NEIGHBORHOODS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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By making use of the familiar concept of neighborhoods of analytic functions, the author proves several inclusion relations associated with the (n,δ)-neighborhoods of various subclasses defined by Salagean operator. Special cases of some of these inclusion relations are shown to yield known results.

1. Introduction

Let $T(j)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0;\ j \in \mathbb{N} = \{1,2,\ldots\}) \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$.

Following [5, 8], we define the (j,δ)-neighborhood of a function $f(z) \in A(j)$ by

$$N_{j,\delta}(f) = \left\{ g \in T(j) : g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k,\ \sum_{k=j+1}^{\infty} k\,|\,a_k - b_k\,| \leq \delta \right\}. \tag{1.2}$$

In particular, for the identity function $e(z) = z$, we immediately have

$$N_{j,\delta}(e) = \left\{ g \in T(j) : g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k,\ \sum_{k=j+1}^{\infty} k\,|\,b_k\,| \leq \delta \right\}. \tag{1.3}$$

The main object of this paper is to investigate the (j,δ)-neighborhoods of the following subclasses of the class $T(j)$ of normalized analytic functions in U with negative coefficients.

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For a function $f(z) \in T(j)$, we define

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = Df(z) = zf'(z), \tag{1.4}$$
$$D^n f(z) = D\big(D^{n-1} f(z)\big) \quad (n \in \mathbb{N}).$$

The differential operator D^n was introduced by Sălăgean [9]. With the help of the differential operator D^n, we say that a function $f(z) \in T(j)$ is in the class $T_j(n,m,\alpha)$ if and only if

$$\operatorname{Re}\left\{\frac{D^{n+m} f(z)}{D^n f(z)}\right\} > \alpha \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ m \in \mathbb{N}) \tag{1.5}$$

for some α $(0 \le \alpha < 1)$, and for all $z \in U$.

The operator D^{n+m} was studied by Sekine [11], Aouf et al. [2], Aouf et al. [3], and Hossen et al. [6]. We note that $T_j(0,1,\alpha) = S_j^*(\alpha)$, the class of starlike functions of order α, and $T_j(1,1,\alpha) = C_j(\alpha)$, the class of convex functions of order α (Chatterjea [4] and Srivastava et al. [12]).

2. Neighborhood for the class $T_j(n,m,\alpha)$

For the class $T_j(n,m,\alpha)$, we need the following lemma given by Sekine [11].

Lemma 2.1. *A function $f(z) \in T(j)$ is in the class $T_j(n,m,\alpha)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n (k^m - \alpha) a_k \le 1 - \alpha \tag{2.1}$$

for $n, m \in \mathbb{N}_0$ and $0 \le \alpha < 1$. The result is sharp.

Applying the above lemma, we prove the following.

Theorem 2.2. *$T_j(n,m,\alpha) \subset N_{j,\delta}(e)$, where*

$$\delta = \frac{(1-\alpha)}{(j+1)^{n-1}\big[(j+1)^m - \alpha\big]}. \tag{2.2}$$

Proof. It follows from (2.1) that if $f(z) \in T_j(n,m,\alpha)$, then

$$(j+1)^{n-1}\big[(j+1)^m - \alpha\big] \sum_{k=j+1}^{\infty} k a_k \le 1 - \alpha, \tag{2.3}$$

that is, that

$$\sum_{k=j+1}^{\infty} k a_k \le \frac{1-\alpha}{(j+1)^n\big[(j+1)^m - 1\big]} = \delta, \tag{2.4}$$

which, in view of definition (1.3), proves Theorem 2.2. □

Putting $j = 1$ in Theorem 2.2, we have the following.

Corollary 2.3. *$T_1(n,m,\alpha) \subset N_{1,\delta}(e)$, where $\delta = (1-\alpha)/2^{n-1}[2^m - \alpha]$.*

Remark 2.4. (i) Putting $n = 0$ and $m = 1$ in Theorem 2.2 and Corollary 2.3, we obtain the results obtained by Altintas and Owa [1].

(ii) Putting $n = m = 1$ in Theorem 2.2 and Corollary 2.3, we obtain the results obtained by Altintas and Owa [1].

3. Neighborhoods for the classes $R_j(n,\alpha)$ and $P_j(n,\alpha)$

A function $f(z) \in T(j)$ is said to be in the class $R_j(n,\alpha)$ if it satisfies

$$\operatorname{Re}\left(D^n f(z)\right)' > \alpha \quad (z \in U) \tag{3.1}$$

for some α $(0 \le \alpha < 1)$ and $n \in \mathbb{N}_0$. The class $R_1(n,\alpha)$ was studied by Yaguchi and Aouf [13]. We note that $R_j(0,\alpha) = R_j(\alpha)$ (Sarangi and Uralegaddi [10]).

Further, a function $f(z) \in T(j)$ is said to be a member of the class $P_j(n,\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{D^n f(z)}{z}\right\} > \alpha \quad (z \in U) \tag{3.2}$$

for some α $(0 \le \alpha < 1)$ and $z \in U$. The class $P_1(n,\alpha)$ was studied by Nunokawa and Aouf [7].

It is easy to see the following.

Lemma 3.1. *A function $f(z) \in T(j)$ is in the class $R_j(n,\alpha)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^{n+1} a_k \le 1 - \alpha. \tag{3.3}$$

The result is sharp.

Lemma 3.2. *A function $f(z) \in T(j)$ is in the class $P_j(n,\alpha)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^{n} a_k \le 1 - \alpha. \tag{3.4}$$

The result is sharp.

From the above lemmas, we see that $R_j(n,\alpha) \subset P_j(n,\alpha)$.

Theorem 3.3. *$R_j(n,\alpha) \subset N_{j,\delta}(e)$, where*

$$\delta = \frac{1-\alpha}{(j+1)^n}. \tag{3.5}$$

Proof. If $f(z) \in R_j(n,\alpha)$, we have

$$(j+1)^n \sum_{k=j+1}^{\infty} k a_k \le 1 - \alpha, \tag{3.6}$$

which gives

$$\sum_{k=j+1}^{\infty} ka_k \leq \frac{1-\alpha}{(j+1)^n} = \delta, \tag{3.7}$$

which, in view of definition (1.3), proves Theorem 3.3. □

Putting $j = 1$ in Theorem 2.2, we have the following.

Corollary 3.4. *$R_1(n,\alpha) \subset N_{1,\delta}(e)$, where $\delta = (1-\alpha)/2^n$.*

Theorem 3.5. *$P_j(n,\alpha) \subset N_{j,\delta}(e)$, where*

$$\delta = \frac{1-\alpha}{(j+1)^{n-1}}. \tag{3.8}$$

Proof. If $f(z) \in P_j(n,\alpha)$, we have

$$(j+1)^{n-1} \sum_{k=j+1}^{\infty} ka_k \leq 1-\alpha, \tag{3.9}$$

which gives

$$\sum_{k=j+1}^{\infty} ka_k \leq \frac{1-\alpha}{(j+1)^{n-1}} = \delta, \tag{3.10}$$

which, in view of definition (1.3), proves Theorem 3.5. □

Putting $j = 1$ in Theorem 3.5, we have the following.

Corollary 3.6. *$P_1(n,\alpha) \subset N_{1,\delta}(e)$, where $\delta = (1-\alpha)/2^{n-1}$.*

4. Neighborhood for the class $K_j(n,m,\alpha,\beta)$

A function $f(z) \in T(j)$ is said to be in the class $K_j(n,m,\alpha,\beta)$ if it satisfies

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1-\alpha \quad (z \in U) \tag{4.1}$$

for some α $(0 \leq \alpha < 1)$ and $g(z) \in T_j(n,m,\beta)$ $(0 \leq \beta < 1)$.

Theorem 4.1. *$N_{j,\delta}(g) \subset K_j(n,m,\alpha,\beta)$, where $g(z) \in T_j(n,m,\beta)$ and*

$$\alpha = 1 - \frac{(j+1)^{n-1}[(j+1)^m - \beta]\delta}{(j+1)^n[(j+1)^m - \beta] - 1 + \beta}, \tag{4.2}$$

where $\delta \leq (j+1) - (1-\beta)(j+1)^{1-n}[(j+1)^m - \beta]^{-1}$.

Proof. Let $f(z)$ be in $N_{j,\delta}(g)$ for $g(z) \in T_j(n,m,\beta)$. Then we know that

$$\sum_{k=j+1}^{\infty} k\,|a_k - b_k| \le \delta,$$
$$\sum_{k=j+1}^{\infty} b_k \le \frac{1-\beta}{(j+1)^n[(j+1)^m-\beta]}. \tag{4.3}$$

Thus we have

$$\begin{aligned} \left|\frac{f(z)}{g(z)} - 1\right| &\le \frac{\sum_{k=j+1}^{\infty} |a_k - b_k|}{1-\sum_{k=j+1}^{\infty} b_k} \\ &\le \frac{\delta}{j+1}\cdot\frac{(j+1)^n[(j+1)^m-\beta]}{(j+1)^n[(j+1)^m-\beta]-1+\beta} \\ &= \frac{(j+1)^{n-1}[(j+1)^m-\beta]\delta}{(j+1)^n[(j+1)^m-\beta]-1+\beta} = 1-\alpha. \end{aligned} \tag{4.4}$$

This implies that $f(z) \in K_j(n,m,\alpha,\beta)$. □

Putting $j = 1$ in Theorem 4.1, we have the following.

COROLLARY 4.2. *$N_{1,\delta}(g) \subset K_1(n,m,\alpha,\beta)$, where $g(z) \in T_1(n,m,\beta)$ and*

$$\alpha = 1 - \frac{2^{n-1}[2^m-\beta]\delta}{2^n[2^m-\beta]-1+\beta}. \tag{4.5}$$

Remark 4.3. Putting $n = 0$ and $m = 1$ in Theorem 4.1 and Corollary 4.2, we obtain the results obtained by Altintas and Owa [1].

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