SOME RESULTS ON (δ-PRE, s)-CONTINUOUS FUNCTIONS

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We study some properties of (δ-pre, s)-continuous functions. Basic characterizations and several properties concerning (δ-pre, s)-continuous functions are studied. The general cases for the composition of functions under specific conditions which yield (δ-pre, s)-continuous functions are also studied and we obtained some results.

1. Introduction

One of the important and basic topics in general topology and several branches of mathematics which have been researched by many authors is the continuity of functions. In this paper, we study (δ-pre, s)-continuous functions as a new weaker form of continuity. In 1996, Dontchev [3] introduced contra-continuous functions, and Jafari and Noiri [10] introduced contra-precontinuous functions (in 2002). Ekici [8] studied the notion of almost contra-precontinuous functions. Recently, Ekici [7] introduced and studied the notion of (δ-pre, s)-continuous functions as a new weaker form of almost contra-precontinuous functions. The aim of this paper is to study some properties of (δ-pre, s)-continuous functions and modification of the results due to Ekici [7]. Basic characterizations concerning (δ-pre, s)-continuous functions are investigated and some results are obtained. Moreover, we obtain some properties in general cases concerning composition of functions under specific conditions, where the composition would yield a (δ-pre, s)-continuous function. Finally, if given a composition of functions, which are (δ-pre, s)-continuous, we obtain the first function in the composition, which will be (δ-pre, s)-continuous.

2. Preliminaries

Throughout this paper, all spaces X and Y (or (X,τ) and (Y,υ)) are always mean topological spaces. Let A be a subset of a space X. For a subset A of (X,τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ represent the closure and interior of A with respect to τ, respectively.

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A subset A of a space X is said to be regular open (resp., regular closed) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp., $A = \mathrm{Cl}(\mathrm{Int}(A))$). The family of all regular open (resp., regular closed) sets of X is denoted by $RO(X)$ (resp., $RC(X)$). We put $RO(X,x) = \{U \in RO(X) : x \in U\}$ and $RC(X,x) = \{F \in RC(X) : x \in F\}$.

The δ-interior [17] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by $\delta - \mathrm{Int}(A)$.

Definition 2.1. A subset A of a space X is called
(1) δ-open [17] if $A = \delta - \mathrm{Int}(A)$,
(2) preopen [13] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$,
(3) δ-preopen [16] if $A \subseteq \mathrm{Int}(\delta - \mathrm{Cl}(A))$,
(4) semi-open [12] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$.

The semiinterior [7] (resp., δ-preinterior [16]) of A is defined by the union of all semiopen (resp., δ-preopen) sets contained in A and is denoted by $s - \mathrm{Int}(A)$ (resp., $\delta - p\,\mathrm{Int}(A)$). Note that $\delta - p\,\mathrm{Int}(A) = A \cap \mathrm{Int}(\delta - \mathrm{Cl}(A))$ [7].

The complement of a δ-open (resp., preopen, δ-preopen, and semiopen) set is said to be δ-closed [17] (resp., preclosed [9], δ -preclosed [7], and semiclosed [2]). Alternatively, a subset A of (X,τ) is called δ-closed if $A = \delta - \mathrm{Cl}(A)$ [17], where $\delta - \mathrm{Cl}(A) = \{x \in X : A \cap \mathrm{Int}(\mathrm{Cl}(U)) \neq \emptyset,\ U \in \tau \text{ and } x \in U\}$, and semiclosed if $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq A$ [6]. The intersection of all semiclosed (resp., δ -preclosed) sets containing A is called the semiclosure [2] (resp., δ-preclosure [16]) of A and is denoted by $s - \mathrm{Cl}(A)$ (resp., $\delta - p\,\mathrm{Cl}(A)$). Note that $\delta - p\,\mathrm{Cl}(A) = A \cup \mathrm{Cl}(\delta - \mathrm{Int}(A))$ [7].

The family of all δ-open (resp., preopen, δ-preopen, δ -preclosed, semiopen, and semiclosed) sets of X is denoted by $\delta O(X)$ (resp., $PO(X)$, $\delta PO(X)$, $\delta PC(X)$, $SO(X)$, $SC(X)$).

The family of all δ-open (resp., preopen, δ-preopen, δ -preclosed, semiopen, and semiclosed) sets of X containing a point $x \in X$ is denoted by $\delta O(X,x)$ (resp., $PO(X,x)$, $\delta PO(X,x)$, $\delta PC(X,x)$, $SO(X,x)$, $SC(X,x)$), that is, $\delta O(X,x) = \{U \in \delta O(X) : x \in U\}$ (resp., $PO(X,x) = \{U \in PO(X) : x \in U\}$, $\delta PO(X,x) = \{U \in \delta PO(X) : x \in U\}$, $\delta PC(X,x) = \{F \in \delta PC(X) : x \in F\}$, $SO(X,x) = \{U \in SO(X) : x \in U\}$, $PC(X,x) = \{F \in PC(X) : x \in F\}$).

Definition 2.2. A function $f : X \to Y$ is said to be
(1) perfectly continuous [14] if $f^{-1}(V)$ is clopen in X for every open set V of Y;
(2) contra-continuous [3] if $f^{-1}(V)$ is closed in X for every open set V of Y;
(3) regular set-connected [4] if $f^{-1}(V)$ is clopen in X for every $V \in RO(Y)$;
(4) s-continuous [1] if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists an open set U in X containing x such that $f(U) \subseteq V$;
(5) almost s-continuous [15] if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists an open set U in X containing x such that $f(U) \subseteq s - \mathrm{Cl}(V)$;
(6) contra-precontinuous [10] if $f^{-1}(V) \in PC(X)$ for each open set V of Y;
(7) almost contra-precontinuous [8] if $f^{-1}(V) \in PC(X)$ for each $V \in RO(Y)$.

Definition 2.3. A function $f : X \to Y$ is called (δ-pre, s)-continuous [7] if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists a δ-preopen set U in X containing x such that $f(U) \subseteq \mathrm{Cl}(V)$.

Remark 2.4. The following diagram holds:

$$
\begin{tikzcd}
\text{Perfectly continuous} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Contra-continuous} \arrow[r, Rightarrow] & \text{Contra-precontinuous} \arrow[d, Rightarrow] \\
\text{Regular set-connected} \arrow[rr, Rightarrow] & & \text{Almost contra-precontinuous} \arrow[d, Rightarrow] \\
\text{Almost } s\text{-continuous} \arrow[u, Rightarrow] & & (\delta\text{-pre, } s)\text{-continuous} \\
s\text{-continuous} \arrow[u, Rightarrow] & &
\end{tikzcd}
\tag{2.1}
$$

None of these implications is reversible as shown in [4–7, 10, 14].

3. Some results

In this section, the modification of results due to Ekici [7] is investigated. Basic characterizations and some properties of $(\delta$-pre, $s)$-continuous functions are also investigated.

Lemma 3.1. *Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of δ-preopen sets in topological space X. Then $\bigcup_{\alpha\in\Delta} A_\alpha$ is δ-preopen in X.*

Proof. For each $\alpha \in \Delta$, since A_α is δ-preopen in X, we have $A_\alpha \subseteq \mathrm{Int}(\delta-\mathrm{Cl}(A_\alpha))$. Then

$$\bigcup_{\alpha\in\Delta} A_\alpha \subseteq \bigcup_{\alpha\in\Delta} \mathrm{Int}\left(\delta-\mathrm{Cl}\left(A_\alpha\right)\right) \subseteq \mathrm{Int}\left(\bigcup_{\alpha\in\Delta} \delta-\mathrm{Cl}\left(A_\alpha\right)\right) = \mathrm{Int}\left(\delta-\mathrm{Cl}\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)\right). \tag{3.1}$$

Therefore, $\bigcup_{\alpha\in\Delta} A_\alpha$ is δ-preopen in X. □

The following theorem is obtained by modification and extending the results from [7, Theorem 1].

Theorem 3.2. *The following are equivalent for a function $f: X \to Y$:*

(1) *f is $(\delta$-pre, $s)$-continuous;*
(2) *for each $x \in X$ and each $F \in SC(Y)$ noncontaining $f(x)$, there exists a δ-preclosed set K in X noncontaining x such that $f^{-1}(\mathrm{Int}(F)) \subseteq K$;*
(3) *$f^{-1}(V) \in \delta PO(X)$ for every $V \in RC(Y)$;*
(4) *$f^{-1}(V) \in \delta PC(X)$ for every $V \in RO(Y)$;*
(5) *$f^{-1}(\mathrm{Cl}(V)) \in \delta PO(X)$ for every $V \in SO(Y)$;*
(6) *$f^{-1}(\mathrm{Int}(V)) \in \delta PC(X)$ for every $V \in SC(Y)$;*
(7) *$f^{-1}(\mathrm{Int}(\mathrm{Cl}(G))) \in \delta PC(X)$ for every open subset G of Y;*
(8) *$f^{-1}(\mathrm{Cl}(\mathrm{Int}(F))) \in \delta PO(X)$ for every closed subset F of Y.*

Proof. (1)⇔(2): let F be any semiclosed set in Y not containing $f(x)$. Then $Y \setminus F$ is a semiopen set in Y containing $f(x)$. By (1), there exists a δ-preopen set U in X containing x such that $f(U) \subseteq \mathrm{Cl}(Y \setminus F)$. Hence, $U \subseteq f^{-1}(\mathrm{Cl}(Y \setminus F)) = X \setminus f^{-1}(\mathrm{Int}(F))$ and then

$f^{-1}(\mathrm{Int}(F)) \subseteq X \setminus U$. Take $K = X \setminus U$. We obtain that K is a δ -preclosed set in X noncontaining x such that $f^{-1}(\mathrm{Int}(F)) \subseteq K$.

The converse can be shown similarly.

(1)$\Leftrightarrow$(3): let $V \in RC(Y)$ and $f(x) \in V$. It follows that $V \in SO(Y)$ containing $f(x)$. By (1), there exists a δ-preopen set U_x in X containing x such that $f(U_x) \subseteq \mathrm{Cl}(V)$. Then $x \in U_x \subseteq f^{-1}(\mathrm{Cl}(V))$ and $f^{-1}(\mathrm{Cl}(V)) = \bigcup_{x \in f^{-1}(\mathrm{Cl}(V))} U_x$. This shows that $f^{-1}(\mathrm{Cl}(V)) \in \delta PO(X)$ by Lemma 3.1. Since $V \in RC(Y)$, then also $\mathrm{Cl}(V) \in RC(Y)$. So, $\mathrm{Cl}(V) = V$ and we have $f^{-1}(V) \in \delta PO(X)$.

Conversely, let $V \in RC(Y)$ and $f(x) \in V$. Then $x \in f^{-1}(V)$ and by (3), $f^{-1}(V) \in \delta PO(X)$. Since $V \in RC(Y)$, it follows that $V \in SO(Y)$ containing $f(x)$. Take $U = f^{-1}(V)$, then

$$x \in f^{-1}(V) = U, \qquad f(U) = f\left(f^{-1}(V)\right) \subseteq V \subseteq \mathrm{Cl}(V). \tag{3.2}$$

This shows that f is (δ-pre, s)-continuous.

(2)$\Leftrightarrow$(4): let $V \in RO(Y)$ and $f(x) \notin V$. It follows that $V \in SC(Y)$ is noncontaining $f(x)$. By (2), there exists a δ -preclosed set K in X noncontaining x such that $f^{-1}(\mathrm{Int}(V)) \subseteq K$. Hence, $X \setminus K$ is a δ-preopen set in X containing x, that is, $x \in X \setminus K \subseteq X \setminus f^{-1}(\mathrm{Int}(V))$. Thus, $X \setminus f^{-1}(\mathrm{Int}(V)) = \bigcup_{x \in X \setminus f^{-1}(\mathrm{Int}(V))} X \setminus K$ is a δ-preopen set in X containing x by Lemma 3.1. Therefore, $f^{-1}(\mathrm{Int}(V))$ is a δ -preclosed set in X noncontaining x. Since $V \in RO(Y)$, then $\mathrm{Int}(V) \in RO(Y)$. So $\mathrm{Int}(V) = V$ and we have $f^{-1}(V)$ is a δ -preclosed set in X noncontaining x.

Conversely, let $V \in RO(Y)$ and $f(x) \notin V$. Then $x \notin f^{-1}(V)$ and by (4), $f^{-1}(V) \in \delta PC(X)$. Since $V \in RO(Y)$, it follows that $V \in SC(Y)$ is noncontaining $f(x)$. Take $K = f^{-1}(V)$. We obtain that K is a δ-preclosed set in X noncontaining x such that $f^{-1}(\mathrm{Int}(V)) \subseteq f^{-1}(V) = K$.

(3)$\Leftrightarrow$(4): let $V \in RO(Y)$. Then $Y \setminus V \in RC(Y)$. By (3), $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V) \in \delta PO(X)$. We have $f^{-1}(V) \in \delta PC(X)$.

The converse can be obtained similarly.

(3)$\Leftrightarrow$(5): let $V \in SO(Y)$. Then $\mathrm{Cl}(V) \in RC(Y)$. By (3), $f^{-1}(\mathrm{Cl}(V)) \in \delta PO(X)$.

Conversely, let $V \in RC(Y)$. It follows that $V \in SO(Y)$. By (5), $f^{-1}(\mathrm{Cl}(V)) \in \delta PO(X)$. Since $V \in RC(Y)$, then $\mathrm{Cl}(V) \in RC(Y)$. So $\mathrm{Cl}(V) = V$ and we have $f^{-1}(V) \in \delta PO(X)$.

(4)$\Leftrightarrow$(6): let $V \in SC(Y)$. Then $\mathrm{Int}(V) \in RO(Y)$. By (4), $f^{-1}(\mathrm{Int}(V)) \in \delta PC(X)$.

Conversely, let $V \in RO(Y)$. It follows that $V \in SC(Y)$. By (6), $f^{-1}(\mathrm{Int}(V)) \in \delta PC(X)$. Since $V \in RO(Y)$, then $\mathrm{Int}(V) \in RO(Y)$. So $\mathrm{Int}(V) = V$ and we have $f^{-1}(V) \in \delta PC(X)$.

(1)$\Leftrightarrow$(5): let $V \in SO(Y)$ and $f(p) \in V$. Since f is (δ-pre, s)-continuous, there exists a $U_p \in \delta PO(X)$ containing p such that $f(U_p) \subseteq \mathrm{Cl}(V)$. Then $p \in U_p \subseteq f^{-1}(\mathrm{Cl}(V))$ and $f^{-1}(\mathrm{Cl}(V)) = \bigcup_{p \in f^{-1}(\mathrm{Cl}(V))} U_p$. This shows that $f^{-1}(\mathrm{Cl}(V)) \in \delta PO(X)$ by Lemma 3.1.

Conversely, let $V \in SO(Y)$ and $f(p) \in V$. Then $p \in f^{-1}(V)$ and by (5), $f^{-1}(\mathrm{Cl}(V)) \in \delta PO(X)$. Let $U = f^{-1}(\mathrm{Cl}(V))$, then

$$p \in f^{-1}(V) \subseteq U, \qquad f(U) = f\left(f^{-1}\left(\mathrm{Cl}(V)\right)\right) \subseteq \mathrm{Cl}(V). \tag{3.3}$$

This shows that f is (δ-pre, s)-continuous.

(2)$\Leftrightarrow$(6): let $V \in SC(Y)$ be a noncontaining $f(x)$. By (2), there exists a δ -preclosed set K in X noncontaining x such that $f^{-1}(\mathrm{Int}(V)) \subseteq K$. Hence, $X \setminus K$ is a δ-preopen set in X containing x, that is, $x \in X \setminus K \subseteq X \setminus f^{-1}(\mathrm{Int}(V))$. Thus, $X \setminus f^{-1}(\mathrm{Int}(V)) = \bigcup_{x\in X\setminus f^{-1}(\mathrm{Int}(V))} X \setminus K$ is a δ-preopen set in X containing x by Lemma 3.1. Therefore, $f^{-1}(\mathrm{Int}(V))$ is a δ -preclosed set in X noncontaining x.

Conversely, let $V \in SC(Y)$ and $f(x) \notin V$. Then $x \notin f^{-1}(V)$ and by (6), $f^{-1}(V) \in \delta PC(X)$. Take $K = f^{-1}(V)$. We obtain that K is a δ -preclosed set in X noncontaining x such that $f^{-1}(\mathrm{Int}(V)) \subseteq f^{-1}(V) = K$.

(5)$\Leftrightarrow$(6): let $V \in SC(Y)$. Then $Y \setminus V \in SO(Y)$. By (5), $f^{-1}(\mathrm{Cl}(Y \setminus V)) = X \setminus f^{-1}(\mathrm{Int}(V)) \in \delta PO(X)$. We have $f^{-1}(\mathrm{Int}(V)) \in \delta PC(X)$.

The converse can be obtained similarly.

(6)$\Leftrightarrow$(7): let G be any open subset of Y. Since $\mathrm{Int}(\mathrm{Cl}(G))$ is regular open, then it is semiclosed in Y. By (6), it follows that $f^{-1}(\mathrm{Int}(\mathrm{Int}(\mathrm{Cl}(G))))$ is δ -preclosed, that is, $f^{-1}(\mathrm{Int}(\mathrm{Cl}(G))) \in \delta PC(X)$.

Conversely, let $V \in SC(Y)$. Then $\mathrm{Int}(V) \in RO(Y)$ and $\mathrm{Int}(V)$ is an open subset of Y. Hence, by (7), $f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(V))))$ is δ -preclosed. Since $\mathrm{Int}(V) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(V)))$, it follows that $f^{-1}(\mathrm{Int}(V)) \in \delta PC(X)$.

(5)$\Leftrightarrow$(8): let F be any closed subset of Y. Since $\mathrm{Cl}(\mathrm{Int}(F))$ is regular closed, then it is semiopen in Y. By (5), it follows that $f^{-1}(\mathrm{Cl}(\mathrm{Cl}(\mathrm{Int}(F))))$ is δ-preopen, that is, $f^{-1}(\mathrm{Cl}(\mathrm{Int}(F))) \in \delta PO(X)$.

Conversely, let $V \in SO(Y)$. Then $\mathrm{Cl}(V) \in RC(Y)$ and $\mathrm{Cl}(V)$ is a closed subset of Y. Hence, by (8), $f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V))))$ is δ-preopen. Since $\mathrm{Cl}(V) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V)))$, it follows that $f^{-1}(\mathrm{Cl}(V)) \in \delta PO(X)$.

(7)$\Leftrightarrow$(8): this is obvious, by taking complement, respectively.

(3)$\Leftrightarrow$(8): let F be any closed subset of Y. Since $\mathrm{Cl}(\mathrm{Int}(F))$ is regular closed subset of Y, then by (3), it follows that $f^{-1}(\mathrm{Cl}(\mathrm{Int}(F))) \in \delta PO(X)$.

Conversely, let $V \in RC(Y)$. Then V is a closed subset of Y. By (8),

$$f^{-1}\big(\mathrm{Cl}\big(\mathrm{Int}(V)\big)\big) \in \delta PO(X). \tag{3.4}$$

Since $V = \mathrm{Cl}(\mathrm{Int}(V))$, it follows that $f^{-1}(V) \in \delta PO(X)$.

(4)$\Leftrightarrow$(7): see [7, Theorem1], (4)$\Leftrightarrow$(6).

(7)$\Leftrightarrow$(2): let $x \in X$ and $F \in SC(Y)$ be noncontaining $f(x)$. Then $\mathrm{Int}(F) \in RO(Y)$ and $\mathrm{Int}(F)$ is an open subset of Y. By (7), it follows that

$$f^{-1}\big(\mathrm{Int}\big(\mathrm{Cl}(\mathrm{Int}(F))\big)\big) \tag{3.5}$$

is a δ -preclosed set in X noncontaining x. Since $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F))) = \mathrm{Int}(F)$, it follows that $f^{-1}(\mathrm{Int}(F))$ is a δ -preclosed set in X noncontaining x. Let $K = f^{-1}(\mathrm{Int}(F))$. We obtain that K is a δ -preclosed set in X noncontaining x such that $f^{-1}(\mathrm{Int}(F)) \subseteq K$.

Conversely, let $x \in X$ and let G be any open subset of Y noncontaining $f(x)$. Since $\mathrm{Int}(\mathrm{Cl}(G))$ is regular open, then it is semiclosed in Y noncontaining $f(x)$. By (2), there exists a δ -preclosed set K in X noncontaining x such that

$$f^{-1}\big(\mathrm{Int}\big(\mathrm{Int}\big(\mathrm{Cl}(G)\big)\big)\big) \subseteq K, \tag{3.6}$$

that is, $f^{-1}(\mathrm{Int}(\mathrm{Cl}(G))) \subseteq K$. Hence, $X \setminus K$ is a δ-preopen set in X containing x, that is, $x \in X \setminus K \subseteq X \setminus f^{-1}(\mathrm{Int}(\mathrm{Cl}(G)))$. Thus, $X \setminus f^{-1}(\mathrm{Int}(\mathrm{Cl}(G))) = \bigcup_{x\in X\setminus f^{-1}(\mathrm{Int}(\mathrm{Cl}(G)))} X \setminus$

K is a δ-preopen set in X containing x by Lemma 3.1. Therefore, $f^{-1}(\mathrm{Int}(\mathrm{Cl}(G)))$ is a δ -preclosed set in X noncontaining x.

(8)$\Leftrightarrow$(1): let $x \in X$ and $V \in SO(Y, f(x))$. Then $\mathrm{Cl}(V) \in RC(Y)$ and clearly $\mathrm{Cl}(V)$ is a closed subset of Y. By (8), it follows that $f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V))))$ is a δ-preopen set in X containing x. Since $\mathrm{Cl}(V) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V)))$, it follows that $f^{-1}(\mathrm{Cl}(V))$ is a δ-preopen set in X containing x. Let $U = f^{-1}(\mathrm{Cl}(V))$, then $f(U) \subseteq \mathrm{Cl}(V)$. This implies that f is (δ-pre, s)-continuous.

Conversely, let $x \in X$ and let F be a closed subset of Y containing $f(x)$. Since $\mathrm{Cl}(\mathrm{Int}(F))$ is regular closed, then it is semiopen in Y containing $f(x)$. By (1), there exists a δ-preopen set U_x in X containing x such that

$$f(U_x) \subseteq \mathrm{Cl}(\mathrm{Cl}(\mathrm{Int}(F))) = \mathrm{Cl}(\mathrm{Int}(F)). \tag{3.7}$$

Hence, $x \in U_x \subseteq f^{-1}(\mathrm{Cl}(\mathrm{Int}(F)))$ and $f^{-1}(\mathrm{Cl}(\mathrm{Int}(F))) = \bigcup_{x\in f^{-1}(\mathrm{Cl}(\mathrm{Int}(F)))} U_x$. This shows that $f^{-1}(\mathrm{Cl}(\mathrm{Int}(F))) \in \delta PO(X)$ by Lemma 3.1. □

Remark 3.3. It is known in [7, Theorem 1] that (1), (3), (4), (7), and (8) are all equivalent. Therefore, (1), (2), (5), and (6) are valuable in Theorem 3.2.

The following example shows that (δ-pre, s)-continuous function does not imply almost contraprecontinuous function.

Example 3.4. Let $X = \{a,b,c\}$, let $\sigma = \{X,\emptyset,\{a\}\}$, and let $\tau = \{X,\emptyset,\{a\},\{b,c\}\}$. Then the identity function $f : (X,\sigma) \to (X,\tau)$ is (δ -pre, s)-continuous but not almost contraprecontinuous, since $\{b,c\}$ is regular closed in (X,τ) but $f^{-1}(\{b,c\}) = \{b,c\}$ is not preopen in (X,σ), that is, $\{b,c\} \nsubseteq \mathrm{Int}(\mathrm{Cl}(\{b,c\})) = \mathrm{Int}(\{b,c\}) = \emptyset$.

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is called the graph of f. The following theorems are obtained in [7] and proved by using [7, Theorem 1(3)]. We prove here by using different technique, that is, by using Theorem 3.2(5) in this paper.

THEOREM 3.5. *Let $f : X \to Y$ be a function and let $g : X \to X \times Y$ be the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is (δ-pre, s)-continuous, then f is (δ-pre, s)-continuous.*

Proof. Let $W \in SO(Y)$, then $X \times W \subseteq X \times \mathrm{Cl}(\mathrm{Int}(W)) = \mathrm{Cl}(\mathrm{Int}(X)) \times \mathrm{Cl}(\mathrm{Int}(W)) = \mathrm{Cl}(\mathrm{Int}(X \times W))$. Hence, $X \times W \in SO(X \times Y)$. Since g is (δ-pre, s)-continuous, it follows from Theorem 3.2(5) that

$$f^{-1}(\mathrm{Cl}(W)) = g^{-1}(\mathrm{Cl}(X \times W)) \in \delta PO(X). \tag{3.8}$$

Thus, f is (δ-pre, s)-continuous by Theorem 3.2. □

LEMMA 3.6 (see [16]). *Let A and X_0 be subsets of a space (X,τ). If $A \in \delta PO(X)$ and $X_0 \in \delta O(X)$, then $A \cap X_0 \in \delta PO(X_0)$.*

Lemma 3.7 (see [16]). *Let $A \subseteq X_0 \subseteq X$. If $X_0 \in \delta O(X)$ and $A \in \delta PO(X_0)$, then $A \in \delta PO(X)$.*

Theorem 3.8. *If $f : X \to Y$ is a (δ-pre, s)-continuous function and A is any δ-open subset of X, then the restriction $f|_A : A \to Y$ is (δ-pre, s)-continuous.*

Proof. Let $G \in SO(Y)$. Since f is (δ-pre, s)-continuous, then $f^{-1}(\mathrm{Cl}(G)) \in \delta PO(X)$ by Theorem 3.2(5). Since A is δ-open subset of X, it follows from Lemma 3.6 that $(f|_A)^{-1}(\mathrm{Cl}(G)) = A \cap f^{-1}(\mathrm{Cl}(G)) \in \delta PO(A)$. Therefore, $f|_A$ is a (δ-pre, s)-continuous function by Theorem 3.2. □

Theorem 3.9. *Let $f : X \to Y$ be a function and let $\{U_\alpha : \alpha \in \Delta\}$ be a δ-open cover of X. If for each $\alpha \in \Delta$, $f|_{U_\alpha}$ is (δ-pre, s)-continuous, then f is a (δ-pre, s)-continuous function.*

Proof. Let $V \in SO(Y)$. Since $f|_{U_\alpha}$ is (δ-pre, s)-continuous for each $\alpha \in \Delta$, $(f|_{U_\alpha})^{-1}(\mathrm{Cl}(V)) \in \delta PO(U_\alpha)$ by Theorem 3.2(5). Since $U_\alpha \in \delta O(X)$, by Lemma 3.7, $(f|_{U_\alpha})^{-1}(\mathrm{Cl}(V)) \in \delta PO(X)$ for each $\alpha \in \Delta$. Then

$$f^{-1}(\mathrm{Cl}(V)) = \bigcup_{\alpha \in \Delta} \left[\left(f|_{U_\alpha} \right)^{-1} (\mathrm{Cl}(V)) \right] \in \delta PO(X) \tag{3.9}$$

by Lemma 3.1. This gives that f is a (δ-pre, s)-continuous function. □

Theorem 3.10. *Let $f : X \to Y$ be a function. If there exists $U \in \delta O(X)$ and the restriction of f to U is a (δ-pre, s)-continuous, then f is (δ-pre, s)-continuous function.*

Proof. Suppose that $x \in X$ and $F \in SO(Y, f(x))$. Since $f|_U$ is (δ-pre, s)-continuous, there exists a $V \in \delta PO(U, x)$ such that $f(V) = (f|_U)(V) \subseteq \mathrm{Cl}(F)$ because $V \subseteq U$. Since $U \in \delta O(X, x)$, it follows from Lemma 3.7 that $V \in \delta PO(X, x)$. Since $x \in X$ is arbitrary, this shows that f is (δ-pre, s)-continuous function. □

Definition 3.11. A function $f : X \to Y$ is said to be

(1) *θ-irresolute* [11] if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists $U \in SO(X, x)$ such that $f(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$,

(2) *δ-preirresolute* [7] if for each $x \in X$ and each $V \in \delta PO(Y, f(x))$, there exists a δ-preopen set U in X containing x such that $f(U) \subseteq V$.

In [7, Theorem 10], Ekici has proved that composition of two functions with specific condition would yield the (δ-pre, s)-continuous function. For the composition of three functions, we have the following results.

Proposition 3.12. *Let $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ be functions. Then the following properties hold.*

(1) *If f and g are δ-preirresolute, and h is (δ -pre, s)-continuous, then $h \circ g \circ f : X \to W$ is (δ-pre, s)-continuous.*

(2) *If f is (δ-pre, s)-continuous, and g and h are θ-irresolute, then $h \circ g \circ f : X \to W$ is (δ-pre, s)-continuous.*

(3) *If f is δ-preirresolute, g is (δ-pre, s)-continuous, and h is θ-irresolute, then $h \circ g \circ f : X \to W$ is (δ-pre, s)-continuous.*

Proof. (1) Let $x \in X$ and $V \in SO(W,(h \circ g \circ f)(x))$. Since h is (δ-pre, s)-continuous, there exists a δ-preopen set G in Z containing $(g \circ f)(x)$ such that $h(G) \subseteq \mathrm{Cl}(V)$. Since g is δ-preirresolute, there exists a δ-preopen set F in Y containing $f(x)$ such that $g(F) \subseteq G$. Since f is δ-preirresolute, there exists a δ-preopen set U in X containing x such that $f(U) \subseteq F$. This shows that $(h \circ g \circ f)(U) \subseteq (h \circ g)(F) \subseteq h(G) \subseteq \mathrm{Cl}(V)$. Therefore, $h \circ g \circ f$ is (δ-pre, s)-continuous.

(2) Let $x \in X$ and $V \in SO(W,(h \circ g \circ f)(x))$. Since h is θ-irresolute, there exists $G \in SO(Z,(g \circ f)(x))$ such that $h(\mathrm{Cl}(G)) \subseteq \mathrm{Cl}(V)$. Since g is θ-irresolute, there exists $F \in SO(Y, f(x))$ such that $g(\mathrm{Cl}(F)) \subseteq \mathrm{Cl}(G)$. Since f is (δ-pre, s)-continuous, there exists a δ-preopen set U in X containing x such that $f(U) \subseteq \mathrm{Cl}(F)$. This shows that $(h \circ g \circ f)(U) \subseteq (h \circ g)(\mathrm{Cl}(F)) \subseteq h(\mathrm{Cl}(G)) \subseteq \mathrm{Cl}(V)$. Therefore, $h \circ g \circ f$ is (δ-pre, s)-continuous.

(3) Let $x \in X$ and $V \in SO(W,(h \circ g \circ f)(x))$. Since h is θ-irresolute, there exists $G \in SO(Z,(g \circ f)(x))$ such that $h(\mathrm{Cl}(G)) \subseteq \mathrm{Cl}(V)$. Since g is (δ-pre, s)-continuous, there exists a δ-preopen set F in Y containing $f(x)$ such that $g(F) \subseteq \mathrm{Cl}(G)$. Since f is δ-preirresolute, there exists a δ-preopen set U in X containing x such that $f(U) \subseteq F$. This shows that $(h \circ g \circ f)(U) \subseteq (h \circ g)(F) \subseteq h(\mathrm{Cl}(G)) \subseteq \mathrm{Cl}(V)$. Therefore, $h \circ g \circ f$ is (δ-pre, s)-continuous. □

Next, we obtained Corollaries 3.13 and 3.14 as general cases, obvious from [7, Theorem 10] and Propositions 3.12(1) and 3.12(2), by repeating application of δ-preirresolute and θ-irresolute functions, respectively.

Corollary 3.13. *If $f_i : X_i \to X_{i+1}$, $i = 1,2,\dots,n$, are δ-preirresolute functions and $g : X_{n+1} \to Y$ is (δ-pre, s)-continuous, then $g \circ f_n \circ \cdots \circ f_2 \circ f_1 : X_1 \to Y$ is (δ-pre, s)-continuous.*

Corollary 3.14. *If $f : X \to Y_1$ is (δ-pre, s)-continuous and $g_i : Y_i \to Y_{i+1}$, $i = 1,2,\dots,n$, are θ-irresolute functions, then $g_n \circ \cdots \circ g_2 \circ g_1 \circ f : X \to Y_{n+1}$ is (δ-pre, s)-continuous.*

Observe that, in Corollary 3.13, the (δ-pre, s)-continuous function lies at the beginning of the composition function, while in Corollary 3.14, the (δ-pre, s)-continuous function lies at the end. How about, if the (δ-pre, s)-continuous function lies inside of the composition function? We have the following results.

Proposition 3.15. *Let $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$, and $p : W \to V$ be functions. Then the following properties hold.*

(1) *If f and g are δ-preirresolute, h is (δ-pre, s)-continuous, and p is θ-irresolute, then $p \circ h \circ g \circ f : X \to V$ is (δ-pre, s)-continuous.*

(2) *If f is δ-preirresolute, g is (δ-pre, s)-continuous, and h and p are θ-irresolute, then $p \circ h \circ g \circ f : X \to V$ is (δ-pre, s)-continuous.*

Proof. (1) Let $x \in X$ and $G \in SO(V,(p \circ h \circ g \circ f)(x))$. Since p is θ-irresolute, there exists $F \in SO(W,(h \circ g \circ f)(x))$ such that $p(\mathrm{Cl}(F)) \subseteq \mathrm{Cl}(G)$. Since h is (δ-pre, s)-continuous, there exists a δ-preopen set N in Z containing $(g \circ f)(x)$ such that $h(N) \subseteq \mathrm{Cl}(F)$. Since g is δ-preirresolute, there exists a δ-preopen set M in Y containing $f(x)$ such that $g(M) \subseteq N$. Since f is δ-preirresolute, there exists a δ-preopen set U in X containing x such that

$f(U) \subseteq M$. This shows that $(p \circ h \circ g \circ f)(U) \subseteq (p \circ h \circ g)(M) \subseteq (p \circ h)(N) \subseteq p(\mathrm{Cl}(F)) \subseteq \mathrm{Cl}(G)$. Therefore, $p \circ h \circ g \circ f$ is (δ-pre, s)-continuous.

(2) Let $x \in X$ and $G \in SO(V, (p \circ h \circ g \circ f)(x))$. Since p is θ-irresolute, there exists $F \in SO(W, (h \circ g \circ f)(x))$ such that $p(\mathrm{Cl}(F)) \subseteq \mathrm{Cl}(G)$. Since h is θ-irresolute, there exists $N \in SO(Z, (g \circ f)(x))$ such that $h(\mathrm{Cl}(N)) \subseteq \mathrm{Cl}(F)$. Since g is (δ-pre, s)-continuous, there exists a δ-preopen set M in Y containing $f(x)$ such that $g(M) \subseteq \mathrm{Cl}(N)$. Since f is δ-preirresolute, there exists a δ-preopen set U in X containing x such that $f(U) \subseteq M$. This shows that $(p \circ h \circ g \circ f)(U) \subseteq (p \circ h \circ g)(M) \subseteq (p \circ h)(\mathrm{Cl}(N)) \subseteq p(\mathrm{Cl}(F)) \subseteq \mathrm{Cl}(G)$. Therefore, $p \circ h \circ g \circ f$ is (δ-pre, s)-continuous. □

Clearly, from Propositions 3.12(3) and 3.15, we obtain the following corollary.

Corollary 3.16. *If for $i = 1, 2, \dots, n$, $f_i : X_i \to X_{i+1}$ are δ-preirresolute functions, $g : X_{i+1} \to Y_1$ is (δ-pre, s)-continuous, and $h_j : Y_j \to Y_{j+1}$, $j = 1, 2, \dots, m$, are θ-irresolute functions, then $h_m \circ \cdots \circ h_1 \circ g \circ f_n \circ \cdots \circ f_1 : X_1 \to Y_{m+1}$ is (δ-pre, s)-continuous.*

Definition 3.17. A function $f : X \to Y$ is called δ-preopen [7] if the image of each δ-preopen set is δ-preopen.

In [7, Theorem 11], Ekici has also proved that, given a composition of two functions with specific conditions where the (δ-pre, s)-continuous function would be yield, the first function in the composition is (δ-pre, s)-continuous. For the composition of three functions, we give the following proposition.

Proposition 3.18. *If $f : X \to Y$ and $g : Y \to Z$ are surjective δ-preopen functions and $h : Z \to W$ is a function such that $h \circ g \circ f : X \to W$ is (δ-pre, s)-continuous, then h is (δ-pre, s)-continuous.*

Proof. Suppose that x, y, and z are three points in X, Y, and Z, respectively, such that $f(x) = y$ and $g(y) = z$. Let $V \in SO(W, (h \circ g \circ f)(x))$. Since $h \circ g \circ f$ is (δ-pre, s)-continuous, there exists a δ-preopen set U in X containing x such that $(h \circ g \circ f)(U) \subseteq \mathrm{Cl}(V)$. Since f is δ-preopen, $f(U)$ is a δ-preopen set in Y containing y such that $(h \circ g)(f(U)) \subseteq \mathrm{Cl}(V)$. Since g is also δ-preopen, $g(f(U))$ is a δ-preopen set in Z containing z such that $h(g(f(U))) \subseteq \mathrm{Cl}(V)$. This implies that h is (δ-pre, s)-continuous. □

As in [7, Corollary 1], we obtained the following corollary.

Corollary 3.19. *Let $f : X \to Y$ and $g : Y \to Z$ be surjective, δ-preirresolute, and δ-preopen functions and let $h : Z \to W$ be a function. Then, $h \circ g \circ f : X \to W$ is (δ-pre, s)-continuous if and only if h is (δ-pre, s)-continuous.*

Proof. It can be obtained from Propositions 3.12(1) and 3.18. □

The following corollaries are considered as general cases obtained from the above discussions.

Corollary 3.20. *If $f_i : X_i \to X_{i+1}$, $i = 1,2,\dots,n$, are surjective δ-preopen functions and $g : X_{n+1} \to Y$ is a function such that $g \circ f_n \circ \cdots \circ f_2 \circ f_1 : X_1 \to Y$ is (δ-pre, s)-continuous, then g is (δ-pre, s)-continuous.*

The proof of Corollary 3.20 is obvious from [7, Theorem 11] and Proposition 3.18.

Corollary 3.21. *Let $f_i : X_i \to X_{i+1}$, $i = 1,2,\dots,n$ be surjective, δ-preirresolute, and δ-preopen functions and let $g : X_{n+1} \to Y$ be a function. Then $g \circ f_n \circ \cdots \circ f_2 \circ f_1 : X_1 \to Y$ is (δ-pre, s)-continuous if and only if g is (δ-pre, s)-continuous.*

The proof of Corollary 3.21 can be obtained from Corollaries 3.13 and 3.20.

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