r) = \infty$.

We now prove a useful lemma for p.p.-rings.

Lemma 3.1. *Let R be a p.p.-ring with 1 such that $ef = 0$ but $fe \neq 0$ for some $e, f \in E(R)$. Then, $o(e) = o(f) = o(fe)$, and if $o(e) < \infty$, then there exist $u, v \in E(R)$ and a prime p such that $o(u) = o(v) = o(vu) = p$ with $uv = 0$ but $vu \neq 0$.*

Proof. Since R is a p.p.-ring, by Theorem 2.2, R is clearly not reduced. Also, since $1 \in R$, by Lemma 2.1, there exists some $g, h \in E(R)$ such that $\ell(fe) = R(1-g)$ and $r(fe) = (1-h)R$. These lead to $\ell(fe) = \ell(g)$ and $r(fe) = r(h)$. Since $1 - f \in \ell(fe)$, we have $(1-f)g = 0$ and so $g = fg$. Since $g = fg$, we see that $gf \in E(R)$ and $\ell(g) = \ell(gf)$. Thus, $(1-gf)fe = 0$ since $(1-gf)g = 0$ and $\ell(g) = \ell(fe)$, that is, $fe - gfe = 0$. Thereby, we have $gfe = fe$. Similarly, we can prove that there exists $h \in E(R)$ such that $h = he$, $eh \in E(R)$, $r(eh) = r(fe)$, and $fe = feh$. Hence, $fe = gfeh = (gf)(eh)$. On the other hand, we have $(eh)(gf) = e(he)(fg)f = 0$. Because $\ell(fe) = \ell(gf)$ and $r(fe) = r(eh)$, we can easily see that $o(gf) = o(eh) = o(fe)$.

Now two cases arise.

(i) $o(gf) = \infty$. In this case, there is nothing to prove.

(ii) $o(gf) < \infty$. Without loss of generality, let $o(gf) = pk$, where p is a prime number. Then, we can easily check that $o(kfe) = p$. By using similar arguments as above, we also have $u, v \in E(R)$ such that $o(u) = o(v) = o(kfe)$ with $uv = 0$ but $vu \neq 0$. Hence, u and v are the required idempotents in R. The proof is completed. □

We now formulate the following main theorem.

Theorem 3.2. *Let R be a p.p.-ring. Then R is reduced if and only if R has no subrings which are isomorphic either to* $\mathrm{UTM}_2(\mathbb{Z})$ *or to* $\mathrm{UTM}_2(\mathbb{Z}_p)$, *where p is a prime.*

Proof. The necessity part of the theorem follows from Theorem 2.2 since $\mathrm{UTM}_2(\mathbb{Z})$ and $\mathrm{UTM}_2(\mathbb{Z}_p)$ both contain some noncommutating idempotents.

To prove the sufficiency part of the theorem, we suppose that R is not reduced. Then we can let $i, j \in E(R)$ such that $ij = 0$, $ji \neq 0$, and $o(i) = o(j) = o(ji)$; and $o(i) = o(j) = o(ji) = p$ if $o(i) < \infty$, where p is a prime. Consider the subring of R generated by i and j. Clearly, $\{0, i, j, ji\}$ forms a subsemigroup of R under ring multiplication and so $S = \{ai + bji + ci : a, b, c \in \mathbb{Z}\}$ forms a subring of R, under the ring multiplication and addition.

Now, we define a mapping $\theta : \mathrm{UTM}_2(\mathbb{Z}) \to S$ by

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \longmapsto aj + (b - c)ji + ci. \tag{3.1}$$

Then, we can easily verify that θ is a surjective homomorphism of $\mathrm{UTM}_2(\mathbb{Z})$ onto S.

We now consider the kernel of θ. Suppose that $A = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in \ker\theta$. Then we have $aj + (b - c)ji + ci = 0$. Multiplying i on the left gives $ci = 0$, and multiplying j on the right gives $aj = 0$. Hence, we have $(b - c)ji = 0$.

The following cases arise.

(i) $o(i) = o(j) = o(ji) = \infty$. Then $a = 0$, $c = 0$, and $(b - c) = 0$. Thus $a = b = c = 0$ and thereby $A = 0$. Hence $\ker\theta = \{0\}$ and θ is an isomorphism.

(ii) $o(i) = o(j) = o(ji) = p$. In this case, we have $p \mid a$, $p \mid c$, and $p \mid (b - c)$. Hence $p \mid a$, $p \mid c$, and $p \mid b$. Consequently $\ker\theta = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) : p \mid a, p \mid b, \text{ and } p \mid c\}$. Observing that $\mathrm{UTM}_2(Z)/\ker\theta \cong \mathrm{UTM}_2(Z_p)$, we have $S \cong \mathrm{UTM}_2(Z_p)$. This contradicts our assumption and therefore our proof is completed. □

As an application of our main theorem, we give a new criterion for a p.p.-ring to be reduced.

Theorem 3.3. *Let R be a p.p.-ring having no subrings isomorphic to* $\mathrm{UTM}_2(Z_p)$ *for prime p. If $o(e) < \infty$ for all $e \in E(R)$, then R is reduced.*

In fact, Theorem 3.3 follows from the following lemma.

Lemma 3.4. *Let R be a p.p.-ring having no subring isomorphic to* $\mathrm{UTM}_2(\mathbb{Z}_p)$. *Suppose that at least one of the idempotents $e, f \in E(R)$ has a prime order p. Then $ef = 0$ if and only if $fe = 0$.*

Proof. Suppose that $ef = 0$ but $fe \neq 0$. Also, suppose that e or f has a prime order p. Then, fe must have an order p. Now, by using the arguments in the proof of Lemma 3.1, we can construct some idempotents $g, h \in R$ and that $o(g) = o(h) = o(hg) = p$ such that $hg = fe$ but $gh = 0$. By using the arguments in the proof of Theorem 3.2, we can show similarly that the subring $S = \langle g, h\rangle$ of the ring R (the subring of R generated by f and g) is isomorphic to $\mathrm{UTM}_2(\mathbb{Z}_p)$. However, this is clearly a contradiction. Thus, we have $fe = 0$. This proves Lemma 3.4. □

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