APPROXIMATION OF BOUNDED VARIATION FUNCTIONS BY A BÉZIER VARIANT OF THE BLEIMANN, BUTZER, AND HAHN OPERATORS

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We give a sharp estimate on the rate of convergence for the Bézier variant of Bleimann, Butzer, and Hahn operators for functions of bounded variation. We consider the case when $\alpha \geq 1$ and our result improves the recently established results of Srivastava and Gupta (2005) and de la Cal and Gupta (2005).

1. Introduction

Bleimann et al. [3] introduced an interesting sequence of positive linear operators defined on the space of real functions on the infinite interval $[0,\infty)$ by

$$L_n(f,x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n-k+1}\right), \quad x \in [0,\infty),\ n \in \mathbb{N}, \tag{1.1}$$

where

$$b_{n,k}(x) = \binom{n}{k} \frac{x^k}{(1+x)^n}. \tag{1.2}$$

The Bézier variant of these operators for $\alpha \geq 1$ is defined in [6] as

$$L_{n,\alpha}(f,x) = \sum_{k=0}^{n} Q_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n-k+1}\right), \quad x \in [0,\infty),\ n \in \mathbb{N}, \tag{1.3}$$

where $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$ and $J_{n,k}(x) = \sum_{j=k}^{n} b_{n,j}(x)$.

As a special case $\alpha = 1$, $L_{n,\alpha}(f,x)$ reduce to the operators $L_{n,1}(f,x) \equiv L_n(f,x)$, defined by (1.1). Some approximation properties of the Bleimann, Butzer, and Hahn operators were discussed in [1, 2], and so forth. Very recently, de la Cal and Gupta [4] and Srivastava

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and Gupta [6] studied the rate of approximation for the Bleimann, Butzer, and Hahn operators and its Bézier variant ($\alpha \geq 1$), respectively.

We recall the Lebesgue-Stieltjes integral representation

$$L_{n,\alpha}(f,x) = \int_0^\infty f(t)d_t(K_{n,\alpha}(x,t)), \tag{1.4}$$

where

$$K_{n,\alpha}(x,t) = \begin{cases} \sum\limits_{k \leq (n-k+1)t} Q_{n,k}^{(\alpha)}(x), & 0 < t < \infty, \\ 0, & t = 0. \end{cases} \tag{1.5}$$

In this paper, we give a different and improved estimate on the rate of approximation for functions of bounded variation on the Bézier variant of Bleimann, Butzer, and Hahn operators.

2. Auxiliary results

In this section, we recall two lemmas, which are essential for our main theorem.

Lemma 2.1 [6, Lemma 3]. *For all $x \in (0,\infty)$, $\alpha \geq 1$, and $k \in \mathbb{N}$, there holds*

$$Q_{n,k}^{(\alpha)}(x) \leq \alpha b_{n,k}(x) < \frac{\alpha(1+x)}{\sqrt{2enx}}. \tag{2.1}$$

Lemma 2.2 [5, Lemma 3]. *For $x \in (0,\infty)$,*

$$\left| \sum_{k/(n-k+1)>x} b_{n,k}(x) - \frac{1}{2} \right| \leq \frac{|1-x|}{6\sqrt{2\pi(n+1)x}} + O(n^{-3/2}). \tag{2.2}$$

3. Rate of convergence

Our main result is stated as follows.

Theorem 3.1. *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$. Let $f(t) = O(t^r)$ for some $r \in \mathbb{N}$ as $t \to \infty$. Then for $x \in (0,\infty)$, $\alpha \geq 1$, and for $n \to \infty$,*

$$\begin{aligned} &\left| L_{n,\alpha}(f,x) - 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-) \right| \\ &\quad \leq \frac{9\alpha(1+x)^2}{(n+2)x} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(f_x) + \frac{\alpha|1-x|}{6\sqrt{2\pi(n+1)x}} \left| f(x+) - f(x-) \right| \\ &\qquad + \frac{\alpha(1+x)}{\sqrt{2enx}} \varepsilon_n(x) \left| f(x) - f(x-) \right| + O(n^{-1}), \end{aligned} \tag{3.1}$$

where

$$\varepsilon_n(x) = \begin{cases} 1, & \text{if } \dfrac{x(n+1)}{1+x} \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases}$$
$$f_x(t) = \begin{cases} f(t) - f(x-), & \text{if } 0 \le t < x, \\ 0, & \text{if } t = x, \\ f(t) - f(x+), & \text{if } x < t < \infty, \end{cases} \tag{3.2}$$

and $V_a^b(f_x)$ is the total variation of f_x on $[a,b]$.

Proof. We have

$$\begin{aligned} f(t) &- 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-) \\ &= f_x(t) + 2^{-\alpha}(f(x+) - f(x-))\operatorname{sign}^{(\alpha)}(t-x) \\ &\quad + (f(x) - 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-))\delta_x(t), \end{aligned} \tag{3.3}$$

where

$$\operatorname{sign}^{(\alpha)}(t-x) := \begin{cases} 2^\alpha - 1, & \text{if } t > x, \\ 0, & \text{if } t = x, \\ -1, & \text{if } t < x, \end{cases} \qquad \delta_x(t) = \begin{cases} 1, & \text{if } x = t, \\ 0, & \text{if } x \neq t. \end{cases} \tag{3.4}$$

Therefore, we can write

$$\begin{aligned} &\left| L_{n,\alpha}(f,x) - 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-) \right| \\ &\quad \le \left| L_{n,\alpha}(f_x,x) \right| + \left| 2^{-\alpha}(f(x+) - f(x-)) L_{n,\alpha}(\operatorname{sign}^{(\alpha)}(t-x),x) \right. \\ &\qquad\qquad \left. + [f(x) - 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-)] L_{n,\alpha}(\delta_x,x) \right|, \end{aligned} \tag{3.5}$$

and our first estimates are

$$\begin{aligned} L_{n,\alpha}(\operatorname{sign}^{(\alpha)}(t-x),x) &= 2^\alpha \sum_{k>(n-k+1)x} Q_{n,k}^{(\alpha)}(x) - 1 + \varepsilon_n(x) Q_{n,k'}^{(\alpha)}(x) \\ &= 2^\alpha \Bigg(\sum_{k>(n-k+1)x} b_{n,k}(x) \Bigg)^\alpha - 1 + \varepsilon_n(x) Q_{n,k'}^{(\alpha)}(x), \\ L_{n,\alpha}(\delta_x,x) &= \varepsilon_n(x) Q_{n,k'}^{(\alpha)}(x). \end{aligned} \tag{3.6}$$

Then we have

$$
\begin{aligned}
G := & \,\big|2^{-\alpha}(f(x+)-f(x-))L_{n,\alpha}(\operatorname{sign}^{(\alpha)}(t-x),x) \\
& + [f(x)-2^{-\alpha}f(x+)-(1-2^{-\alpha})f(x-)]L_{n,\alpha}(\delta_x,x)\big| \\
= & \left|2^{-\alpha}(f(x+)-f(x-))\left[2^{\alpha}\sum_{k>(n-k+1)x} Q_{n,k}^{(\alpha)}(x)-1\right]+(f(x)-f(x-))\varepsilon_n(x)Q_{n,k'}^{(\alpha)}(x)\right|.
\end{aligned}
\tag{3.7}
$$

Using the mean value theorem, we get

$$
\left|\sum_{k>(n-k+1)x} Q_{n,k}^{(\alpha)}(x)-2^{-\alpha}\right| = \alpha\big(\xi_{n,k}(x)\big)^{\alpha-1}\left|\sum_{k>(n-k+1)x} b_{n,k}(x)-2^{-1}\right|, \tag{3.8}
$$

where $\xi_{n,k}(x)$ lies between 2^{-1} and $\sum_{k>(n-k+1)x} b_{n,k}(x)$. Because of Lemma 2.2, it is easily seen that the intermediate point $\xi_{n,k}(x)$ is close to 2^{-1} for sufficiently large n. Then we can write $\xi_{n,k}(x)=(2+\varepsilon)^{-1}$ for each $\varepsilon>0$. Thus, we have

$$
\big(\xi_{n,k}(x)\big)^{\alpha-1}=(2+\varepsilon)^{1-\alpha}\le 1 \tag{3.9}
$$

for each $\alpha\ge 1$. By using (3.9) and Lemma 2.2 in (3.8), we obtain

$$
\left|\sum_{k>(n-k+1)x} Q_{n,k}^{(\alpha)}(x)-2^{-\alpha}\right| \le \frac{\alpha|1-x|}{6\sqrt{2\pi(n+1)x}}+O(n^{-3/2}). \tag{3.10}
$$

Hence, by using (3.10) in (3.7) and Lemma 2.1, we obtain

$$
G\le \frac{\alpha|1-x|}{6\sqrt{2\pi(n+1)x}}\,|f(x+)-f(x-)|+\frac{\alpha(1+x)}{\sqrt{2enx}}\varepsilon_n(x)\,|f(x)-f(x-)|+O(n^{-3/2}). \tag{3.11}
$$

On the other hand, to estimate $L_{n,\alpha}(f_x,x)$, we break the Lebesgue-Stieltjes integral into four parts as follows:

$$
L_{n,\alpha}(f_x,x)=\left(\int_0^{x-x/\sqrt{n}}+\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}+\int_{x+x/\sqrt{n}}^{2x}+\int_{2x}^{\infty}\right)f_x(t)\,d_t\big(K_{n,\alpha}(x,t)\big) \tag{3.12}
$$

then, by proceeding along the lines of [6], we get

$$
|L_{n,\alpha}(f_x,x)|\le \frac{9\alpha(1+x)^2}{(n+2)x}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(f_x)+O(n^{-1}). \tag{3.13}
$$

Using (3.11) and (3.13) in (3.5), we get the desired result. This completes the proof of Theorem 3.1. □

Notice that for the case $0<\alpha<1$, these results can be found in [5].

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