EXISTENCE OF POSITIVE PERIODIC SOLUTION OF A PERIODIC COOPERATIVE MODEL WITH DELAYS AND IMPULSES

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Sufficient conditions are obtained for the existence of at least one positive periodic solution of a periodic cooperative model with delays and impulses by using Mawhin's continuation theorem of coincidence degree theory.

1. Introduction

In 1974 May [10] suggested the following cooperative species model [3]:

$$\begin{aligned} \dot{x}_1(t) &= r_1 x_1(t)\left[1 - \frac{x_1(t)}{a_1 + b_1 x_2(t)} - c_1 x_1(t)\right], \\ \dot{x}_2(t) &= r_2 x_2(t)\left[1 - \frac{x_2(t)}{a_2 + b_2 x_1(t)} - c_2 x_2(t)\right], \end{aligned} \tag{1.1}$$

where a_i, b_i, and c_i, $i = 1,2$, are positive constants. Recently, paper [11] has studied the existence of positive periodic solutions of the following system:

$$\begin{aligned} \dot{x}_1(t) &= r_1(t) x_1(t)\left[1 - \frac{x_1(t)}{a_1(t) + b_1(t) x_2(t)} - c_1(t) x_1(t)\right], \\ \dot{x}_2(t) &= r_2(t) x_2(t)\left[1 - \frac{x_2(t)}{a_2(t) + b_2(t) x_1(t)} - c_2(t) x_2(t)\right], \end{aligned} \tag{1.2}$$

where a_i, b_i, and c_i $(i = 1,2)$ are nonnegative ω-periodic continuous functions. It is well known that more realistic and interesting species models should take into account both the seasonality of the changing environment and time delays [4, 8, 9], and that the birth of many species is not continuous but occurs at fixed time intervals (some wild animals have seasonal births), in the long run; the birth of these species can be considered as an impulse to the system [1, 2, 5, 7]. To describe this phenomenon exactly, we proposed

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the following periodic cooperative species model with delays and impulses, which is a generalization of (1.1) and (1.2),

$$\begin{gathered}
\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = r_1(t)x_1(t)\left[1 - \frac{x_1(t-\tau_{11}(t))}{a_1(t)+b_1(t)x_2(t-\tau_{12}(t))} - c_1(t)x_1(t-\tau_{13}(t))\right], \quad t>0,\ t\neq t_k, \\
\Delta x_1(t_k) = -\gamma_{1k}x_1(t_k), \quad k=1,2,\ldots, \\
\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = r_2(t)x_2(t)\left[1 - \frac{x_2(t-\tau_{21}(t))}{a_2(t)+b_2(t)x_1(t-\tau_{22}(t))} - c_2(t)x_2(t-\tau_{23}(t))\right], \quad t>0,\ t\neq t_k, \\
\Delta x_2(t_k) = -\gamma_{2k}x_2(t_k), \quad k=1,2,\ldots,
\end{gathered} \tag{1.3}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ are the impulses at moment t_k and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$ and there exists a positive integer q such that $t_{k+q} = t_k + \omega$, $\gamma_{i(k+q)} = \gamma_{ik} < 1$, $k = 1,2,\ldots$, $i = 1,2$, $r_i(t)$, $a_i(t)$, $c_i(t)$, $i = 1,2$, are positive continuous ω-periodic functions, $b_i(t)$, $\tau_{ij}(t)$, $i = 1,2$, $j = 1,2,3$, are nonnegative continuous ω-periodic functions.

As usual in the theory of impulsive differential equations, at the points of discontinuity t_k of the solution $t \mapsto x_i(t)$ we assume that $x_i(t_k) \equiv x_i(t_k^-)$. It is clear that, in general, the derivatives $x_i'(t_k)$ do not exist. On the other hand, according to the first equality (1.3) there exist the limits $x_i'(t_k^{\mp})$. According to the above convention, we assume $x_i'(t_k) \equiv x_i'(t_k^-)$.

Throughout this paper, we assume that

$$\prod_{r_\sigma \le t_k < t} (1-\gamma_{ik}), \quad i = 1,2 \tag{1.4}$$

are ω-periodic functions.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions, and state some preliminary results needed in later sections. We then study, in Section 3, the existence of periodic solutions of system (1.3) by using the continuation theorem of coincidence degree theory proposed by Gaines and Mawhin [6].

2. Preliminaries

In order to obtain the existence of a positive periodic solution of system (1.3), we first make the following preparations.

Consider the impulsive system

$$\begin{gathered}
x'(t) = f(t,x(t),x(t-\tau_1(t)),\ldots,x_n(t-\tau_n(t))), \quad t\neq t_k,\ k=1,2,\ldots, \\
\Delta x(t)\big|_{t=t_k} = I_k(x(t_k^-)),
\end{gathered} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous, and f is ω-periodic with respect to its first argument; $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous, and there exists a positive integer q such that $t_{k+q} = t_k + \omega$, $I_{k+q}(x) = I_k(x)$ with $t_k \in \mathbb{R}$, $t_{k+1} > t_k$, $\lim_{k\to\infty} t_k = \infty$, $\Delta x(t)\,|_{t=t_k} = x(t_k^+) - x(t_k^-)$. For $t_k \neq 0$ $(k = 1,2,\dots)$, $[0,\omega] \cap \{t_k\} = \{t_1, t_2, \dots, t_q\}$. As we know, $\{t_k\}$ are called points of jump.

For any $\sigma \geq t_0$, let

$$r_\sigma = \min_{1\leq i\leq n} \inf_{t\geq\sigma} \{t - \tau_i(t)\} \tag{2.2}$$

and let PC_σ denote the set of functions $\phi : [r_\sigma, \sigma] \to \mathbb{R}$ which are real-valued absolutely continuous in $[t_k, t_{k+1}) \cap (r_\sigma, \sigma)$ and at t_k situated in $(r_\sigma, \sigma]$ may have discontinuity of the first kind.

Definition 2.1. For any $\sigma \geq 0$ and $\phi \in PC_\sigma$, a function $x \in ([r_\sigma, \infty), \mathbb{R})$ denoted by $x(t, \sigma, \phi)$ is said to be a solution of system (2.1) on $(\sigma, \infty]$ satisfying the initial value conditions

$$x(t) = \phi(t), \quad \phi(\sigma) > 0, \quad t \in [r_\sigma, \sigma] \tag{2.3}$$

if the following conditions are satisfied:

(i) $x(t)$ is absolutely continuous on each interval $(t_k, t_{k+1}) \subset [r_\sigma, \infty)$;

(ii) for any $t_k \in [\sigma, \infty)$, $k = 1, 2, \dots, x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$;

(iii) $x(t)$ satisfies (2.1) for almost everywhere in $[\sigma, \infty)$ and at impulsive points t_k situated in $[\sigma, \infty)$ may have discontinuity of the first kind.

Consider the following nonimpulsive delay differential system

$$\begin{aligned}
\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} &= r_1(t)y_1(t)\Bigg[1 - \frac{\prod_{0\leq t_k<t-\tau_{11}(t)}(1-\gamma_{1k})y_1(t-\tau_{11}(t))}{a_1(t) + b_1(t)\prod_{0\leq t_k<t-\tau_{12}(t)}(1-\gamma_{2k})y_2(t-\tau_{12}(t))} \\
&\qquad - c_1(t)\prod_{0\leq t_k<t-\tau_{13}(t)}(1-\gamma_{1k})y_1(t-\tau_{13}(t))\Bigg], \\
\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} &= r_2(t)y_2(t)\Bigg[1 - \frac{\prod_{0\leq t_k<t-\tau_{21}(t)}(1-\gamma_{2k})y_2(t-\tau_{21}(t))}{a_2(t) + b_2(t)\prod_{0\leq t_k<t-\tau_{22}(t)}(1-\gamma_{1k})y_1(t-\tau_{22}(t))} \\
&\qquad - c_2(t)\prod_{0\leq t_k<t-\tau_{23}(t)}(1-\gamma_{2k})y_2(t-\tau_{23}(t))\Bigg],
\end{aligned} \tag{2.4}$$

with initial condition $y_i(t) = \phi_i(t)$, $t \leq 0$, where $\phi_i(t)$ is defined as above.

In the following, we will establish a fundamental theorem that enables us to reduce the existence of solution of system (1.3) to the corresponding problem for the nonimpulsive delay differential system (2.4).

Theorem 2.2. *Assume that (1.4) holds. Then*

(i) *if $y = (y_1, y_2)^T$ is a solution of (2.4), then*

$$x = \left(\prod_{0 \le t_k < t} (1 - \gamma_{1k}) y_1, \prod_{0 \le t_k < t} (1 - \gamma_{2k}) y_2 \right)^T \tag{2.5}$$

is a solution of (1.3);

(ii) *if $x = (x_1, x_2)^T$ is a solution of (1.3), then*

$$y = \left(\prod_{0 \le t_k < t} (1 - \gamma_{1k})^{-1} x_1, \prod_{0 \le t_k < t} \ln (1 - \gamma_{2k})^{-1} x_2 \right)^T \tag{2.6}$$

is a solution of (2.4).

Proof. First, we prove (i). It is easy to see that $x_i = \prod_{0 \le t_k < t}(1 - \gamma_{ik}) y_i$, $i = 1,2$, are absolutely continuous on the interval $(t_k, t_{k+1}]$ and that for any $t \neq t_k$, $k = 1,2\dots$,

$$x = \left(\prod_{0 \le t_k < t} (1 - \gamma_{1k}) y_1, \prod_{0 \le t_k < t} (1 - \gamma_{2k}) y_2 \right)^T \tag{2.7}$$

satisfies system (1.3).

On the other hand, for every $t_k \in \{t_k\}$,

$$\begin{aligned} x_i(t_k^+) &= \lim_{t \to t_k^+} \prod_{0 \le t_j < t} (1 - \gamma_{ij}) y_i(t) = \prod_{0 \le t_j \le t_k} (1 - \gamma_{ij}) y_i(t_k), \quad i = 1,2, \\ x_i(t_k) &= \prod_{0 \le t_j < t_k} (1 - \gamma_{ij}) y_i(t_k), \quad i = 1,2. \end{aligned} \tag{2.8}$$

Thus, for every $k = 1,2,\dots$,

$$x_i(t_k^+) = (1 - \gamma_{ik}) x_i(t_k), \quad i = 1,2. \tag{2.9}$$

The proof is complete.

Next, we prove (ii). Since $x_i(t)$ is absolutely continuous on each interval $(t_k, t_{k+1}]$ and, in view of (2.9), it follows that, for any $k = 1,2,\dots$,

$$\begin{aligned} y_i(t_k^+) &= \prod_{0 \le t_j \le t_k} (1 - \gamma_{ij})^{-1} x_i(t_k^+) = \prod_{0 \le t_j < t_k} (1 - \gamma_{ij})^{-1} x_i(t_k) = y_i(t_k), \quad i = 1,2, \\ y_i(t_k^-) &= \prod_{0 \le t_j \le t_{k-1}} (1 - \gamma_{ij})^{-1} x_i(t_k^-) = y_i(t_k), \quad k = 1,2,\dots, \end{aligned} \tag{2.10}$$

which implies that $y_i(t)$, $i = 1,2$, are continuous on $[0, \infty)$. It is easy to prove that $y_i(t)$ are absolutely continuous on $[0, \infty)$. Now, one can easily check that

$$y = \left(\prod_{0 \le t_k < t} (1 - \gamma_{1k})^{-1} x_1, \prod_{0 \le t_k < t} (1 - \gamma_{2k})^{-1} x_2 \right)^T \tag{2.11}$$

is a solution of (2.9). The proof is complete. $\square$

3. Existence of periodic solutions

In this section, based on Mawhin's continuation theorem, we will study the existence of at least one periodic solution of (1.3). To do so, we will make some preparations.

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L : \operatorname{Dom} L \subset \mathbb{X} \to \mathbb{Y}$ a linear mapping, and $N : \mathbb{X} \to \mathbb{Y}$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $\mathbb{Y}$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im}(I - Q)$, it follows that mapping $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I - P)\mathbb{X} \to \operatorname{Im} L$ is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to \mathbb{X}$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Now, we introduce Mawhin's continuation theorem [6, page 40] as follows.

Lemma 3.1. *Let $\Omega \subset \mathbb{X}$ be an open bounded set and let $N : \mathbb{X} \to \mathbb{Y}$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume*

(a) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;*

(b) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$, and* $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

In what follows, we will use the following notations:

$$\bar{h} = \frac{1}{\omega}\int_0^\omega h(t)\mathrm{d}t, \qquad h^m = \min_{t\in[0,\omega]}\{h(t)\}, \qquad h^M = \max_{t\in[0,\omega]}\{h(t)\}, \tag{3.1}$$

where $h(t)$ is a periodic continuous function with period ω,

$$\begin{aligned}
A_1^s &= \sup_{t\in[0,\omega]}\left\{\prod_{0\le t_k<t}\left[\ln(1-\gamma_{1k})\right]^{-1}\right\}, & B_1^s &= \sup_{t\in[0,\omega]}\left\{\prod_{0\le t_k<t}\left[\ln(1-\gamma_{2k})\right]^{-1}\right\},\\
A_1^f &= \inf_{t\in[0,\omega]}\left\{\prod_{0\le t_k<t}\left[\ln(1-\gamma_{1k})\right]^{-1}\right\}, & B_1^f &= \inf_{t\in[0,\omega]}\left\{\prod_{0\le t_k<t}\left[\ln(1-\gamma_{2k})\right]^{-1}\right\},\\
A_2^s &= \sup_{t\in[0,\omega]}\left\{\prod_{0\le t_k<t}\ln(1-\gamma_{1k})\right\}, & B_2^s &= \sup_{t\in[0,\omega]}\left\{\prod_{0\le t_k<t}\ln(1-\gamma_{2k})\right\},\\
A_2^f &= \inf_{t\in[0,\omega]}\left\{\prod_{0\le t_k<t}\ln(1-\gamma_{1k})\right\}, & B_2^f &= \inf_{t\in[0,\omega]}\left\{\prod_{0\le t_k<t}\ln(1-\gamma_{2k})\right\}.
\end{aligned} \tag{3.2}$$

Before we proceed to state and prove our main result, we introduce a lemma which is useful in the proof of our main result.

Let

$$y_1(t) = \exp\{z_1(t)\}, \qquad y_2(t) = \exp\{z_2(t)\}, \tag{3.3}$$

then (2.4) is transformed into

$$\begin{aligned}
\frac{dz_1(t)}{dt} &= r_1(t)\Bigg[1-\frac{\prod_{0\le t_k<t-\tau_{11}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{11}(t))\}}{a_1(t)+b_1(t)\prod_{0\le t_k<t-\tau_{12}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{12}(t))\}}\\
&\qquad -c_1(t)\prod_{0\le t_k<t-\tau_{13}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{13}(t))\}\Bigg],\\
\frac{dz_2(t)}{dt} &= r_2(t)\Bigg[1-\frac{\prod_{0\le t_k<t-\tau_{21}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{21}(t))\}}{a_2(t)+b_2(t)\prod_{0\le t_k<t-\tau_{22}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{22}(t))\}}\\
&\qquad -c_2(t)\prod_{0\le t_k<t-\tau_{23}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{23}(t))\}\Bigg].
\end{aligned}\tag{3.4}$$

One can easily check that if system (3.4) has an ω-periodic solution $(z_1^*(t),z_2^*(t))^T$, then $(e^{z_1^*(t)},e^{z_2^*(t)})^T$ is a positive ω-periodic solution of system (2.4).

Lemma 3.2. *Let*

$$\begin{aligned}
f(z_1,z_2) = \Bigg(&\bar{r}_1-\frac{1}{\omega}\int_0^\omega \frac{r_1(t)\prod_{0\le t_k<t-\tau_{11}(t)}(1-\gamma_{1k})\exp\{z_1\}}{a_1(t)+b_1(t)\prod_{0\le t_k<t-\tau_{12}(t)}(1-\gamma_{2k})\exp\{z_2\}}dt\\
&-\frac{1}{\omega}\int_0^\omega c_1(t)r_1(t)\prod_{0\le t_k<t-\tau_{13}(t)}(1-\gamma_{1k})\exp\{z_1\}dt,\\
&\bar{r}_2-\frac{1}{\omega}\int_0^\omega \frac{r_2(t)\prod_{0\le t_k<t-\tau_{21}(t)}(1-\gamma_{2k})\exp\{z_2\}}{a_2(t)+b_2(t)\prod_{0\le t_k<t-\tau_{22}(t)}(1-\gamma_{1k})\exp\{z_1\}}dt\\
&-\frac{1}{\omega}\int_0^\omega c_2(t)r_2(t)\prod_{0\le t_k<t-\tau_{23}(t)}(1-\gamma_{2k})\exp\{z_2\}dt\Bigg)^T,
\end{aligned}\tag{3.5}$$

and $\Omega=\{(z_1,z_2)^T\in\mathbb{R}^2:\|(z_1,z_2)^T\|<H_0\}$, *where* r_i,a_i,b_i,c_i, $i=1,2$, *are the same as those in system (1.3) and*

$$\begin{aligned}
H_0>\max\Bigg\{&\left|\ln\frac{\bar{r}_1}{c_1^m r_1^m A_2^f}\right|,\left|\ln\frac{\bar{r}_2}{c_2^m r_2^m B_2^f}\right|,\left|\ln\frac{\bar{r}_1}{[(a_1^m)^{-1}+c_1^M]r_1^M A_2^s}\right|,\\
&\left|\ln\frac{\bar{r}_2}{[(a_2^m)^{-1}+c_2^M]r_2^M B_2^s}\right|\Bigg\}
\end{aligned}\tag{3.6}$$

is a constant. Then

$$\deg\{f,\Omega,(0,0)\}\neq 0.\tag{3.7}$$

Proof. Set

$$\Phi(z_1,z_2,\delta) = \left(\bar{r}_1 - \frac{1}{\omega}\int_0^\omega \frac{r_1(t)\prod_{0\le t_k<t-\tau_{11}(t)}(1-\gamma_{1k})\exp\{z_1\}}{a_1(t)+\delta b_1(t)\prod_{0\le t_k<t-\tau_{12}(t)}(1-\gamma_{2k})\exp\{z_2\}}\mathrm{d}t \right.$$
$$- \frac{1}{\omega}\int_0^\omega c_1(t)r_1(t)\prod_{0\le t_k<t-\tau_{13}(t)}(1-\gamma_{1k})\exp\{z_1\}\,\mathrm{d}t,$$
$$\bar{r}_2 - \frac{1}{\omega}\int_0^\omega \frac{r_2(t)\prod_{0\le t_k<t-\tau_{21}(t)}(1-\gamma_{2k})\exp\{z_2\}}{a_2(t)+\delta b_2(t)\prod_{0\le t_k<t-\tau_{22}(t)}(1-\gamma_{1k})\exp\{z_1\}}\mathrm{d}t \tag{3.8}$$
$$\left. - \frac{1}{\omega}\int_0^\omega c_2(t)r_2(t)\prod_{0\le t_k<t-\tau_{23}(t)}(1-\gamma_{2k})\exp\{z_2\}\mathrm{d}t\right)^T,$$

then it is easy to see that, for $(z_1,z_2,\delta)\in\mathbb{R}^2\times[0,1]$,

$$\bar{r}_1 - \frac{1}{\omega}\int_0^\omega \frac{r_1(t)\prod_{0\le t_k<t-\tau_{11}(t)}(1-\gamma_{1k})\exp\{z_1\}}{a_1(t)+\delta b_1(t)\prod_{0\le t_k<t-\tau_{12}(t)}(1-\gamma_{2k})\exp\{z_2\}}\mathrm{d}t$$
$$- \frac{1}{\omega}\int_0^\omega c_1(t)r_1(t)\prod_{0\le t_k<t-\tau_{13}(t)}(1-\gamma_{1k})\exp\{z_1\}\mathrm{d}t$$
$$< \bar{r}_1 - \frac{1}{\omega}\int_0^\omega c_1(t)r_1(t)\prod_{0\le t_k<t-\tau_{13}(t)}(1-\gamma_{1k})\exp\{z_1\}\mathrm{d}t$$
$$< \bar{r}_1 - c_1^m r_1^m A_2^f\exp\{z_1\} < 0 \quad \text{as } z_1 \ge \frac{H_0}{2},$$
$$\bar{r}_2 - \frac{1}{\omega}\int_0^\omega \frac{r_2(t)\prod_{0\le t_k<t-\tau_{21}(t)}(1-\gamma_{2k})\exp\{z_2\}}{a_2(t)+\delta b_2(t)\prod_{0\le t_k<t-\tau_{22}(t)}(1-\gamma_{1k})\exp\{z_1\}}\mathrm{d}t$$
$$- \frac{1}{\omega}\int_0^\omega c_2(t)r_2(t)\prod_{0\le t_k<t-\tau_{23}(t)}(1-\gamma_{2k})\exp\{z_2\}\mathrm{d}t$$
$$< \bar{r}_2 - \frac{1}{\omega}\int_0^\omega c_2(t)r_2(t)\prod_{0\le t_k<t-\tau_{23}(t)}(1-\gamma_{2k})\exp\{z_2\}\mathrm{d}t$$
$$< \bar{r}_2 - c_2^m r_2^m B_2^f\exp\{z_2\} < 0 \quad \text{as } z_2 \ge \frac{H_0}{2},$$

$$
\begin{aligned}
\bar{r}_1 - \frac{1}{\omega}\int_0^{\omega} & \frac{r_1(t)\prod_{0\le t_k<t-\tau_{11}(t)}(1-\gamma_{1k})\exp\{z_1\}}{a_1(t)+\delta b_1(t)\prod_{0\le t_k<t-\tau_{12}(t)}(1-\gamma_{2k})\exp\{z_2\}}\mathrm{d}t \\
& -\frac{1}{\omega}\int_0^{\omega} c_1(t)r_1(t)\prod_{0\le t_k<t-\tau_{13}(t)}(1-\gamma_{1k})\exp\{z_1\}\mathrm{d}t \\
& \ge \bar{r}_1 - \frac{1}{\omega}\int_0^{\omega}\frac{r_1^M A_2^s\exp\{z_1\}}{a_1^m}dt - \frac{1}{\omega}\int_0^{\omega} c_1^M r_1^M A_2^s \exp\{z_1\}\mathrm{d}t \\
& = \bar{r}_1 - \left[\left(a_1^m\right)^{-1} + c_1^M\right] r_1^M A_2^s \exp\{z_1\} > 0 \quad \text{as } z_1 \le -\frac{H_0}{2}, \\
\bar{r}_2 - \frac{1}{\omega}\int_0^{\omega} & \frac{r_2(t)\prod_{0\le t_k<t-\tau_{21}(t)}(1-\gamma_{2k})\exp\{z_2\}}{a_2(t)+\delta b_2(t)\prod_{0\le t_k<t-\tau_{22}(t)}(1-\gamma_{1k})\exp\{z_1\}}\mathrm{d}t \\
& -\frac{1}{\omega}\int_0^{\omega} c_2(t)r_2(t)\prod_{0\le t_k<t-\tau_{23}(t)}(1-\gamma_{2k})\exp\{z_2\}\mathrm{d}t \\
& \ge \bar{r}_2 - \frac{1}{\omega}\int_0^{\omega}\frac{r_2^M B_2^s\exp\{z_2\}}{a_2^m}dt - \frac{1}{\omega}\int_0^{\omega} c_2^M r_2^M B_2^s \exp\{z_2\}\mathrm{d}t \\
& = \bar{r}_2 - \left[\left(a_2^m\right)^{-1} + c_2^M\right] r_2^M B_2^s \exp\{z_2\} > 0 \quad \text{as } z_2 \le -\frac{H_0}{2}.
\end{aligned}
\tag{3.9}
$$

Therefore,

$$
\Phi(z_1,z_2,\delta) \ne 0 \quad \text{for}(z_1,z_2,\delta) \in \partial\Omega\times[0,1]. \tag{3.10}
$$

From the property of invariance under a homotopy, we have

$$
\deg\{f(z_1,z_2),\Omega,(0,0)\} = \deg\{\Phi(z_1,z_2,0),\Omega,(0,0)\}. \tag{3.11}
$$

By a straightforward computation, we find

$$
\deg\{\Phi(z_1,z_2,0),\Omega,(0,0)\} = -1 \ne 0. \tag{3.12}
$$

This completes the proof. □

We are now in a position to state and prove the existence of periodic solutions of (1.3).

Theorem 3.3. *Assume that (1.4) holds. Suppose further that*

(i) $a_1^m > A_2^s\exp\{M_1\}$;

(ii) $a_2^m > B_2^s\exp\{M_2\}$;

where $M_1 = \ln(A_1^s/c_1^m) + 2\omega r_1^M$, $M_2 = \ln(B_1^s/c_2^m) + 2\omega r_2^M$. *Then system (1.3) has at least one positive ω-periodic solution.*

Proof. According to the discussion made in Section 2, we need only to prove that the nonimpulsive delay differential system (3.4) has an ω-periodic solution. In order to use the continuation theorem of coincidence degree theory to establish the existence of ω-periodic solutions of (3.4), we take $\mathbb{X} = \mathbb{Y} = \{(z_1(t), z_2(t))^T \in C(\mathbb{R}, \mathbb{R}^2) : z_1(t+\omega) = z_1(t), z_2(t+\omega) = z_2(t)\}$, and $\|(z_1(t), z_2(t))^T\| = \max_{t\in[0,\omega]} |z_1(t)| + \max_{t\in[0,\omega]} |z_2(t)|$. With this norm, $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces. Set

$$L\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix}, \qquad P\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Q\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega z_1(t)\mathrm{d}t \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega z_2(t)\mathrm{d}t \end{bmatrix}, \qquad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{X},$$
$$N\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} lG_1(t, z_1(t), z_2(t)) \\ G_2(t, z_1(t), z_2(t)) \end{bmatrix}, \tag{3.13}$$

where

$$\begin{aligned} G_1(t, z_1(t), z_2(t)) &= r_1(t)\Bigg[1 - \frac{\prod_{0\le t_k<t-\tau_{11}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{11}(t))\}}{a_1(t) + b_1(t)\prod_{0\le t_k<t-\tau_{12}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{12}(t))\}} \\ &\qquad - c_1(t)\prod_{0\le t_k<t-\tau_{13}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{13}(t))\}\Bigg], \\ G_2(t, z_1(t), z_2(t)) &= r_2(t)\Bigg[1 - \frac{\prod_{0\le t_k<t-\tau_{21}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{21}(t))\}}{a_2(t) + b_2(t)\prod_{0\le t_k<t-\tau_{22}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{22}(t))\}} \\ &\qquad - c_2(t)\prod_{0\le t_k<t-\tau_{23}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{23}(t))\}\Bigg]. \end{aligned} \tag{3.14}$$

Obviously $\operatorname{Ker} L = \mathbb{R}^2$ and

$$\dim \operatorname{Ker} L = 2 = \operatorname{co}\dim \operatorname{Im} L. \tag{3.15}$$

So, $\operatorname{Im} L$ is closed in $\mathbb{X}$ and L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im}(I - Q). \tag{3.16}$$

Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ is given by

$$K_P\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \displaystyle\int_0^\omega z_1(s)\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \int_0^t z_1(s)\mathrm{d}s\,\mathrm{d}t \\ \displaystyle\int_0^\omega z_2(s)\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \int_0^t z_2(s)\mathrm{d}s\,\mathrm{d}t \end{bmatrix}. \tag{3.17}$$

Thus

$$QN\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\omega}\displaystyle\int_0^{\omega} G_1\left(t, z_1(t), z_2(t)\right)\mathrm{d}t \\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega} G_2\left(t, z_1(t), z_2(t)\right)\mathrm{d}t \end{bmatrix}; \tag{3.18}$$

hence

$$K_P(I-Q)N : \mathbb{X} \longrightarrow \mathbb{X},$$

$$\begin{aligned} K_P(I-Q)N\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} \displaystyle\int_0^t G_1\left(s, z_1(s), z_2(s)\right)\mathrm{d}s \\ \displaystyle\int_0^t G_2\left(s, z_1(s), z_2(s)\right)\mathrm{d}s \end{bmatrix} - \begin{bmatrix} \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\int_0^t G_1\left(s, z_1(s), z_2(s)\right)\mathrm{d}s\,\mathrm{d}t \\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\int_0^t G_2\left(s, z_1(s), z_2(s)\right)\mathrm{d}s\,\mathrm{d}t \end{bmatrix} \\ &\quad - \begin{bmatrix} \left(\dfrac{t}{\omega} - \dfrac{1}{2}\right)\displaystyle\int_0^{\omega} G_1\left(s, z_1(s), z_2(s)\right)\mathrm{d}s \\ \left(\dfrac{t}{\omega} - \dfrac{1}{2}\right)\displaystyle\int_0^{\omega} G_2\left(s, z_1(s), z_2(s)\right)\mathrm{d}s \end{bmatrix}. \end{aligned} \tag{3.19}$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Using the Arzela-Ascoli theorem, it is not difficult to show that $K_P(I-Q)N$ is compact for any open bounded set $\Omega \subset \mathbb{X}$. Moreover $QN(\bar{\Omega})$ is bounded. Thus N is L-compact on Ω with any open bounded set $\Omega \subset \mathbb{X}$. Then isomorphism J of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$ can be the identity mapping since $\operatorname{Im} Q = \operatorname{Ker} L$.

Now we reach the position to search for an appropriate open bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $Lx = \lambda N x$, $\lambda \in (0,1)$, we have

$$\begin{aligned} \frac{\mathrm{d}z_1(t)}{\mathrm{d}t} &= \lambda r_1(t)\Bigg[1 - \frac{\prod_{0 \le t_k < t-\tau_{11}(t)}\left(1-\gamma_{1k}\right)\exp\left\{z_1\left(t-\tau_{11}(t)\right)\right\}}{a_1(t) + b_1(t)\prod_{0 \le t_k < t-\tau_{12}(t)}\left(1-\gamma_{2k}\right)\exp\left\{z_2\left(t-\tau_{12}(t)\right)\right\}} \\ &\qquad - c_1(t)\prod_{0 \le t_k < t-\tau_{13}(t)}\left(1-\gamma_{1k}\right)\exp\left\{z_1\left(t-\tau_{13}(t)\right)\right\}\Bigg], \\ \frac{\mathrm{d}z_2(t)}{\mathrm{d}t} &= \lambda r_2(t)\Bigg[1 - \frac{\prod_{0 \le t_k < t-\tau_{21}(t)}\left(1-\gamma_{2k}\right)\exp\left\{z_2\left(t-\tau_{21}(t)\right)\right\}}{a_2(t) + b_2(t)\prod_{0 \le t_k < t-\tau_{22}(t)}\left(1-\gamma_{1k}\right)\exp\left\{z_1\left(t-\tau_{22}(t)\right)\right\}} \\ &\qquad - c_2(t)\prod_{0 \le t_k < t-\tau_{23}(t)}\left(1-\gamma_{2k}\right)\exp\left\{z_2\left(t-\tau_{23}(t)\right)\right\}\Bigg]. \end{aligned} \tag{3.20}$$

Suppose that $z(t)=(z_1(t),z_2(t))^T\in\mathbb{X}$ is a solution of system (3.20) for some $\lambda\in(0,1)$. Integrating (3.20) over the interval $[0,\omega]$, we obtain

$$\begin{aligned}\int_0^\omega r_1(t)\mathrm{d}t&=\int_0^\omega r_1(t)\bigg[\frac{\prod_{0\le t_k<t-\tau_{11}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{11}(t))\}}{a_1(t)+b_1(t)\prod_{0\le t_k<t-\tau_{12}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{12}(t))\}}\\&\qquad+c_1(t)\prod_{0\le t_k<t-\tau_{13}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{13}(t))\}\bigg]\mathrm{d}t,\\ \int_0^\omega r_2(t)\mathrm{d}t&=\int_0^\omega r_2(t)\bigg[\frac{\prod_{0\le t_k<t-\tau_{21}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{21}(t))\}}{a_2(t)+b_2(t)\prod_{0\le t_k<t-\tau_{22}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{22}(t))\}}\\&\qquad+c_2(t)\prod_{0\le t_k<t-\tau_{23}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{23}(t))\}\bigg]\mathrm{d}t.\end{aligned}\tag{3.21}$$

From (3.20) and (3.21), we have

$$\begin{aligned}\int_0^\omega|\dot z_1(t)|\mathrm{d}t&=\lambda\int_0^\omega\bigg|r_1(t)\bigg[1-\frac{\prod_{0\le t_k<t-\tau_{11}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{11}(t))\}}{a_1(t)+b_1(t)\prod_{0\le t_k<t-\tau_{12}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{12}(t))\}}\\&\qquad-c_1(t)\prod_{0\le t_k<t-\tau_{13}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{13}(t))\}\bigg]\bigg|\mathrm{d}t\\&\le\lambda\int_0^\omega\bigg|r_1(t)\bigg[1+\frac{\prod_{0\le t_k<t-\tau_{11}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{11}(t))\}}{a_1(t)+b_1(t)\prod_{0\le t_k<t-\tau_{12}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{12}(t))\}}\\&\qquad+c_1(t)\prod_{0\le t_k<t-\tau_{13}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{13}(t))\}\bigg]\bigg|\mathrm{d}t\\&\le2\int_0^\omega r_1(t)\mathrm{d}t\le2\omega r_1^M,\\ \int_0^\omega|\dot z_2(t)|\mathrm{d}t&=\lambda\int_0^\omega\bigg|r_2(t)\bigg[1-\frac{\prod_{0\le t_k<t-\tau_{21}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{21}(t))\}}{a_2(t)+b_2(t)\prod_{0\le t_k<t-\tau_{22}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{22}(t))\}}\\&\qquad-c_2(t)\prod_{0\le t_k<t-\tau_{23}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{23}(t))\}\bigg]\bigg|\mathrm{d}t\\&\le\lambda\int_0^\omega\bigg|r_2(t)\bigg[1+\frac{\prod_{0\le t_k<t-\tau_{21}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{21}(t))\}}{a_2(t)+b_2(t)\prod_{0\le t_k<t-\tau_{22}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{22}(t))\}}\\&\qquad+c_2(t)\prod_{0\le t_k<t-\tau_{23}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{23}(t))\}\bigg]\bigg|\mathrm{d}t\\&\le2\int_0^\omega r_2(t)\mathrm{d}t\le2\omega r_2^M.\end{aligned}\tag{3.22}$$

That is,

$$\int_0^\omega |\dot{z}_1(t)|\,\mathrm{d}t \le 2\omega r_1^M, \tag{3.23}$$

$$\int_0^\omega |\dot{z}_2(t)|\,\mathrm{d}t \le 2\omega r_2^M. \tag{3.24}$$

Let $t_1 \in [0,\omega]$ such that $z_1(t_1) = \max_{t\in[0,\omega]}\{z_1(t)\}$, since $r_1(t) > 0$; the first equation of (3.20) implies

$$\begin{aligned} &\frac{\prod_{0\le t_k<t_1-\tau_{11}(t_1)}(1-\gamma_{1k})\exp\{z_1(t_1-\tau_{11}(t_1))\}}{a_1(t_1)+b_1(t_1)\prod_{0\le t_k<t_1-\tau_{12}(t_1)}(1-\gamma_{2k})\exp\{z_2(t_1-\tau_{12}(t_1))\}} \\ &\quad + c_1(t_1)\prod_{0\le t_k<t_1-\tau_{13}(t_1)}(1-\gamma_{1k})\exp\{z_1(t_1-\tau_{13}(t_1))\} = 1; \end{aligned} \tag{3.25}$$

hence,

$$c_1(t_1)\prod_{0\le t_k<t_1-\tau_{13}(t_1)}(1-\gamma_{1k})\exp\{z_1(t_1-\tau_{13}(t_1))\} < 1; \tag{3.26}$$

moreover,

$$\prod_{0\le t_k<t_1-\tau_{13}(t_1)}(1-\gamma_{1k})\exp\{z_1(t_1-\tau_{13}(t_1))\} < \frac{1}{c_1(t_1)}, \tag{3.27}$$

then

$$z_1(t_1-\tau_{13}(t_1)) < \ln\frac{A_1^s}{c_1^m}. \tag{3.28}$$

We denote $t_1-\tau_{13}(t_1) = t_1^* + l_1\omega$, $t_1^* \in [0,\omega]$ and l_1 is an integer, then

$$z_1(t_1^*) < \ln\frac{A_1^s}{c_1^m}; \tag{3.29}$$

in view of this and (3.23), we have

$$z_1(t) = z_1(t_1^*) + \int_{t_1^*}^t \dot{z}_1(s)\mathrm{d}s \le z_1(t_1^*) + \int_0^\omega |\dot{z}_1(s)|\,\mathrm{d}s < \ln\frac{A_1^s}{c_1^m} + 2\omega r_1^M := M_1. \tag{3.30}$$

From (3.20), (3.30), and condition (i), it follows that

$$\begin{aligned} c_1(t_1) & \prod_{0\le t_k<t_1-\tau_{13}(t_1)}(1-\gamma_{1k})\exp\{z_1(t_1-\tau_{13}(t_1))\} \\ &= 1-\frac{\prod_{0\le t_k<t_1-\tau_{11}(t_1)}(1-\gamma_{1k})\exp\{z_1(t_1-\tau_{11}(t_1))\}}{a_1(t_1)+b_1(t_1)\prod_{0\le t_k<t_1-\tau_{12}(t_1)}(1-\gamma_{2k})\exp\{z_2(t_1-\tau_{12}(t_1))\}} \\ &> \frac{a_1(t_1)-\prod_{0\le t_k<t_1-\tau_{11}(t_1)}(1-\gamma_{1k})\exp\{M_1\}}{a_1(t_1)} \\ &\ge \frac{a_1(t_1)-A_2^s\exp\{M_1\}}{a_1(t_1)}>0; \end{aligned} \tag{3.31}$$

hence,

$$\prod_{0\le t_k<t_1-\tau_{13}(t_1)}(1-\gamma_{1k})\exp\{z_1(t_1-\tau_{13}(t_1))\}\ge\frac{a_1(t_1)-A_2^s\exp\{M_1\}}{a_1(t_1)c_1(t_1)} \tag{3.32}$$

or

$$z_1(t_1-\tau_{13}(t_1))>\ln\left[A_1^f\frac{a_1(t_1)-A_2^s\exp\{M_1\}}{a_1(t_1)c_1(t_1)}\right]. \tag{3.33}$$

Therefore,

$$\begin{aligned} z_1(t_1^*) &> \ln\left[A_1^f\frac{a_1(t_1)-A_2^s\exp\{M_1\}}{a_1(t_1)c_1(t_1)}\right], \\ z_1(t) &= z(t_1^*)+\int_{t_1^*}^t \dot{z}_1(s)\mathrm{d}s \\ &> \ln\left[A_1^f\frac{a_1(t_1)-A_2^s\exp\{M_1\}}{a_1(t_1)c_1(t_1)}\right]-\int_0^\omega|\dot{z}_1(t)|\mathrm{d}t \\ &> \ln\left[A_1^f\frac{a_1(t_1)-A_2^s\exp\{M_1\}}{a_1(t_1)c_1(t_1)}\right]-2\omega r_1^M:=M_1', \end{aligned} \tag{3.34}$$

that is,

$$z_1(t)>M_1'. \tag{3.35}$$

From (3.35) and (3.30), we have

$$|z_1(t)|<\max\{|M_1|,|M_1'|\}:=H_1. \tag{3.36}$$

Let $t_2 \in [0,\omega]$ such that $z_2(t_2) = \max_{t\in[0,\omega]}\{z_2(t)\}$; since $r_2(t) > 0$, the second equation of (3.20) implies that

$$\frac{\prod_{0\le t_k<t_2-\tau_{21}(t_2)}(1-\gamma_{2k})\exp\{z_2(t_2-\tau_{21}(t_2))\}}{a_2(t_2)+b_2(t_2)\prod_{0\le t_k<t_2-\tau_{22}(t_2)}(1-\gamma_{1k})\exp\{z_1(t_2-\tau_{22}(t_2))\}} + c_2(t_2)\prod_{0\le t_k<t_2-\tau_{23}(t_2)}(1-\gamma_{2k})\exp\{z_2(t_2-\tau_{23}(t_2))\} = 1; \tag{3.37}$$

thus,

$$c_2(t_2)\prod_{0\le t_k<t_2-\tau_{23}(t_2)}(1-\gamma_{2k})\exp\{z_2(t_2-\tau_{23}(t_2))\} < 1 \tag{3.38}$$

or

$$\prod_{0\le t_k<t_2-\tau_{23}(t_2)}(1-\gamma_{2k})\exp\{z_2(t_2-\tau_{23}(t_2))\} < \frac{1}{c_2(t_2)}, \tag{3.39}$$

then

$$z_2(t_2-\tau_{23}(t_2)) < \ln\frac{B_1^s}{c_2^m}. \tag{3.40}$$

We denote $t_2-\tau_{23}(t_2) = t_2^* + l_2\omega$, $t_2^* \in [0,\omega]$ and l_2 is an integer, then

$$z_2(t_2^*) < \ln\frac{B_1^s}{c_2^m}; \tag{3.41}$$

in view of this and (3.24), we have

$$z_2(t) = z_2(t_2^*) + \int_{t_2^*}^{t}\dot z_2(s)\,\mathrm{d}s < \ln\frac{B_1^s}{c_2^m} + 2\omega r_2^M = M_2. \tag{3.42}$$

It follows from (3.20), (3.42), and condition (ii) that

$$\begin{aligned}
c_2(t)&\prod_{0\le t_k<t-\tau_{23}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{23}(t))\}\\
&= 1 - \frac{\prod_{0\le t_k<t-\tau_{21}(t)}(1-\gamma_{2k})\exp\{z_2(t-\tau_{21}(t))\}}{a_2(t)+b_2(t)\prod_{0\le t_k<t-\tau_{22}(t)}(1-\gamma_{1k})\exp\{z_1(t-\tau_{22}(t))\}}\\
&> \frac{a_2(t_2)-\prod_{0\le t_k<t-\tau_{21}(t)}(1-\gamma_{2k})\exp\{M_2\}}{a_2(t_2)}\\
&\ge \frac{a_2(t_2)-B_2^s\exp\{M_2\}}{a_2(t_2)} > 0;
\end{aligned} \tag{3.43}$$

hence,

$$\prod_{0\le t_k<t-\tau_{23}(t)}(1-\gamma_{2k})z_2(t-\tau_{23}(t)) > \frac{a_2(t_2)-B_2^s\exp\{M_2\}}{a_2(t_2)c_2(t_2)}, \tag{3.44}$$

then

$$z_2(t_2-\tau_{23}(t_2)) > \ln\left[B_1^f\frac{a_2^m-B_2^s\exp\{M_2\}}{a_2^Mc_2^M}\right]. \tag{3.45}$$

Therefore,

$$z_2(t_2^*) > \ln\left[B_1^f\frac{a_2^m-B_2^s\exp\{M_2\}}{a_2^Mc_2^M}\right], \tag{3.46}$$

from this and (3.23), we obtain

$$z_2(t) = z_2(t_2^*)+\int_{t_2^*}^t \dot{z}_2(s)\mathrm{d}s > \ln\left[B_1^f\frac{a_2^m-B_2^s\exp\{M_2\}}{a_2^Mc_2^M}\right]-2\omega r_2^M := M_2', \tag{3.47}$$

that is,

$$z_2(t) > M_2'. \tag{3.48}$$

From (3.35) and (3.42) we have

$$|z_2(t)| < \max\{|M_2|,|M_2'|\} := H_2. \tag{3.49}$$

Denote $H = H_1+H_2+H_0$, clearly H is independent of λ. Now we take $\Omega = \{(z_1(t), z_2(t))^T \in \mathbb{X} : \|(z_1,z_2)^T\| < H\}$. This Ω satisfied the condition (a) of Lemma 3.1. While $(z_1(t),z_2(t))^T \in \partial\Omega\cap\mathbb{R}^2$, $(z_1,z_2)^T$ is a constant vector with $|z_1|+|z_2| = H$. Then

$$QN\begin{bmatrix} z_1\\ z_2\end{bmatrix} = \begin{bmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega G_1(t,z_1(t),z_2(t))\mathrm{d}t\\ \dfrac{1}{\omega}\displaystyle\int_0^\omega G_2(t,z_1(t),z_2(t))\mathrm{d}t\end{bmatrix} \ne 0. \tag{3.50}$$

Furthermore, take $J = I : \operatorname{Im}Q \to \operatorname{Ker}L$. By Lemma 3.2, we have

$$\deg\{JQN,\operatorname{Ker}L\cap\Omega,(0,0)^T\} \ne 0. \tag{3.51}$$

According to Lemma 3.1, system (3.4) has at least one ω-periodic solution. As a consequence, system (1.3) has at least one positive ω-periodic solution. The proof is complete. □

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