INVERTIBILITY OF MATRIX WIENER-HOPF PLUS HANKEL OPERATORS WITH APW FOURIER SYMBOLS

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A characterization of the invertibility of a class of matrix Wiener-Hopf plus Hankel operators is obtained based on a factorization of the Fourier symbols which belong to the Wiener subclass of the almost periodic matrix functions. Additionally, a representation of the inverse, lateral inverses, and generalized inverses is presented for each corresponding possible case.

1. Introduction

Operators of Wiener-Hopf plus Hankel type have been receiving an increasing attention in the last years (see [1, 2, 4, 6, 10, 12–16]). Some of the interest in their study arises directly from concrete applications where these kind of operators appear. This is the case in problems of wave diffraction by some particular rectangular geometries which originate specific boundary-transmission value problems that may be equivalently translated by systems of integral equations that lead to such kind of operators (see, e.g., [5, 7, 8]).

A great part of the study in this kind of operators is concentrated in the description of their Fredholm and invertibility properties. In particular, for some classes of the so-called *Fourier symbols of the operators*, their invertibility properties are already known (cf. the above references). Despite those advances, for some other classes of Fourier symbols, a complete description of the Fredholm and invertibility properties is still missing. In this way, some of the ongoing researches try to achieve the best possible factorization procedures of the involved Fourier symbols in such a way that a representation of the (generalized) inverses of the Wiener-Hopf plus Hankel operators will be possible to obtain when in the presence of a convenient factorization.

Within this spirit, the main aim of the present work is to provide an invertibility criterion for the Wiener-Hopf plus Hankel operators of the form

$$WH_\Phi = W_\Phi + H_\Phi : \left[L^2_+(\mathbb{R})\right]^N \longrightarrow \left[L^2(\mathbb{R}_+)\right]^N, \tag{1.1}$$

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where W_Φ stands for the Wiener-Hopf operator defined by

$$W_\Phi = r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F} : \left[L^2_+(\mathbb{R})\right]^N \longrightarrow \left[L^2(\mathbb{R}_+)\right]^N, \tag{1.2}$$

H_Φ denotes the Hankel operator given by

$$H_\Phi = r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}J : \left[L^2_+(\mathbb{R})\right]^N \longrightarrow \left[L^2(\mathbb{R}_+)\right]^N, \tag{1.3}$$

and the Fourier symbol Φ belongs to the so-called APW subclass of $[L^\infty(\mathbb{R})]^{N\times N}$. This will therefore extend some of the results of [15] to the matrix case.

As for the notations in (1.1)–(1.3), and in what follows, $[L^2_+(\mathbb{R})]^N$ denotes the subspace of $[L^2(\mathbb{R})]^N$ formed by all vectors of functions supported in the closure of $\mathbb{R}_+$, r_+ represents the operator of restriction from $[L^2(\mathbb{R})]^N$ into $[L^2(\mathbb{R}_+)]^N$, $\mathcal{F}$ stands for the Fourier transformation, and J is the reflection operator given by the rule $J\Phi(x) = \widetilde{\Phi}(x) := \Phi(-x)$, $x\in\mathbb{R}$.

In view of defining the APW functions, let us first consider the algebra of *almost periodic functions*, usually denoted by AP. When endowed with the usual norm and multiplicative operation, AP is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ ($\lambda\in\mathbb{R}$), where

$$e_\lambda(x) := e^{i\lambda x}, \quad x\in\mathbb{R}. \tag{1.4}$$

In this framework, it turns out that the elements of APW are those from AP which allow a representation by an absolutely convergent series. In fact, APW is precisely the (proper) subclass of all functions $\varphi\in AP$ which can be written in an absolutely convergent series of the form

$$\varphi = \sum_j \varphi_j e_{\lambda_j}, \quad \lambda_j\in\mathbb{R}, \ \sum_j |\varphi_j| < \infty. \tag{1.5}$$

Let us agree on the notation $\mathcal{G}B$ for the group of all invertible elements of a Banach algebra B. To end with the notation, we will say that a matrix function Φ belongs to $APW^{N\times N}$, and write $\Phi\in APW^{N\times N}$, if all entries of the matrix Φ belong to APW.

As mentioned above, the representation of the (generalized/lateral/both-sided) inverses of WH_Φ based on some factorization of the Fourier symbol Φ is an important goal, and will be obtained in the final part of the paper. In this way, the main contributions of the present work are described in Theorems 5.1, 4.1, and 3.3.

In the next section we will recall some useful particular known results which are anyway presented with complete proofs for the reader's convenience.

2. Initial multiplicative decompositions

According to (1.1)–(1.3), we have

$$WH_\Phi = r_+\left(\mathcal{F}^{-1}\Phi\cdot\mathcal{F} + \mathcal{F}^{-1}\Phi\cdot\mathcal{F}J\right) = r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}\left(I_{[L^2_+(\mathbb{R})]^N} + J\right), \tag{2.1}$$

where $I_{[L^2_+(\mathbb{R})]^N}$ denotes the identity operator in $[L^2_+(\mathbb{R})]^N$. Furthermore, since

$$I_{[L^2_+(\mathbb{R})]^N} + J = \ell^e r_+, \tag{2.2}$$

where $\ell^e : [L^2(\mathbb{R}_+)]^N \to [L^2(\mathbb{R})]^N$ denotes the even extension operator, we may also rewrite the Wiener-Hopf plus Hankel operator in (1.1) as

$$WH_\Phi = r_+ \mathcal{F}^{-1} \Phi \cdot \mathcal{F} \ell^e r_+. \tag{2.3}$$

From the theory of Wiener-Hopf and Hankel operators, it is well known that

$$\begin{aligned} W_{\Psi\Phi} &= W_\Psi \ell_0 W_\Phi + H_\Psi \ell_0 H_{\tilde{\Phi}}, \\ H_{\Psi\Phi} &= W_\Psi \ell_0 H_\Phi + H_\Psi \ell_0 W_{\tilde{\Phi}}, \end{aligned} \tag{2.4}$$

where $\ell_0 : [L^2(\mathbb{R}_+)]^N \to [L^2(\mathbb{R})]^N$ denotes the zero extension operator. Additionally, from the last two identities, it follows that

$$WH_{\Psi\Phi} = W_\Psi \ell_0 WH_\Phi + H_\Psi \ell_0 WH_{\tilde{\Phi}}, \tag{2.5}$$

$$WH_{\Psi\Phi} = WH_\Psi \ell_0 WH_\Phi + H_\Psi \ell_0 WH_{\tilde{\Phi}-\Phi}. \tag{2.6}$$

Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, $\mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$, and $H^\infty(\mathbb{C}_\pm)$ be the set of all bounded and analytic functions in $\mathbb{C}_\pm$. Fatou's theorem ensures that functions in $H^\infty(\mathbb{C}_\pm)$ have nontangential limits on $\mathbb{R}$ almost everywhere, and it is usually denoted by $H^\infty_\pm(\mathbb{R})$ the set of all elements in $L^\infty(\mathbb{R})$ that are nontangential limits of functions in $H^\infty(\mathbb{C}_\pm)$. Below, we will use the matrix versions $[H^\infty(\mathbb{C}_\pm)]^{N\times N}$ and $[H^\infty_\pm(\mathbb{R})]^{N\times N}$ of those Hardy spaces.

It is already interesting to observe that, due to (2.6), if we consider Φ being an even function or $\Psi \in [H^\infty_-(\mathbb{R})]^{N\times N}$, we will then obtain the multiplicative relation

$$WH_{\Psi\Phi} = WH_\Psi \; \ell_0 \; WH_\Phi \tag{2.7}$$

of two corresponding Wiener-Hopf plus Hankel operators. A more general property in this direction is formulated in the next result.

Proposition 2.1. *If* $\Psi \in [H^\infty_-(\mathbb{R})]^{N\times N}$ *and* $\Phi, \Xi \in [L^\infty(\mathbb{R})]^{N\times N}$, *such that* $\Xi = \tilde{\Xi}$, *then*

$$WH_{\Psi\Phi\Xi} = WH_\Psi \ell_0 WH_\Phi \ell_0 WH_\Xi = W_\Psi \ell_0 WH_\Phi \ell_0 WH_\Xi. \tag{2.8}$$

Proof. Since $\Psi \in [H^\infty_-(\mathbb{R})]^{N\times N}$, we may apply the first presented multiplicative relation for Wiener-Hopf plus Hankel operators; see (2.7). This leads us to

$$WH_{\Psi\Phi\Xi} = WH_\Psi \ell_0 WH_{\Phi\Xi}. \tag{2.9}$$

In addition, since $\Xi = \tilde{\Xi}$, it also follows from (2.7) that

$$WH_{\Phi\Xi} = WH_\Phi \ell_0 WH_\Xi. \tag{2.10}$$

From (2.9) and (2.10), we have that

$$WH_{\Psi\Phi\Xi} = WH_{\Psi}\ell_0 WH_{\Phi}\ell_0 WH_{\Xi}. \tag{2.11}$$

Since $\Psi \in [H^{\infty}_{-}(\mathbb{R})]^{N\times N}$, we have $H_\Psi = 0$ due to the structure of the Hankel operators. Therefore $WH_\Psi = W_\Psi$, and it follows from (2.11) that $WH_{\Psi\Phi\Xi} = W_\Psi \ell_0 WH_\Phi \ell_0 WH_\Xi$. □

Proposition 2.2. *If $\Phi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ and $\widetilde{\Phi_e} = \Phi_e$, then WH_{Φ_e} is invertible and its inverse is the operator $\ell_0 WH_{\Phi_e^{-1}}\ell_0 : [L^2(\mathbb{R}_+)]^N \to [L^2_+(\mathbb{R})]^N$.*

Proof. Since both Φ_e and Φ_e^{-1} are even functions, we directly obtain from the above multiplicative relations of Wiener-Hopf plus Hankel operators that

$$\begin{gathered} WH_{\Phi_e}\ell_0 WH_{\Phi_e^{-1}}\ell_0 = WH_{\Phi_e\cdot\Phi_e^{-1}}\ell_0 = WH_1\ell_0 = W_1\ell_0 = I_{[L^2(\mathbb{R}_+)]^N}, \\ \ell_0 WH_{\Phi_e^{-1}}\ell_0 WH_{\Phi_e} = \ell_0 WH_{\Phi_e\cdot\Phi_e^{-1}} = \ell_0 W_1 = I_{[L^2_+(\mathbb{R})]^N}. \end{gathered} \tag{2.12}$$

This obviously shows that WH_{ϕ_e} is invertible and its inverse is $\ell_0 WH_{\phi_e^{-1}}\ell_0$. □

3. Matrix APW asymmetric factorization

The *(Bohr) mean value* of $\phi \in AP$ is defined by

$$M(\phi) = \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|}\int_{I_\alpha} \phi(x)dx, \tag{3.1}$$

where $\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ is a family of intervals $I_\alpha \subset \mathbb{R}$ such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$ as $\alpha \to \infty$ (for an unbounded set $A \subset \mathbb{R}_+$). The mean value of an element in AP always exists, is finite, and is independent of the particular choice of the family $\{I_\alpha\}_{\alpha\in A}$.

Let $\Omega(\psi) := \{\lambda \in \mathbb{R} : M(\psi e_{-\lambda}) \neq 0\}$ be the *Bohr-Fourier spectrum* of ψ. We will denote by AP^- (AP^+) the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ, $\lambda \le 0$ ($\lambda \ge 0$), and consider APW^- (APW^+) to be the set of all functions $\psi \in APW$ such that $\Omega(\psi) \subset (-\infty, 0]$ ($\Omega(\psi) \subset [0, +\infty)$, resp.). It is therefore clear that $APW^- \subset AP^-$ and $APW^+ \subset AP^+$.

Definition 3.1. Say that a matrix function $\Phi \in \mathcal{G}APW^{N\times N}$ admits an *APW asymmetric factorization* if it can be represented in the form

$$\Phi = \Phi_- \operatorname{diag}[e_{\lambda_1}, \dots, e_{\lambda_N}]\Phi_e, \tag{3.2}$$

where $\lambda_k \in \mathbb{R}$, $e_{\lambda_k}(x) = e^{i\lambda_k x}$, $x \in \mathbb{R}$, $\Phi_- \in \mathcal{G}APW^{N\times N}_-$, $\Phi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$, and $\widetilde{\Phi_e} = \Phi_e$.

Remark 3.2. We would like to remark that an APW asymmetric factorization, if it exists, is not unique. Anyway, the partial indices of two APW asymmetric factorizations of the same matrix function are unique up to a change in their order (cf. Theorem 3.3). Consequently the λ_k partial indices can be rearranged in any desired way. Namely, if (3.2) is an APW asymmetric factorization of Φ and Π is a permutation constant matrix, then by considering $\Pi^{-1}\operatorname{diag}[e_{\lambda_1}, \dots, e_{\lambda_N}]\Pi =: \widehat{\operatorname{diag}}[e_{\lambda_1}, \dots, e_{\lambda_N}]$, $\overrightarrow{\Phi_-} := \Phi_-\Pi$, and $\overleftarrow{\Phi_e} := \Pi^{-1}\Phi_e$, we obtain a second asymmetric APW factorization of Φ given by

$$\Phi = \overrightarrow{\Phi_-}\,\widehat{\operatorname{diag}}[e_{\lambda_1},\ldots,e_{\lambda_N}]\overleftarrow{\Phi_e}. \tag{3.3}$$

Besides this last fact, we have the following general result about the uniqueness of these factorizations.

Theorem 3.3. *Let $\Phi \in \mathcal{G}APW^{N\times N}$. Suppose that*

$$\Phi = \Phi_-^{(1)}D^{(1)}\Phi_e^{(1)}, \tag{3.4}$$

with $D^{(1)} = \operatorname{diag}[e_{\lambda_1},\ldots,e_{\lambda_N}]$ and $\lambda_1 \geq \cdots \geq \lambda_N$, is an APW asymmetric factorization of Φ and assume additionally that

$$\Phi = \Phi_-^{(2)}D^{(2)}\Phi_e^{(2)}, \tag{3.5}$$

with $D^{(2)} = \operatorname{diag}[e_{\mu_1},\ldots,e_{\mu_N}]$ and $\mu_1 \geq \cdots \geq \mu_N$, represents any other APW asymmetric factorization of Φ. Then

$$\begin{aligned} \Phi_-^{(2)} &= \Phi_-^{(1)}\Psi^{-1}, \\ D^{(1)} &= D^{(2)} =: D, \\ \Phi_e^{(2)} &= D^{-1}\Psi D\Phi_e^{(1)}, \end{aligned} \tag{3.6}$$

where $\Psi(x) = (\psi_{jk}(x))_{j,k=1}^N$ is a matrix function with nonzero and constant determinant, having entries which are entire functions, and

$$\psi_{jk}(z) = \begin{cases} 0, & \text{if } \lambda_j > \lambda_k, \\ c_{jk} = \text{const} \neq 0, & \text{if } \lambda_j = \lambda_k. \end{cases} \tag{3.7}$$

Proof. If Φ admits the above-mentioned two APW asymmetric factorizations, then we can write

$$\Phi = \Phi_-^{(1)}D^{(1)}\Phi_e^{(1)} = \Phi_-^{(2)}D^{(2)}\Phi_e^{(2)}, \tag{3.8}$$

which leads to

$$\left(\Phi_-^{(2)}\right)^{-1}\Phi_-^{(1)}D^{(1)} = D^{(2)}\Phi_e^{(2)}\left(\Phi_e^{(1)}\right)^{-1}. \tag{3.9}$$

We now define $\Phi_- := (\Phi_-^{(2)})^{-1}\Phi_-^{(1)}$ and $\Phi_e := \Phi_e^{(2)}(\Phi_e^{(1)})^{-1}$. Thus, we have $\Phi_- \in \mathcal{G}APW_-^{N\times N}$, $\Phi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$, and $\widetilde{\Phi}_e = \Phi_e$. From (3.9), we obtain the following identity for each (j,k) element of that matrix:

$$\left(\Phi_-\right)_{jk}(x)e^{i\lambda_k x} = e^{i\mu_j x}\left(\Phi_e\right)_{jk}(x); \tag{3.10}$$

whence

$$\left(\Phi_e\right)_{jk}(x) = \left(\Phi_-\right)_{jk}(x)e^{i(\lambda_k-\mu_j)x}, \tag{3.11}$$

and recall that Φ_e is an even function. Thus

$$(\Phi_-)_{jk}(x)e^{i(\lambda_k-\mu_j)x} = \widetilde{(\Phi_-)}_{jk}(x)e^{i(\mu_j-\lambda_k)x}, \tag{3.12}$$

and finally we infer from (3.12) that

$$(\Phi_-)_{jk}(x) = e^{2i(\mu_j-\lambda_k)x}\widetilde{(\Phi_-)}_{jk}(x). \tag{3.13}$$

If $\mu_j \geq \lambda_k$, then the element in the left-hand side of (3.13) is in the class APW^-, and the function in the right-hand side belongs to APW^+, which implies that there exist constants c_{jk} such that

$$(\Phi_-)_{jk}(x) = c_{jk} = \widetilde{(\Phi_-)}_{jk}(x)e^{2i(\mu_j-\lambda_k)x}. \tag{3.14}$$

Therefore, $c_{jk} = c_{jk}e^{2i(\mu_j-\lambda_k)x}$. Thus, if $\mu_j > \lambda_k$, we obtain $c_{jk} = 0$, and in the case where $\mu_j = \lambda_k$, we conclude that c_{jk} are nonzero constants. Altogether, we have

$$(\Phi_-)_{jk}(x) = \begin{cases} 0, & \text{if } \mu_j > \lambda_k, \\ c_{jk} = \text{const} \neq 0, & \text{if } \mu_j = \lambda_k. \end{cases} \tag{3.15}$$

Let us now assume that $\mu_j < \lambda_k$. By the hypothesis, we know that $(\Phi_-)_{jk} \in APW_-$ and so $(\Phi_-)_{jk}$ can be represented in the following form:

$$(\Phi_-)_{jk}(x) = \sum_m (a_m)_{jk}e^{i(\nu_m)_{jk}x}, \tag{3.16}$$

with $\sum_m |(a_m)_{jk}| < \infty$ for all $j,k = \overline{1,N}$. From (3.16) we directly have

$$\widetilde{(\Phi_-)}_{jk}(x) = \sum_m (a_m)_{jk}e^{-i(\nu_m)_{jk}x}. \tag{3.17}$$

Combining (3.13), (3.16), and (3.17) we obtain

$$\sum_m (a_m)_{jk}e^{i(\nu_m)_{jk}x} = e^{2i(\mu_j-\lambda_k)x}\sum_m (a_m)_{jk}e^{-i(\nu_m)_{jk}x}, \tag{3.18}$$

or equivalently

$$\sum_m (a_m)_{jk}e^{i(\nu_m)_{jk}x} = \sum_m (a_m)_{jk}e^{i(2(\mu_j-\lambda_k)-(\nu_m)_{jk})x}, \tag{3.19}$$

and this leads us to the following identity:

$$(\nu_m)_{jk} = 2(\mu_j-\lambda_k) - (\nu_m)_{jk}. \tag{3.20}$$

In conclusion, we have in the present case

$$(\nu_m)_{jk} = \mu_j - \lambda_k < 0. \tag{3.21}$$

So, for any couple (j,k), we will obtain only one real number $(\nu_m)_{jk}$, which is precisely the difference $\mu_j - \lambda_k$ and this means that in the representation of $(\Phi_-)_{jk}$ (cf. (3.16)) we need to have $(\Phi_-)_{jk}(x) = c_{jk}e^{i(\nu_m)_{jk}x}$, with some constant $c_{jk} = \text{const}$, for all $j,k = \overline{1,N}$. Thus, we arrive at the conclusion that $(\Phi_-)_{jk}$ are entire functions when $\mu_j < \lambda_k$.

We will now prove that $D^{(1)} = D^{(2)}$, that is, $\mu_j = \lambda_j$ for all j. Let us first assume that $\mu_j > \lambda_j$, for some j. Then $\mu_l > \lambda_k$ for $l \leq j \leq k$ and from (3.15) we infer that $(\Phi_-)_{lk} = 0$ for $l \leq j \leq k$. This and the Laplace expansion theorem show that $\det \Phi_-(x) = 0$ for all $x \in \mathbb{R}$, which is impossible simply because Φ_- is invertible. If for some j we would assume $\mu_j < \lambda_j$, we can repeat the above reasoning starting from (3.8) with $D^{(1)}\Phi_e^{(1)}(\Phi_e^{(2)})^{-1} = (\Phi_-^{(1)})^{-1}\Phi_-^{(2)}D^{(2)}$ instead of (3.9) and obtain once again a contradiction. Thus, $\mu_j = \lambda_j$ for all j.

Letting $\Psi := \Phi_-$ we immediately have that Ψ is an entire function. Additionally, by virtue of the equality $D^{(1)} = D^{(2)} =: D$ and (3.15), Ψ satisfies (3.7). The block-triangular structure of Ψ implies that $\det\Psi$ is a constant, and since $\Psi = (\Phi_-^{(2)})^{-1}\Phi_-^{(1)}$ this constant cannot be zero. Finally, identity (3.9) gives that $\Phi_e = D^{-1}\Psi D$, and therefore $\Phi_e^{(2)} = D^{-1}\Psi D\Phi_e^{(1)}$. This together with the identity $\Phi_-^{(2)} = \Phi_-^{(1)}\Psi^{-1}$ concludes the proof. □

4. Invertibility characterization

Let $S : X \to Y$ be a bounded linear operator acting between Banach spaces. If $\operatorname{Im} S$ is closed, the cokernel of S is defined as $\operatorname{Coker} S = Y/\operatorname{Im} S$. Then, in this case, S is said to be *properly d-normal* if $\dim \operatorname{Coker} S$ is finite and $\dim \operatorname{Ker} S$ is infinite, *properly n-normal* if $\dim \operatorname{Ker} S$ is finite and $\dim \operatorname{Coker} S$ is infinite, and *Fredholm* if both $\dim \operatorname{Ker} S$ and $\dim \operatorname{Coker} S$ are finite. An operator is called *semi-Fredholm* if it is properly n-normal, or properly d-normal, or Fredholm.

For further purposes let us also recall that two linear operators T and S are said to be *equivalent* operators if there exist two bounded invertible operators E and F such that $T = ESF$.

THEOREM 4.1. *Let Φ have an APW asymmetric factorization, with partial indices $\lambda_1,\ldots,\lambda_N$.*

(a) *If there exist positive and negative partial indices, then WH_Φ is not semi-Fredholm.*
(b) *If $\lambda_i \leq 0$, $i = \overline{1,N}$, and if for at least one i, $\lambda_i < 0$, then WH_Φ is properly d-normal and right-invertible.*
(c) *If $\lambda_i \geq 0$, $i = \overline{1,N}$, and if for at least one index i, $\lambda_i > 0$, then WH_Φ is properly n-normal and left-invertible.*
(d) *If $\lambda_i = 0$, $i = \overline{1,N}$, then WH_Φ is invertible.*

Proof. Since by hypothesis Φ admits an APW asymmetric factorization, we have

$$\Phi = \Phi_- D\Phi_e, \tag{4.1}$$

where $\Phi_- \in \mathcal{G}APW_-^{N\times N}$, $D = \operatorname{diag}[e_{\lambda_1},\ldots,e_{\lambda_N}]$, and Φ_e is an invertible even element. Without lost of generality (cf. Remark 3.2), we will assume that $\lambda_1 \geq \cdots \geq \lambda_N$. As previously observed, from (4.1) we therefore obtain the operator factorization

$$WH_\Phi = W_{\Phi_-}\ell_0 WH_D \ell_0 WH_{\Phi_e}. \tag{4.2}$$

We know that W_{Φ_-} is invertible because $\Phi_- \in \mathcal{G}APW_-^{N\times N}$ (and its inverse is given by $\ell_0 W_{\Phi_-^{-1}}\ell_0$). Additionally, WH_{Φ_e} is also invertible because Φ_e is an even element (cf. Proposition 2.2). Thus, (4.2) shows us an operator equivalence relation between WH_Φ and WH_D (note that $\ell_0 : [L^2(\mathbb{R}_+)]^N \to [L_+^2(\mathbb{R})]^N$ is invertible by $r_+ : [L_+^2(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N$). We will therefore analize the regularity properties of WH_D.

Suppose that at least some of the partial indices are greater than zero, some of them may be equal to zero, and that some of them are less than zero; for instance, $\lambda_1,\dots,\lambda_i > 0$, $\lambda_{i+1} = \cdots = \lambda_j = 0$, and $\lambda_{j+1},\dots,\lambda_N < 0$. This means that

$$\begin{aligned}\ell_0 WH_D &= \operatorname{diag}[\ell_0 WH_{e_{\lambda_1}},\dots,\ell_0 WH_{e_{\lambda_i}},\ell_0 WH_{e_{\lambda_{i+1}}},\dots,\ell_0 WH_{e_{\lambda_j}},\ell_0 WH_{e_{\lambda_{j+1}}},\dots,\ell_0 WH_{e_{\lambda_N}}]\\ &= \operatorname{diag}[\ell_0 WH_{e_{\lambda_1}},\dots,\ell_0 WH_{e_{\lambda_i}},I,\dots,I,\ell_0 W_{e_{\lambda_{j+1}}},\dots,\ell_0 W_{e_{\lambda_N}}],\end{aligned} \tag{4.3}$$

because $WH_{e_{\lambda_k}} = W_{e_{\lambda_k}}$, for $k = \overline{j+1,N}$, due to the condition $\lambda_{j+1} < 0,\dots,\lambda_N < 0$ and due to the structure of the Hankel operators (and also because $\ell_0 WH_{e_{\lambda_k}} = I$, $k = \overline{i+1,j}$ due to the condition $\lambda_{i+1} = \cdots = \lambda_j = 0$). The nonzero scalar operators in the diagonal matrix operator (4.3) are such that $WH_{e_{\lambda_1}},\dots,WH_{e_{\lambda_i}}$ are properly n-normal and left-invertible (cf. [15, Theorem 6]); $W_{e_{\lambda_{j+1}}},\dots,W_{e_{\lambda_N}}$ are d-normal and right-invertible (cf. the Gohberg-Feldman-Coburn-Douglas theorem [9, 11], [3, Theorem 2.28]). Therefore, WH_D cannot be semi-Fredholm, hence WH_Φ cannot be semi-Fredholm. This proves part (a) of the theorem.

Suppose now that $\lambda_i \le 0$, $i = \overline{1,N}$. This implies that $D \in AP_-^{N\times N}$. Since $AP_-^{N\times N} = AP^{N\times N} \cap [H_-^\infty(\mathbb{R})]^{N\times N}$, it holds that $D \in [H_-^\infty(\mathbb{R})]^{N\times N}$ and hence $WH_D = W_D$. So, in this case, WH_Φ is equivalent to W_D. If we employ again the Gohberg-Feldman-Coburn-Douglas theorem to the each one of the operators in the main diagonal of the operator W_D, it follows the assertion (b) of the theorem.

Part (c) can be deduced from the assertion (b) by passing to adjoints.

If all partial indices are zero, we have that $\ell_0 WH_D$ is just the identity operator. This, together with the operator equivalence relation (4.2) presented in the first part of the proof, leads us to the last assertion (d). □

5. Inverses representation

We now reach to our final goal: the representation of generalized/lateral/both-sided inverses of WH_Φ based on a factorization of the Fourier symbol. This result extends the scalar version obtained in [15, Theorem 7].

Let us first recall that a bounded linear operator $S^- : Y \to X$ (acting between Banach spaces) is called a reflexive generalized inverse of a bounded linear operator $S : X \to Y$ if (i) S^- is a generalized inverse (or an inner pseudoinverse) of S, that is, $SS^-S = S$; (ii) S^- is an outer pseudoinverse of S, that is, $S^-SS^- = S^-$.

Theorem 5.1. *Suppose Φ admits an APW asymmetric factorization and*

$$T = \ell_0 r_+ \mathcal{F}^{-1}\Phi_e^{-1} \cdot \mathcal{F}\ell^e r_+ \mathcal{F}^{-1}\operatorname{diag}[e_{-\lambda_1},\dots,e_{-\lambda_N}] \cdot \mathcal{F}\ell^e r_+ \mathcal{F}^{-1}\Phi_-^{-1} \cdot \mathcal{F}\ell : [L^2(\mathbb{R}_+)]^N \longrightarrow [L^2_+(\mathbb{R})]^N, \tag{5.1}$$

where Φ_e^{-1} and Φ_-^{-1} are the inverses of the corresponding factors of an APW asymmetric factorization of Φ, $\Phi = \Phi_- D \Phi_e$, and the operator $\ell : [L^2(\mathbb{R}_+)]^N \to [L^2_+(\mathbb{R})]^N$ denotes an arbitrary extension operator (i.e., T is independent of the particular choice of the extension ℓ). Then the operator T is a reflexive generalized inverse of WH_Φ and, in the following special cases, T is additionally

(a) *the right inverse of WH_Φ, if $\lambda_i \le 0$ for all $i = \overline{1,N}$;*
(b) *the left inverse of WH_Φ, if $\lambda_i \ge 0$ for all $i = \overline{1,N}$;*
(c) *the both-sided inverse of WH_Φ, if $\lambda_i = 0$ for all $i = \overline{1,N}$.*

In the case when there exist partial indices with different signs, the operator WH_Φ is not Fredholm but T is still a (reflexive) generalized inverse of WH_Φ.

Proof. We start with the cases (a), (b), and (c). Since Φ admits an *APW* asymmetric factorization, we can write

$$\Phi = \Phi_- \operatorname{diag}[e_{\lambda_1},\dots,e_{\lambda_N}]\Phi_e \tag{5.2}$$

(with the corresponding factor properties). Consequently, from (2.3), it follows that

$$WH_\Phi = r_+ A_- E A_e \ell^e r_+, \tag{5.3}$$

where $A_- = \mathcal{F}^{-1}\Phi_- \cdot \mathcal{F}$, $E = \mathcal{F}^{-1}\operatorname{diag}[e_{\lambda_1},\dots,e_{\lambda_N}] \cdot \mathcal{F}$ and $A_e = \mathcal{F}^{-1}\Phi_e \cdot \mathcal{F}$.

(a) If $\lambda_i \le 0$ for all $i = \overline{1,N}$, consider

$$\begin{aligned} WH_\Phi T &= r_+ A_- E A_e \ell^e r_+ \ell_0 r_+ A_e^{-1} \ell^e r_+ E^{-1} \ell^e r_+ A_-^{-1}\ell \\ &= r_+ A_- E A_e \ell^e r_+ A_e^{-1} \ell^e r_+ E^{-1} \ell^e r_+ A_-^{-1}\ell, \end{aligned} \tag{5.4}$$

where the term $\ell_0 r_+$ was omitted due to the fact that $r_+\ell_0 r_+ = r_+$. Since A_e^{-1} preserves the even property of its symbol, we may also drop the first $\ell^e r_+$ term in (5.4), and obtain

$$WH_\Phi T = r_+ A_- E \ell^e r_+ E^{-1} \ell^e r_+ A_-^{-1}\ell. \tag{5.5}$$

Additionally, due to the definition of E and E^{-1} in the present case ($\lambda_i \le 0$ for all $i = \overline{1,N}$), we have $\ell_0 r_+ E \ell^e r_+ E^{-1} \ell^e r_+ = \ell_0 r_+$; also because A_- is a *minus type factor* it follows $r_+ A_- = r_+ A_- \ell_0 r_+$. Therefore, from (5.5), we have

$$WH_\Phi T = r_+ A_- \ell_0 r_+ A_-^{-1}\ell = r_+\ell = I_{[L^2(\mathbb{R}_+)]^N}. \tag{5.6}$$

(b) If $\lambda_i \ge 0$ for all $i = \overline{1,N}$, we will now analyze the composition

$$TWH_\Phi = \ell_0 r_+ A_e^{-1} \ell^e r_+ E^{-1} \ell^e r_+ A_-^{-1} \ell r_+ A_- E A_e \ell^e r_+. \tag{5.7}$$

In the present case, due to the definition of E^{-1}, it follows $\ell^e r_+ E^{-1} \ell^e r_+ = \ell^e r_+ E^{-1}$. The same reasoning applies to the *minus type factor* A_-^{-1}, and therefore the equality (5.7) takes the form

$$TWH_\Phi = \ell_0 r_+ A_e^{-1} \ell^e r_+ A_e \ell^e r_+ = \ell_0 r_+ \ell^e r_+ = \ell_0 r_+ = I_{[L_+^2(\mathbb{R})]^N}, \tag{5.8}$$

where we have used the fact that $\ell^e r_+ A_e \ell^e r_+ = A_e \ell^e r_+$.

(c) From the last two cases (a) and (b), it directly follows that in the case of $\lambda_i = 0$ for all $i = \overline{1,N}$, the operator T is the both-sided inverse of WH_Φ (cf. (5.6) and (5.8)).

Let us now turn to the more general case: assume now that there exist partial indices with different signs.

In this case, we recall that the assertion about the non-Fredholm property was already provided in Theorem 4.1, assertion (a).

As about the generalized inverse, we will start by rewriting the operator E in the following new form:

$$\begin{aligned} E &= \operatorname{diag}\left[\mathcal{F}^{-1} e_{\lambda_{11}} \cdot \mathcal{F}, \ldots, \mathcal{F}^{-1} e_{\lambda_{1N}} \cdot \mathcal{F}\right] \operatorname{diag}\left[\mathcal{F}^{-1} e_{\lambda_{21}} \cdot \mathcal{F}, \ldots, \mathcal{F}^{-1} e_{\lambda_{2N}} \cdot \mathcal{F}\right] \\ &=: E_1 E_2, \end{aligned} \tag{5.9}$$

where

$$\lambda_{1j} = \begin{cases} \lambda_j, & \text{if } \lambda_j \leq 0, \\ 0, & \text{if } \lambda_j \geq 0, \end{cases} \qquad \lambda_{2j} = \begin{cases} \lambda_j, & \text{if } \lambda_j \geq 0, \\ 0, & \text{if } \lambda_j \leq 0, \end{cases} \tag{5.10}$$

for $j = \overline{1,N}$.

We will then directly compute $WH_\Phi TWH_\Phi$, in the following way:

$$WH_\Phi TWH_\Phi \tag{5.11}$$
$$= \left(r_+ A_- E_1 E_2 A_e \ell^e r_+\right)\left(\ell_0 r_+ A_e^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A_-^{-1} \ell\right)\left(r_+ A_- E_1 E_2 A_e \ell^e r_+\right) \tag{5.12}$$
$$= r_+ A_- E_1 E_2 A_e \ell^e r_+ A_e^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A_-^{-1} \ell r_+ A_- E_1 E_2 A_e \ell^e r_+ \tag{5.13}$$
$$= r_+ A_- E_1 E_2 \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ E_1 E_2 A_e \ell^e r_+ \tag{5.14}$$
$$= r_+ A_- E_1 E_2 A_e \ell^e r_+ \tag{5.15}$$
$$= WH_\Phi, \tag{5.16}$$

where in (5.14) we omitted the first term $\ell^e r_+$ of (5.13) due to the factor (invariance) property of A_e^{-1} that yields $A_e \ell^e r_+ A_e^{-1} \ell^e r_+ = \ell^e r_+$. Similarly we dropped the term ℓr_+ in $\ell^e r_+ A_-^{-1} \ell r_+ A_-$ due to a factor property of A_-^{-1}. Analogous arguments apply to the factors E_1^{-1} and E_2^{-1}. In a more detailed way: (i) if one of the factors E_1 or E_2 equals I, then it is clear that $E_2(\ell^e r_+ E_2^{-1} \ell^e r_+) E_2 = E_2(\ell^e r_+ E_2^{-1}) E_2 = E_2 \ell^e r_+$ or $\ell_0 r_+ E_1 \ell^e r_+ E_1^{-1} \ell^e r_+ E_1 = \ell_0 r_+ E_1$ holds, respectively; (ii) in the general diagonal matrix case, the situation is identical just because in each place of the main diagonal we have the last situation. This justifies the simplification made in obtaining (5.15) from (5.14).

As about the composition $TWH_\Phi T$, it follows that

$$
\begin{aligned}
TWH_\Phi T & \\
&= (\ell_0 r_+ A_e^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A_-^{-1} \ell)(r_+ A_- E_1 E_2 A_e \ell^e r_+)(\ell_0 r_+ A_e^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A_-^{-1} \ell) && (5.17)\\
&= \ell_0 r_+ A_e^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A_-^{-1} \ell r_+ A_- E_1 E_2 A_e \ell^e r_+ A_e^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A_-^{-1} \ell && (5.18)\\
&= \ell_0 r_+ A_e^{-1} \ell^e r_+ E_2^{-1} E_1^{-1} \ell^e r_+ A_-^{-1} \ell && (5.19)\\
&= T, && (5.20)
\end{aligned}
$$

where the third $\ell^e r_+$ is unnecessary in (5.18) due to the factor (invariance) property of A_e that yields $A_e \ell^e r_+ A_e^{-1} \ell^e r_+ = \ell^e r_+$, and we also can omit the term ℓr_+ in (5.18) since A_-^{-1} is a *minus* type. Additionally, a similar reasoning as above was also used for obtaining equality (5.19) since due to the definitions of E_1 and E_2 it holds $\ell^e r_+ E_1 \ell^e r_+ = \ell^e r_+ E_1$, and $\ell^e r_+ E_2^{-1} \ell^e r_+ E_2 \ell^e r_+ E_2^{-1} = \ell^e r_+ E_2^{-1}$. □

We end up by mentioning that almost all the above methods also work—without crucial changes—in the case of matrix Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols. However, a corresponding version of Theorem 3.3 for invertible $AP^{N\times N}$ elements is an open problem. This has to do with the difficulties in substituting the arguments in the part of the proof of Theorem 3.3 where some representations of APW elements are used.

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References

[1] E. L. Basor and T. Ehrhardt, *Factorization theory for a class of Toeplitz + Hankel operators*, Journal of Operator Theory **51** (2004), no. 2, 411–433.

[2] G. Bogveradze and L. P. Castro, *Wiener-Hopf plus Hankel operators on the real line with unitary and sectorial symbols*, Contemporary Mathematics **414** (2006), 77–85.

[3] A. Böttcher, Yu. I. Karlovich, and I. M. Spitkovsky, *Convolution Operators and Factorization of Almost Periodic Matrix Functions*, Operator Theory: Advances and Applications, vol. 131, Birkhäuser, Basel, 2002.

[4] L. P. Castro and F.-O. Speck, *Inversion of matrix convolution type operators with symmetry*, Portugaliae Mathematica. Nova Série **62** (2005), no. 2, 193–216.

[5] L. P. Castro, F.-O. Speck, and F. S. Teixeira, *Explicit solution of a Dirichlet-Neumann wedge diffraction problem with a strip*, Journal of Integral Equations and Applications **15** (2003), no. 4, 359–383.

[6] ———, *A direct approach to convolution type operators with symmetry*, Mathematische Nachrichten **269/270** (2004), 73–85.

[7] ———, *On a class of wedge diffraction problems posted by Erhard Meister*, Operator Theoretical Methods and Applications to Mathematical Physics, Oper. Theory Adv. Appl., vol. 147, Birkhäuser, Basel, 2004, pp. 213–240.

[8] ______, *Mixed boundary value problems for the Helmholtz equation in a quadrant*, Integral Equations and Operator Theory **56** (2006), no. 1, 1–44.

[9] L. A. Coburn and R. G. Douglas, *Translation operators on the half-line*, Proceedings of the National Academy of Sciences of the United States of America **62** (1969), no. 4, 1010–1013.

[10] T. Ehrhardt, *Invertibility theory for Toeplitz plus Hankel operators and singular integral operators with flip*, Journal of Functional Analysis **208** (2004), no. 1, 64–106.

[11] I. C. Gohberg and I. A. Fel'dman, *Wiener-Hopf integro-difference equations*, Soviet Mathematics. Doklady **9** (1968), 1312–1316 (Russian), translation from Doklady Akademii Nauk SSSR **183** (1968), 25–28.

[12] N. Karapetiants and S. Samko, *Equations with Involutive Operators*, Birkhäuser Boston, Massachusetts, 2001.

[13] V. G. Kravchenko and G. S. Litvinchuk, *Introduction to the Theory of Singular Integral Operators with Shift*, Mathematics and Its Applications, vol. 289, Kluwer Academic, Dordrecht, 1994.

[14] A. B. Lebre, E. Meister, and F. S. Teixeira, *Some results on the invertibility of Wiener-Hopf-Hankel operators*, Zeitschrift für Analysis und ihre Anwendungen **11** (1992), no. 1, 57–76.

[15] A. P. Nolasco and L. P. Castro, *Factorization of Wiener-Hopf plus Hankel operators with APW Fourier symbols*, International Journal of Pure and Applied Mathematics **14** (2004), no. 4, 537–550.

[16] S. Roch and B. Silbermann, *Algebras of Convolution Operators and Their Image in the Calkin Algebra*, Report MATH, vol. 90–05, Akademie der Wissenschaften der DDR Karl-Weierstrass-Institut für Mathematik, Berlin, 1990.

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