CLOSED CONFORMAL VECTOR FIELDS ON PSEUDO-RIEMANNIAN MANIFOLDS

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We give here a geometric proof of the existence of certain local coordinates on a pseudo-Riemannian manifold admitting a closed conformal vector field.

1. Introduction

A vector field V on a pseudo-Riemannian manifold (M,g) is called *conformal* if

$$\mathscr{L}_V g = 2\lambda g \tag{1.1}$$

for a scalar field λ, where $\mathscr{L}$ denotes the Lie derivative on M. It is easy to see that if V is locally a gradient field, then (1.1) is equivalent to

$$\nabla_X V = \lambda X \quad \text{for every vector field } X. \tag{1.2}$$

Here ∇ denotes the Levi-Civita connection of g. We call vector fields satisfying (1.2) *closed conformal vector fields.* They appear in the work of Fialkow [3] about conformal geodesics, in the works of Yano [7–11] about concircular geometry in Riemannian manifolds, and in the works of Tashiro [6], Kerbrat [4], Kühnel and Rademacher [5], and many other authors.

If V is lightlike on (M,g), then from (1.2), we get

$$Xg(V,V) = 2g(\nabla_X V,V) = 2\lambda g(X,V) = 0 \tag{1.3}$$

for every vector field X. Thus $\lambda \equiv 0$ and V is parallel. About lightlike parallel vector fields, we have the following theorem.

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THEOREM 1.1 (Brinkmann [2]). *If (M,g) admits a lightlike parallel vector field V, then there are local coordinates $u^1,u^2,\dots,u^n$ ($n := \dim M > 2$) such that $V = \partial/\partial u^1$ and*

$$(g_{ij}) = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & (g_{\alpha\beta}) & \\ 0 & 0 & & & \end{array}\right), \tag{1.4}$$

where $\alpha,\beta \in \{3,\dots,n\}$ and $\partial g_{\alpha\beta}/\partial u^1 = 0$.

Brinkmann's proof is purely analytical. We will give, in the next section, geometric tools which will allow us to generalize Brinkmann's theorem.

2. Geometric constructions

Let (M,g) be a connected pseudo-Riemannian manifold of dimension n and signature $(k,n-k)$ with $0<k<n$. Given a vector field W on M, we denote by $W^\flat$ the one-form defined by $W^\flat(X) = g(W,X)$. Then W is locally a gradient field if and only if $dW^\flat = 0$. In the following, a vector field W satisfying $\nabla_W W = 0$ will be called *geodesic*.

LEMMA 2.1. *If W is a geodesic vector field, then $dW^\flat$ is invariant under the flow of W.*

Proof. Let $(\nabla W^\flat)(X,Y) = (\nabla_X W^\flat)(Y) = g(\nabla_X W, Y)$. Then, from the fact that W is geodesic, it follows that

$$\begin{aligned}(\mathcal{L}_W \nabla W^\flat)(X,Y) &= Wg(\nabla_X W, Y) - g(\nabla_{[W,X]} W, Y) - g(\nabla_X W, [W,Y]) \\ &= g(R(W,X)W,Y) + g(\nabla_X W, \nabla_Y W),\end{aligned} \tag{2.1}$$

where R denotes the Riemannian curvature tensor,

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{2.2}$$

Since $g(R(W,X)W,Y)$ is symmetric with respect to X,Y, from

$$dW^\flat(X,Y) = (\nabla W^\flat)(X,Y) - (\nabla W^\flat)(Y,X), \tag{2.3}$$

we get $(\mathcal{L}_W dW^\flat)(X,Y) = (\mathcal{L}_W \nabla W^\flat)(X,Y) - (\mathcal{L}_W \nabla W^\flat)(Y,X) = 0$. □

LEMMA 2.2. *If W is a lightlike geodesic vector field, then $dW^\flat(X,W) = 0$.*

Proof. We have the following.

$$\left.\begin{array}{l} W \text{ lightlike } \Rightarrow (\nabla W^\flat)(X,W) = g(\nabla_X W, W) = 0 \\ W \text{ geodesic } \Rightarrow (\nabla W^\flat)(W,X) = g(\nabla_W W, X) = 0 \end{array}\right\} \Rightarrow dW^\flat(X,W) = 0.$$

□

A nontangent vector field $\widetilde{W}$ on a pseudo-Riemannian hypersurface $\widetilde{M}$ can be extended to a geodesic vector field W in a neighbourhood of $\widetilde{M}$ in the following way. Let $c(s,p)$ be the geodesic starting at $p = c(0,p) \in \widetilde{M}$ with $\dot{c}(0,p) = \widetilde{W}(p)$ and $W(c(s,p)) := \dot{c}(s,p)$. Then, taking into account the fact that $\widetilde{W}$ is transversal (i.e. nontangent) to $\widetilde{M}$, we conclude that W is a geodesic vector field on a neighbourhood of $\widetilde{M}$ extending $\widetilde{W}$. Moreover, if $\widetilde{W}$ is lightlike, then so is W. Denoting with $\widetilde{W}^\top$, $\widetilde{W}^\perp$ the tangent and normal component of $\widetilde{W}$, for vector fields X, Y on $\widetilde{M}$ tangent to $\widetilde{M}$, we have the following lemma.

Lemma 2.3. $dW^\flat(X,Y) = d(\widetilde{W}^\top)^\flat(X,Y)$.

Proof. The statement follows from $g(\nabla_X \widetilde{W}^\perp, Y) - g(\nabla_Y \widetilde{W}^\perp, X) = -g(\widetilde{W}^\perp, [X,Y]) = 0$. □

The following remark will be used in the proof of the next proposition.

Remark 2.4. Let V be a vector field and let φ be a function on M. At a point $p_0 \in M$, the gradient of the solutions of $Vf = \varphi$ span an affine hyperplane H of $T_{p_0}M$. Let $v := V(p_0)$, then $H = \{x \in T_{p_0}M \mid g(x,v) = \varphi(p_0)\}$ and

(a) if $\varphi(p_0) \neq 0$, then H contains lightlike, spacelike, and timelike vectors,

(b) if $\varphi(p_0) = 0$, then H contains only lightlike vectors and the zero vector if and only if $n = 2$ and v is lightlike.

Proposition 2.5. *If V is a closed conformal vector field on (M,g), then in a neighbourhood of a point p_0 where $V(p_0) \neq 0$, there is a lightlike geodesic gradient field W such that $g(V,W) = 1$.*

Proof. We divide the proof into two cases.
Case 1. $n > 2$ or $n = 2$ and $V(p_0)$ is nonlightlike.

Let u be a solution of $Vu = 0$ with $g(p_0)(\nabla u, \nabla u) \neq 0$ (here ∇u denotes the gradient of u). According to Remark 2.4(b), such a solution exists. Let $\mathcal{U}$ be an open neighbourhood of p_0 on which $g(\nabla u, \nabla u) \neq 0$, and let $\widetilde{M}$ be the pseudo-Riemannian hypersurface $u^{-1}(u(p_0)) \cap \mathcal{U}$. Then ∇u is a normal vector field on $\widetilde{M}$ and, from $Vu = 0$, we have that $\tilde{V} := V|_{\widetilde{M}}$ is a tangent vector field on $\widetilde{M}$. Let $\tilde{f} : \widetilde{M} \to \mathbb{R}$ be a solution of $\tilde{V}\tilde{f} = 1$ such that $g(p_0)(\nabla\tilde{f}, \nabla\tilde{f})$ and $g(p_0)(\nabla u, \nabla u)$ have opposite sign (see Remark 2.4(a)). Without loss of generality, we assume that $g(\nabla\tilde{f}, \nabla\tilde{f}) \neq 0$ on $\widetilde{M}$. Setting $\widetilde{W} := \nabla\tilde{f} + h\nabla u$, where $h^2 := -g(\nabla\tilde{f}, \nabla\tilde{f})/g(\nabla u, \nabla u) > 0$, we get

$$g(\widetilde{W}, \widetilde{W}) = g(\nabla\tilde{f}, \nabla\tilde{f}) + h^2 g(\nabla u, \nabla u) = 0, \qquad g(\tilde{V}, \widetilde{W}) = \tilde{V}\tilde{f} = 1. \tag{2.4}$$

Let now W be the geodesic vector field extending $\widetilde{W}$ in a neighbourhood of $\widetilde{M}$. Then W is lightlike. From $Wg(V,W) = g(\nabla_W V, W) + g(V, \nabla_W W) = 0$ and $g(\tilde{V}, \widetilde{W}) = 1$, we conclude that $g(V,W) = 1$. It remains to show that W is locally a gradient.

For vector fields X, Y on $\widetilde{M}$ (not necessarily tangent to $\widetilde{M}$), we can write

$$X = X^\top + \alpha\widetilde{W}, \qquad Y = Y^\top + \beta\widetilde{W}, \tag{2.5}$$

where α and β are certain functions on $\widetilde{M}$ and $X^\top$, $Y^\top$ are tangent to $\widetilde{M}$. Using Lemma 2.2, we get

$$0 = dW^\flat(X,W) = dW^\flat(X^\top + \alpha W, W) = dW^\flat(X^\top, W). \tag{2.6}$$

In the same way, we get $dW^\flat(W,Y^\top) = 0$, and therefore $dW^\flat(X,Y) = dW^\flat(X^\top,Y^\top)$. Now Lemma 2.3 and $\widetilde{W}^\top = \nabla\tilde{f}$ imply that $dW^\flat(X,Y) = 0$ on $\widetilde{M}$. Using Lemma 2.1, we conclude that $dW^\flat = 0$.

Case 2. $n = 2$ and $V(p_0)$ is lightlike.

According to Remark 2.4(b), we cannot proceed as in Case 1 since the gradient at p_0 of a solution of $Vu = 0$ is a lightlike vector. Remarking that along an integral curve α of V through p_0 V is lightlike, we set $\widetilde{M} := Im\alpha$. Let now $\widetilde{W}$ be a lightlike vector field along α such that V and $\widetilde{W}$ are linearly independent. Then, since g is nondegenerate, $g(V,V)g(\widetilde{W},\widetilde{W}) - g(V,\widetilde{W})^2 = -g(V,\widetilde{W})^2 \neq 0$. Therefore we can assume that $g(V,\widetilde{W}) = 1$. Since $\widetilde{W}$ is not tangent to α, we can extend it to a geodesic vector field W on a neighbourhood $\mathcal{U}$ of p_0. Then $Wg(W,W) = 0$ which, together with $\widetilde{W}$ lightlike, implies W lightlike, and $Wg(V,W) = g(\nabla_W V, W) = 0$ which, together with $g(V,\widetilde{W}) = 1$, implies $g(V,W) = 1$. Since every vector field on $\mathcal{U}$ can be written as a linear combination of V and W, we have $g(\nabla_X W, Y) - g(\nabla_Y W, X) = 0$ for every vector field X, Y on $\mathcal{U}$ if and only if $g(\nabla_V W, W) - g(\nabla_W W, V) = 0$.

Thus W being lightlike and geodesic implies that W is a gradient vector field.

It remains to show that V is lightlike along an integral curve α through $p_0 := \alpha(0)$. This follows from $(d/dt)g(V,V) = 2g(\nabla_V V, V) = 2\lambda g(V,V)$, since its general solution is $g(\alpha(t))(V,V) = g(p_0)(V,V)e^{2\int_0^t \lambda(u)du}$. □

For example, let $M = \mathbb{R}^n_k$ be the pseudo-Euclidian space of dimension n and signature $(k, n-k)$ with $0 < k < n$, that is, $\langle x,x\rangle = -(x_1^2 + \cdots + x_k^2) + (x_{k+1}^2 + \cdots + x_n^2)$. The position vector field $V(x) = \sum_{i=1}^n x_i(\partial/\partial x_i)|_x$ satisfies $\nabla_X V = X$, and therefore it is a closed conformal vector field. We will construct, following the proof of Proposition 2.5, a lightlike geodesic gradient field W with $\langle V,W\rangle = 1$ in a neighbourhood of a point $x_0 \neq 0$ ($V(x) = 0$ if and only if $x = 0$). We take for simplicity $x_0 = (1,0,\dots,0)$, then $u(x_1,\dots,x_n) := x_n/x_1$ is a solution of $Vu = 0$ with $\langle\nabla u,\nabla u\rangle|_{x_0} = 1$. The hypersuface $\widetilde{M} := u^{-1}(u(x_0)) = u^{-1}(0)$ is the hyperplane $x_n = 0$. Let $\widetilde{V} := V|_{\widetilde{M}}$, then $\tilde{f}(x_1,\dots,x_{n-1}) := \ln x_1$ is a solution of $\widetilde{V}\tilde{f} = 1$ with $\langle\nabla\tilde{f},\nabla\tilde{f}\rangle|_{x_0} = -1$. Defining for every $x \in \widetilde{M}$ that

$$\widetilde{W}(x) := \nabla\tilde{f}(x) + \nabla u(x) = \frac{1}{x_1}\left(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_n}\right)\bigg|_x, \tag{2.7}$$

it is easy to see that

$$W(x) := \frac{1}{x_1 + x_n}\left(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_n}\right)\bigg|_x \tag{2.8}$$

is a geodesic vector field on M extending $\widetilde{W}$. Moreover W is lightlike, $\langle V, W\rangle = 1$, and $W = \nabla \ln|x_1 + x_n|$. It is clear that W is not unique and not everywhere defined. More generally, for an arbitrary point $x_0 \neq 0$, we have, for instance, that

$$W = \nabla \ln |\langle a,x\rangle|, \quad \text{where } a \text{ is a lightlike vector in } \mathbb{R}^n_k \text{ with } \langle a,x_0\rangle \neq 0, \tag{2.9}$$

is a lightlike geodesic gradient field satisfying $\langle V, W\rangle = 1$.

Finally we remark that a nontrivial conformal vector field (a vector field V is nontrivial if there is a point $p \in M$ with $V(p) \neq 0$) has isolated zeros (see [4]). This is in general not true if the conformal vector field is not closed (see, e.g., an example in [1]).

3. Local coordinates

Let V and W be vector fields as in Proposition 2.5 and let $E_1 = V - g(V,V)W$, $E_2 = W$. It is easy to see that

(i) E_1, E_2 are linearly independent;

(ii) the distribution $\mathfrak{D}$ spanned by E_1, E_2 is integrable and the metric g is nondegenerate on $\mathfrak{D}$;

(iii) the distribution $\mathfrak{D}^\perp$ spanned by the vector fields orthogonal to E_1, E_2 is integrable and g is nondegenerate on $\mathfrak{D}^\perp$;

(iv) $[E_1, E_2] = 0$.

We can now state the following theorem.

Theorem 3.1. *If (M,g) admits a closed conformal vector field V, then in a neighbourhood of a point p_0 where $V(p_0) \neq 0$, there are local coordinates $u^1, u^2, \ldots, u^n$ such that $V = \partial/\partial u^1 + a(\partial/\partial u^2)$, for some function $a = a(u^2)$, and*

$$(g_{ij}) = \left(\begin{array}{cc|ccc} -a & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & (g_{\alpha\beta}) & \\ 0 & 0 & & & \end{array}\right), \tag{3.1}$$

where $\alpha, \beta \in \{3, \ldots, n\}$, $\det(g_{\alpha\beta}) \neq 0$, and $\partial g_{\alpha\beta}/\partial u^1 + a(\partial g_{\alpha\beta}/\partial u^2) = a' g_{\alpha\beta}$ $(a' := da/du^2)$.

Proof. From Frobenius theorem, we know that there are local coordinates $u^1, u^2, \ldots, u^n$ such that

$$\frac{\partial}{\partial u^1} = E_1, \qquad \frac{\partial}{\partial u^2} = E_2, \qquad g_{1\alpha} = g_{2\alpha} = 0, \quad \alpha = 3, \ldots, n. \tag{3.2}$$

Hence $g_{11} = g(E_1, E_1) = g(V,V) - 2g(V,V)g(V,W) = -g(V,V)$, $g_{12} = g(V,W) = 1$, $g_{22} = g(W,W) = 0$ and, setting $E_i = \partial/\partial u^i$, $i = 1,\dots,n$, we have that

$$\begin{aligned}\frac{\partial g_{\alpha\beta}}{\partial u^1} + a\frac{\partial g_{\alpha\beta}}{\partial u^2} &= g(\nabla_{E_1}E_\alpha + g(V,V)\nabla_{E_2}E_\alpha, E_\beta)\\ &\quad + g(E_\alpha, \nabla_{E_1}E_\beta + g(V,V)\nabla_{E_2}E_\beta)\\ &= g(\nabla_{E_\alpha}E_1 + g(V,V)\nabla_{E_\alpha}E_2, E_\beta)\\ &\quad + g(E_\alpha, \nabla_{E_\beta}E_1 + g(V,V)\nabla_{E_\beta}E_2)\\ &= g(\nabla_{E_\alpha}(E_1 + g(V,V)E_2), E_\beta)\\ &\quad + g(E_\alpha, \nabla_{E_\beta}(E_1 + g(V,V)E_2))\\ &= g(\nabla_{E_\alpha}V, E_\beta) + g(E_\alpha, \nabla_{E_\beta}V,) = 2\lambda g_{\alpha\beta},\end{aligned} \tag{3.3}$$

where $a = g(V,V)$. From $Xg(V,V) = 2\lambda g(X,V)$ and $g(E_1,V) = g(E_3,V) = \cdots = g(E_n,V) = 0$, we conclude that $a = a(u^2)$. Furthermore

$$a' = Wg(V,V) = 2\lambda \tag{3.4}$$

and $a = 0$ if and only if V is lightlike (cf. with Brinkmann's theorem). □

On the other hand, we have the following proposition.

Proposition 3.2. *If on a neighbourhood $\mathcal{U}$ of a point $p_0 \in M$, there are local coordinates as in Theorem 3.1, then $V = \partial/\partial u^1 + a(\partial/\partial u^2)$ is a closed conformal vector field on $\mathcal{U}$.*

Proof. The statement follows from

$$\begin{aligned} g(\nabla_{E_i}V, E_j) &= g(\nabla_{E_i}E_1, E_j) + a'\delta_{2i}\delta_{1j} + ag(\nabla_{E_i}E_2, E_j)\\ &= \frac{1}{2}\left(\frac{\partial g_{1j}}{\partial u^i} + \frac{\partial g_{ij}}{\partial u^1} - \frac{\partial g_{1i}}{\partial u^j} + a\frac{\partial g_{ij}}{\partial u^2}\right) + a'\delta_{2i}\delta_{1j}\\ &= \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial u^1} + a\frac{\partial g_{ij}}{\partial u^2}\right) + \frac{1}{2}a'(\delta_{1i}\delta_{2j} + \delta_{2i}\delta_{1j}),\end{aligned} \tag{3.5}$$

where δ is the Kronecker delta. Namely, for every pair (i,j), we get $g(\nabla_{E_i}V, E_j) = (1/2)a'g_{ij}$. Moreover, V is lightlike if and only if $a = 0$. □

Remark 3.3. If in Proposition 3.2 we assume that $a \neq 0$, then according to Fialkow results, see [3, formulas (12.9) and (12.10)], we must be able to prove that $(\mathcal{U}, g)$ is locally isometric to a warped product with a one-dimensional base manifold. This can be seen in the following way: take local coordinates $\overline{u}^1,\dots,\overline{u}^n$ in $\mathcal{U}$ such that

$$\frac{\partial}{\partial\overline{u}^1}=\frac{1}{\sqrt{|a|}}\left(\frac{\partial}{\partial u^1}+a\frac{\partial}{\partial u^2}\right),\qquad \frac{\partial}{\partial\overline{u}^2}=\frac{\partial}{\partial u^1},\qquad \frac{\partial}{\partial\overline{u}^\alpha}=\frac{\partial}{\partial u^\alpha},\quad \alpha=3,\dots,n. \tag{3.6}$$

This is reached by the coordinate transformation

$$\overline{u}^1=\int\frac{\sqrt{|a|}}{a}du^2,\qquad \overline{u}^2=u^1-\int\frac{1}{a}du^2,\qquad \overline{u}^\alpha=u^\alpha,\quad \alpha=3,\dots,n. \tag{3.7}$$

Then it is easy to see that $a=a(\overline{u}^1)$ and that

$$(\overline{g}_{ij}):=\left(g\left(\frac{\partial}{\partial\bar{u}^i},\frac{\partial}{\partial\bar{u}^j}\right)\right)=\left(\begin{array}{c|cccc}\pm1 & 0 & 0 & \cdots & 0\\ \hline 0 & -a & 0 & \cdots & 0\\ 0 & 0 & & & \\ \vdots & \vdots & & (g_{\alpha\beta}) & \\ 0 & 0 & & & \end{array}\right). \tag{3.8}$$

Furthermore, from $\partial g_{\alpha\beta}/\partial u^1+a(\partial g_{\alpha\beta}/\partial u^2)=a' g_{\alpha\beta}$, we get

$$\frac{\partial g_{\alpha\beta}}{\partial\overline{u}^1}=\frac{1}{\sqrt{|a|}}\left(\frac{\partial g_{\alpha\beta}}{\partial u^1}+a\frac{\partial g_{\alpha\beta}}{\partial u^2}\right)=\frac{1}{\sqrt{|a|}}\frac{da}{du^2}g_{\alpha\beta}=\frac{1}{a}\frac{da}{d\overline{u}^1}g_{\alpha\beta}, \tag{3.9}$$

and therefore $g_{\alpha\beta}=a\overline{g}_{\alpha\beta}$, where $\partial\overline{g}_{\alpha\beta}/\partial\overline{u}^1=0$. Thus $(\mathcal{U},g)$ is locally isometric to a warped product with a one-dimensional base manifold and warped factor a. In these local coordinates, the metric of the fiber manifold is given by

$$\left(\begin{array}{c|ccc}-1 & 0 & \cdots & 0\\ \hline 0 & & & \\ \vdots & & (\overline{g}_{\alpha\beta}) & \\ 0 & & & \end{array}\right) \tag{3.10}$$

which means, in other words, that $\overline{u}^2,\dots,\overline{u}^n$ are Fermi coordinates on the fiber manifold.

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