C_0-SEMIGROUPS OF LINEAR OPERATORS ON SOME ULTRAMETRIC BANACH SPACES

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C_0-semigroups of linear operators play a crucial role in the solvability of evolution equations in the classical context. This paper is concerned with a brief conceptualization of C_0-semigroups on (ultrametric) free Banach spaces $\mathbb{E}$. In contrast with the classical setting, the parameter of a given C_0-semigroup belongs to a clopen ball Ω_r of the ground field $\mathbb{K}$. As an illustration, we will discuss the solvability of some homogeneous p-adic differential equations.

1. Introduction

Let $(\mathbb{K},+,\cdot,|\cdot|)$ be a (complete) ultrametric-valued field and let Ω_r be the closed ball of $\mathbb{K}$ centered at 0 with radius $r > 0$, that is, $\Omega_r = \{\kappa \in \mathbb{K} : |\kappa| \leq r\}$. It is well known that Ω_r is also open in $\mathbb{K}$; for this reason, Ω_r is called a *clopen*. Recall that each ball Ω_r is an additive subgroup of $\mathbb{K}$. From now on, the radius r of the ball Ω_r will be suitably chosen so that the series, which defines the p-adic exponential, converges. Indeed, let $\mathbb{K} = \mathbb{Q}_p$ be the field of p-adic numbers ($p \geq 2$ being a prime) equipped with the p-adic valuation $|\cdot|$ and let $\Omega_r = \{q \in \mathbb{Q}_p : |q| \leq r\}$. In contrast with the classical context, the p-adic exponential given by

$$e^q := \sum_{n\geq 0} \frac{q^n}{n!} \tag{1.1}$$

is not always well defined and analytic for each $q \in \mathbb{Q}_p$. However, it does converge for all $q \in \mathbb{Z}_p$ such that $|q| < r = p^{-1/(p-1)}$, where $\mathbb{Z}_p$ denotes the ring of p-adic integers. (The ring of p-adic integers $\mathbb{Z}_p$ is the unit ball of $\mathbb{Q}_p$ centered at zero, that is, the set of all $x \in \mathbb{Q}_p$ such that $|x| \leq 1$, where $|\cdot|$ is the p-adic valuation of $\mathbb{Q}_p$). For more on these and related issues, we refer the reader to [1, 7, 8, 18].

In this paper, we provide the reader with a brief conceptualization of ultrametric counterparts of C_0-*semigroups* in connection with the formalism of linear operators on free Banach and non-Archimedean Hilbert spaces, recently developed in [2–6].

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The present paper is mainly motivated by the solvability of p-adic differential and partial differential equations [9, 11–13, 18] as strong (mild) solutions to the Cauchy problem related to several classes of differential and partial differential equations in the classical setting which can be expressed through C_0-semigroups, see, for example, [15, 16].

As for the p-adic exponential defined above, here, the parameter of a given C_0-semigroup belongs to one of those clopen balls Ω_r whose radius r will be suitably chosen. Let us mention however that the idea of considering one-parameter families of bounded linear operators on balls such as Ω_r was first initiated in [1] for bounded symmetric operators defined on $\mathbb{Q}_p$. Here, we consider those issues within the framework of free Banach and non-Archimedean Hilbert spaces, while a development of a theory of linear operators on those ultrametric spaces is underway. One of the consequences of the ongoing discussion is that if $\mathbb{K} = \mathbb{Q}_p$ and if A is a bounded linear operator on a free Banach space $\mathbb{E}$ satisfying $\|A\| \le r$ with $r = p^{-1/(p-1)}$, then the function defined by

$$v(q) = \left(\sum_{n\ge 0} \frac{(qA)^n}{n!} \right) u_0, \quad q \in \Omega_r, \tag{1.2}$$

for a fixed $u_0 \in \mathbb{E}$ is the solution to the homogeneous p-adic differential equation given by

$$\frac{du}{dq} = Au, \quad u(0) = u_0. \tag{1.3}$$

This paper is organized as follows: Section 2 is devoted to the required background needed in the sequel. In Section 3, we study C_0-semigroups and consider the solvability of homogeneous p-differential equations involving both bounded and unbounded linear operators on a free Banach space $\mathbb{E}$.

2. Preliminaries

2.1. Free Banach spaces

Defintion 2.1. Let $(\mathbb{K}, |\cdot|)$ be a complete non-Archimedean field and let $\mathbb{E}$ be a vector space over $\mathbb{K}$. A nonnegative real-valued function $\|\cdot\|$ over $\mathbb{E}$ is called an ultrametric norm if

(a) $\|x\| = 0$ if and only if $x = 0$;
(b) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{K}$ and $x \in \mathbb{E}$;
(c) $\|x + y\| \le \max(\|x\|, \|y\|)$ for all $x, y \in \mathbb{E}$ with equality holding if $\|x\| \ne \|y\|$.

Defintion 2.2. An ultrametric Banach space is a vector space endowed with an ultrametric norm, which is complete.

For details on ultrametric Banach spaces and related issues, see, for example, [14, 17].

Example 2.3. Let $(\mathbb{K}, |\cdot|)$ be a complete ultrametric field and let $\rho = (\rho_i)_{i\in I} \subset \mathbb{R}^+ - \{0\}$ be real numbers, where I is a given index set.

Define

$$l^{\infty}(I,\mathbb{K},\rho) := \left\{ x = (x_i)_{i\in I} \in \mathbb{K}^I : \sup_{i\in I} |x_i|\rho_i < \infty \right\}. \tag{2.1}$$

One equips $l^{\infty}(I,\mathbb{K},\rho)$ with the ultrametric norm defined by $\|x\| := \sup_{i\in I}|x_i|\rho_i$. It is well known that $(l^{\infty}(I,\mathbb{K},\rho), \|\cdot\|)$ is an ultrametric Banach space, see [5, 6].

Example 2.4. Let $c_0(I,\mathbb{K},\rho) \subset l^{\infty}(I,\mathbb{K},\rho)$ be a subspace defined by

$$c_0(I,\mathbb{K},\rho) := \left\{ x = (x_i)_{i\in I} \in \mathbb{K}^I : \lim_{i\in I} |x_i|\rho_i = 0 \right\}. \tag{2.2}$$

Clearly, $(c_0(I,\mathbb{K},\rho), \|\cdot\|)$, where $\|\cdot\|$ is the ultrametric norm given in Example 2.3, is a closed subspace of $(l^{\infty}(I,\mathbb{K},\rho), \|\cdot\|)$, and hence it is an ultrametric Banach space.

Defintion 2.5. An ultrametric Banach space $(\mathbb{E}, \|\cdot\|)$ over a (complete) field $(\mathbb{K}, |\cdot|)$ is said to be a free Banach space if there exists a family $(e_i)_{i\in I}$ of elements of $\mathbb{E}$ such that each element $x \in \mathbb{E}$ can be written in a unique fashion as a pointwise convergent series defined by $x = \sum_{i\in I} x_i e_i$ with $\lim_{i\in I} x_i e_i = 0$, and $\|x\| = \sup_{i\in I}|x_i|\|e_i\|$.

The family $(e_i)_{i\in I}$ is then called an *orthogonal basis* for $\mathbb{E}$. If $\|e_i\| = 1$, for all $i \in I$, then $(e_i)_{i\in I}$ is called an *orthonormal basis* for $\mathbb{E}$.

Example 2.6. Let $(\mathbb{K}, |\cdot|)$ be a complete ultrametric field and let M be a compact (topological) space. Let $C(M,\mathbb{K})$ denote the space of continuous functions which go from M into $\mathbb{K}$. The space $C(M,\mathbb{K})$ is equipped the with the sup norm defined by

$$\|u\|_{\infty} := \sup_{m\in M} |u(m)|. \tag{2.3}$$

It can be shown that $(C(M,\mathbb{K}), \|\cdot\|_{\infty})$ is an ultrametric Banach space. In particular, when $M = \mathbb{Z}_p$ and $\mathbb{K} = (\mathbb{Q}_p, |\cdot|)$, where $p \geq 2$ is a prime number, then the resulting space $(C(\mathbb{Z}_p,\mathbb{Q}_p), \|\cdot\|_{\infty})$ is a free Banach space. Indeed, consider the sequence of functions defined by

$$f_n(x) = \frac{x(x-1)(x-2)(x-3)\cdots(x-n+1)}{n!}, \quad n \geq 1,\ f_0(x) = 1. \tag{2.4}$$

It is well known [10] that the family $(f_n)_{n\in\mathbb{N}}$ is an orthonormal base, that is, $\|f_n\|_{\infty} = 1$, and that every function $u \in C(\mathbb{Z}_p,\mathbb{Q}_p)$ has a unique uniformly convergent decomposition defined by $u(x) = \sum_{n=0}^{\infty} c_n f_n(x)$, $c_p \in \mathbb{Q}_p$, with $|c_n| \mapsto 0$ as $n \mapsto \infty$ and $\|u\|_{\infty} = \sup_{n\in\mathbb{N}}|c_n|$.

Example 2.7. Let $\mathbb{K}$ be a field which is complete with respect to a non-Archimedean valuation which will be denoted $|\cdot|$. Fix once and for all a sequence $\omega = (\omega_s)_{s\in\mathbb{N}}$ of nonzero elements of $\mathbb{K}$. Set $\mathbb{E}_{\omega} = c_0(\mathbb{N},\mathbb{K},(\|e_i\|_{i\in N}))$, where $\|e_i\| = |\omega_i|^{1/2}$ for each $i \in \mathbb{N}$. As mentioned above, an (ultrametric) norm is defined on $\mathbb{E}_{\omega}$ by

$$x = (x_s)_{s\in\mathbb{N}}, \qquad \|x\| := \sup_{s\in\mathbb{N}} |x_s| |\omega_s|^{1/2}. \tag{2.5}$$

Note that $\mathbb{E}_{\omega}$ is a free Banach space—it has a canonical orthogonal base—namely, $(e_i)_{i\in\mathbb{N}}$, where e_i is the sequence all of whose terms are 0 except the ith term which is 1, and

$\langle e_i, e_j \rangle = \omega_i \delta_{ij}$, where δ_{ij} is the Kronecker symbol. An inner product (symmetric, bilinear, nondegenerate form) is defined by: for all $x = (x_s)_{s\in\mathbb{N}}$, $y = (y_s)_{s\in\mathbb{N}} \in \mathbb{E}_\omega$,

$$\langle x, y\rangle := \sum_{s=0}^{\infty} x_s y_s \omega_s. \tag{2.6}$$

The space $(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle)$ is then called a non-Archimedean (or p-adic) Hilbert space, see, for example, [2, 5, 6].

For a free Banach space $\mathbb{E}$, let $\mathbb{E}^*$ denote its (topological) dual and $B(\mathbb{E})$ the Banach space of all bounded linear operators on $\mathbb{E}$, see [2, 3, 5, 6]. Both $\mathbb{E}^*$ and $B(\mathbb{E})$ are equipped with their respective natural norms. For $(u,v) \in \mathbb{E}\times\mathbb{E}^*$, we define the linear operator $(v\otimes u)$ by setting

$$\forall x \in \mathbb{E}, \quad (v\otimes u)(x) := v(x)u = \langle v, x\rangle u. \tag{2.7}$$

It follows that $\|v\otimes u\| = \|v\|\cdot\|u\|$.

Let $(e_i)_{i\in\mathbb{N}}$ be an orthogonal basis for $\mathbb{E}$. Define $e_i' \in \mathbb{E}^*$ by $x = \sum_{i\in\mathbb{N}} x_i e_i$ with $e_i'(x) = x_i$. It turns out that $\|e_i'\| = 1\|e_i\|$. Furthermore, every $x' \in \mathbb{E}^*$ can be expressed as a pointwise convergent series: $x'e = \sum_{i\in\mathbb{N}} \langle x' e_i\rangle e_i$. Moreover,

$$\|x'\| := \sup_{i\in\mathbb{N}} \left[\frac{|\langle x', e_i\rangle|}{||e_i||}\right]. \tag{2.8}$$

Now let us recall that every bounded linear operator A on $\mathbb{E}$ can be expressed as a pointwise convergent series, that is, there exists an infinite matrix $(a_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ with coefficients in $\mathbb{K}$ such that

$$A = \sum_{ij} a_{ij}(e_j' \otimes e_i), \tag{2.9}$$

and for any $j \in \mathbb{N}$,

$$\lim_{i\to\infty} |a_{ij}|\,||e_i|| = 0. \tag{2.10}$$

Moreover, for each $j \in \mathbb{N}$, $Ae_j = \sum_{i\in\mathbb{N}} a_{ij} e_i$ and its norm is defined by

$$\|A\| := \sup_{i,j}\left[\frac{|a_{ij}|\,||e_i||}{||e_j||}\right]. \tag{2.11}$$

2.2. Unbounded linear operators on free Banach spaces. Let $\mathbb{E}, \mathbb{F}$ be free Banach spaces. Suppose that $(e_i)_{i\in\mathbb{N}}$ and $(h_j)_{j\in\mathbb{N}}$ are, respectively, the canonical orthogonal bases associated with the free Banach spaces $\mathbb{E}$ and $\mathbb{F}$.

For details on the next definition, see [3, 4], in which a similar definition appears on non-Archimedean Hilbert spaces.

Defintion 2.8. An unbounded linear operator A from $\mathbb{E}$ into $\mathbb{F}$ is a pair $(D(A), A)$ consisting of a subspace $D(A) \subset \mathbb{E}$ (called the domain of A), and a (possibly not continuous) linear transformation $A : D(A) \subset \mathbb{E} \mapsto \mathbb{F}$, such that the domain $D(A)$ contains the basis $(e_i)_{i\in\mathbb{N}}$ and consists of all $u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}$, such that $Au = \sum_{i\in\mathbb{N}} u_i A e_i$ converges in $\mathbb{F}$, that is,

$$\begin{aligned} D(A) &:= \Big\{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E} : \lim_{i\to\infty} |u_i|\,\|Ae_i\| = 0\Big\}, \\ A &= \sum_{i,j\in\mathbb{N}} a_{i,j} e'_j \otimes h_i, \quad \forall j \in \mathbb{N},\ \lim_{i\to\infty} |a_{i,j}|\,\|h_i\| = 0. \end{aligned} \tag{2.12}$$

The collection of those unbounded linear operators is denoted by $U(\mathbb{E},\mathbb{F})$.

For more on these and related issues, we refer the reader to [3, 4].

3. p-adic C_0-semigroup of bounded linear operators

Let $(\mathbb{E}, \|\cdot\|)$ be a free Banach space. Throughout the rest of this paper, we consider families $(T(\kappa))_{\kappa\in\Omega_r} : \mathbb{E} \mapsto \mathbb{E}$ of bounded linear operators. We always suppose that r is suitably chosen so that $\kappa \mapsto T(\kappa)$ is well defined.

Defintion 3.1. Let $r > 0$ be a real number. A family $(T(\kappa))_{\kappa\in\Omega_r} : \mathbb{E} \mapsto \mathbb{E}$ of bounded linear operators will be called a *s*emigroup of bounded linear operators on $\mathbb{E}$ if

(i) $T(0) = I_{\mathbb{E}}$, where $I_{\mathbb{E}}$ is the unit operator of $\mathbb{E}$;

(ii) $T(\kappa+\kappa') = T(\kappa)T(\kappa')$ for all $\kappa, \kappa' \in \Omega_r$.

The semigroup $(T(\kappa))_{\kappa\in\Omega_r}$ will be called of class C_0 or strongly continuous if the following additional condition holds:

(iii) $\lim_{\kappa\to 0} \|T(\kappa)x - x\| = 0$ for each $x \in \mathbb{E}$.

Remark 3.2. A semigroup $(T(\kappa))_{\kappa\in\Omega_r}$ will be called uniformly continuous if the following additional condition holds:

(iv) $\lim_{\kappa\to 0} \|T(\kappa) - I_{\mathbb{E}}\| = 0$.

Remark 3.3. One should point out that a semigroup $(T(\kappa))_{\kappa\in\Omega_r}$ of bounded linear operators is not only a semigroup, but also a group. Every $T(\kappa)$ is invertible, the inverse being $T(-\kappa)$, according to Definition 3.1(i) and (ii). Moreover, it is an infinite abelian group.

Defintion 3.4. If $(T(\kappa))_{\kappa\in\Omega_r}$ is a semigroup, then the linear operator A defined by

$$\begin{aligned} D(A) &= \left\{x \in \mathbb{E} : \lim_{\kappa\to 0} \left(\frac{T(\kappa)x - x}{\kappa}\right) \text{ exists}\right\}, \\ Ax &= \lim_{\kappa\to 0} \left(\frac{T(\kappa)x - x}{\kappa}\right), \quad \text{for each } x \in D(A), \end{aligned} \tag{3.1}$$

is called the infinitesimal generator associated with the semigroup $(T(\kappa))_{\kappa\in\Omega_r}$.

Remark 3.5. (i) Note that if $(T(\kappa))_{\kappa\in\Omega_r}$ is a semigroup on $\mathbb{E}$ and if $(e_i)_{i\in\mathbb{N}}$ denotes the orthogonal basis for $\mathbb{E}$, then $T(\kappa)$ for each $\kappa \in \Omega_r$ can be expressed [3, 5, 6], for any $x = \sum_{i\in\mathbb{N}} x_i e_i \in \mathbb{E}$, by $T(\kappa)x = \sum_{i\in\mathbb{N}} x_i T(\kappa) e_i$, where

$$\forall s \in \mathbb{N}, \quad T(\kappa)e_s = \sum_{i\in\mathbb{N}} a_{i,s}(\kappa) e_i \quad \text{with } \lim_{i\to\infty} |a_{i,s}(\kappa)|\,\|e_i\| = 0. \tag{3.2}$$

(ii) Using (i), one can easily see that for each $0 \neq \kappa \in \Omega_r$,

$$\forall s \in \mathbb{N}, \quad \left(\frac{T(\kappa) - I_{\mathbb{E}}}{\kappa}\right) e_s = \left(\frac{a_{s,s}(\kappa) - 1}{\kappa}\right) e_s + \sum_{i \neq s} \frac{a_{i,s}(\kappa)}{\kappa} e_i \tag{3.3}$$

with $\lim_{i \neq s, i \to \infty} |a_{i,s}(\kappa)| \|e_i\| = 0$.

(iii) If $(T(\kappa))_{\kappa \in \Omega_r}$ is a semigroup on $\mathbb{E}$, then its infinitesimal generator A may or may not be a bounded linear operator on $\mathbb{E}$.

In this paper, we mainly focus on general semigroups and strongly continuous semigroups of bounded linear operators on general free Banach spaces.

We begin with the following example.

Example 3.6. Take $\mathbb{K} = \mathbb{Q}_p$ the field of p-adic numbers. Consider the ball Ω_r of $\mathbb{Q}_p$ with $r = p^{-1/(p-1)}$. Let $\mathbb{E}$ be a free Banach space over $\mathbb{Q}_p$ and let $(e_i)_{i \in \mathbb{N}}$ be the canonical orthogonal base. Define for each $q \in \Omega_r$ and for $x = \sum_{i \geq 0} x_i e_i \in \mathbb{E}$ the family of linear operators $T(q)x = \sum_{i \geq 0} x_i e^{\mu_i q} e_i$, where $(\mu_i)_{i \in \mathbb{N}} \subset \Omega_r$ is a sequence of nonzero elements.

It is routine to check that the family $(T(q))_{q \in \Omega_r}$ is well defined.

Proposition 3.7. *The family $(T(q))_{q \in \Omega_r}$ of linear operators given above is a C_0-semigroup of bounded linear operators, whose infinitesimal generator is the (bounded) diagonal operator A defined by $Ax = \sum_{i \geq 0} \mu_i x_i e_i$ for each $x = \sum_{i \geq 0} x_i e_i \in \mathbb{E}$.*

Proof. First, note that $T(q)$ is analytic on the ball Ω_r. It is routine to check that $(T(q))_{q \in \Omega_r}$ is a family of bounded linear operators on $\mathbb{E}$. Indeed, for each $q \in \Omega_r$,

$$T(q)e_i = e^{\mu_i q} e_i = \left(\sum_{n \geq 0} \frac{\mu_i^n q^n}{n!}\right) e_i, \quad \forall i \in \mathbb{N}, \tag{3.4}$$

and hence, $\|T(q)\| = |(\sum_{n \geq 0} (\mu_i^n q^n)/n!)| < \infty$, by the fact that $q\mu_i \in \Omega_r$ for each $i \in \mathbb{N}$. Furthermore, one can easily check that $T(0) = I_{\mathbb{E}}$, $T(q + q') = T(q)T(q')$ for all $q, q' \in \Omega_r$, and that $\lim_{q \to 0} \|T(q)x - x\| = 0$ for each $x \in \mathbb{E}$, and hence, $(T(q))_{q \in \Omega_r}$ is a C_0-semigroup of bounded linear operators.

Now, let B be the infinitesimal generator of $(T(q))_{q \in \Omega_r}$. It remains to show that $A = B$. First of all, let us show that $D(B) = \mathbb{E}$ $(= D(A))$. Clearly, for each $0 \neq q \in \Omega_r$, $(T(q)e_i - e_i)/q = ((e^{\mu_i q} - 1)/q)e_i$ for each $i \in \mathbb{N}$, and hence

$$D(B) = \left\{x = (x_i)_{i \in \mathbb{N}} : \lim_{i \to \infty} |x_i| \cdot \left\|\frac{T(q)e_i - e_i}{q}\right\| = 0\right\} = \mathbb{E} \tag{3.5}$$

by $|x_i| \cdot \|(T(q)e_i - e_i)/q\| \leq (|x_i| \|e_i\|)/|q| \mapsto 0$ as $i \mapsto \infty$, for each $x = \sum_{i \in \mathbb{N}} x_i e_i \in \mathbb{E}$.

To complete the proof, it suffices to prove that

$$\left\|Ae_i - \left(\frac{T(q)e_i - e_i}{q}\right)\right\| \longmapsto 0 \quad \text{as } q \longmapsto 0. \tag{3.6}$$

The latter is actually obvious since $((e^{\mu_i q} - 1)/q) \mapsto \mu_i$ as $q \mapsto 0$, and hence $B = A$ is the infinitesimal generator of the C_0-semigroup $(T(q))_{q \in \Omega_r}$. $\square$

In the next theorem, we suppose $\mathbb{K} = \mathbb{Q}_p$, where $p \geq 2$ is prime. Note also that it is a natural generalization of Example 3.6.

Theorem 3.8. *Let A be a bounded linear operator on $\mathbb{E}$ such that $\|A\| < r = p^{-1/(p-1)}$. Then, A is the infinitesimal generator of a uniformly continuous semigroup of bounded operators $(T(q))_{q\in\Omega_r}$.*

Proof. Suppose that A is a bounded linear operator on $\mathbb{E}$ with $\|A\| < r = p^{-1/(p-1)}$, and set, for each $q \in \Omega_r$,

$$T(q) = \sum_{n\geq 0} \frac{(qA)^n}{n!}. \tag{3.7}$$

Clearly, the series given by (3.7) converges in norm and defines a family of bounded linear operators on $\mathbb{E}$, by $|q| \cdot \|A\| < r$. It is also routine to check that $T(0) = I_{\mathbb{E}}$, $T(q+q') = T(q)T(q')$ for all $q, q' \in \Omega_r$.

It remains to show that $(T(q))_{q\in\Omega_r}$ given above is uniformly continuous. Indeed, $0 \neq q \in \Omega_r$; one has $T(q) - I_{\mathbb{E}} = qA\{\sum_{n\geq 0}((qA)^n/(n+1)!)\}$, and hence

$$\left\|\frac{T(q) - I_{\mathbb{E}}}{q} - A\right\| \leq \|A\| \cdot \|T(q) - I_{\mathbb{E}}\| < \|T(q) - I_{\mathbb{E}}\|. \tag{3.8}$$

Now, $\|T(q) - I_{\mathbb{E}}\| \leq |q| \cdot \|A\| \cdot \|\zeta(q)\|$, where $\zeta(q) = \sum_{n\geq 0}((qA)^n/(n+1)!)$ converges, and hence

$$\lim_{q\mapsto 0} \|T(q) - I_{\mathbb{E}}\| = 0. \tag{3.9}$$

Consequently,

$$\lim_{q\mapsto 0} \left\|\frac{T(q) - I_{\mathbb{E}}}{q} - A\right\| = 0 \tag{3.10}$$

by both (3.8) and (3.9). □

Remark 3.9. (i) Note that the mapping $\Omega_r \mapsto B(\mathbb{E})$, $q \mapsto T(q)$ is analytic. Furthermore, $dT(q)/dt = AT(q) = T(q)A$.

(ii) An abstract version of Theorem 3.8, that is, in a general ultrametric-valued field $\mathbb{K}$, remains an unsolved problem.

Now, let $\mathbb{K}$ be a (complete) ultrametric-valued field and let $\Omega_r \subset \mathbb{K}$ be a clopen, where r is chosen so that $\Omega_r \mapsto B(\mathbb{E})$, $\kappa \mapsto T(\kappa)$ is well defined.

We have the following theorem.

Theorem 3.10. *Let $(T(\kappa))_{\kappa\in\Omega_r}$ be a C_0-semigroup satisfying $\|T(\kappa)\| \leq M$ for each $\kappa \in \Omega_r \subset \mathbb{K}$ with $M > 0$, and let A be its infinitesimal generator. Then, for each $x \in D(A)$, $T(\kappa)x \in D(A)$ for each $\kappa \in \Omega_r$. Furthermore,*

$$\left(\frac{dT(\kappa)}{d\kappa}\right)x = AT(\kappa)x = T(\kappa)Ax. \tag{3.11}$$

Proof. The proof in some extent, is similar to that of the classical one; however, for the sake of clarity, we will provide the reader with all details.

Let $x \in D(A)$ and let $0 \neq \kappa \in \Omega_r$. Using Definition 3.1, Definition 3.4, and the boundedness of the C_0-semigroup $T(\kappa)$, it easily follows that

$$\left(\frac{T(\kappa)-I_{\mathbb{E}}}{\kappa}\right)T(\kappa')x = T(\kappa')\left(\frac{T(\kappa)-I_{\mathbb{E}}}{\kappa}\right)x \longmapsto T(\kappa')Ax \tag{3.12}$$

as $\kappa \mapsto 0$.

Consequently, $T(\kappa')x \in D(A)$ and $AT(\kappa')x = T(\kappa')Ax$, by (3.12). Furthermore, since $T(\kappa')((T(\kappa)-I_{\mathbb{E}})/\kappa)x \mapsto T(\kappa')Ax$ as $\kappa \mapsto 0$, it follows that the right derivative of $T(\kappa')x$ is $T(\kappa')Ax$. Thus, to complete the proof, we have to show that for each $0 \neq \kappa' \in \Omega_r$, the left derivative of $T(\kappa')x$ exists and is $T(\kappa')Ax$. (Note that if $\sigma, \sigma' \in \Omega_r$, so is $\sigma - \sigma'$, by $|\sigma - \sigma'| \leq \max(|\sigma|, |\sigma'|) < r$.) Now

$$\begin{aligned}\lim_{\kappa\to 0}\left(\frac{T(\kappa')x - T(\kappa'-\kappa)x}{\kappa} - T(\kappa')x\right) &= \lim_{\kappa\to 0} T(\kappa'-\kappa)\left(\frac{T(\kappa)x - x}{\kappa} - Ax\right)\\ &\quad + \lim_{\kappa\to 0}\left[T(\kappa'-\kappa)Ax - T(\kappa')Ax\right].\end{aligned} \tag{3.13}$$

Clearly, $\lim_{\kappa\to 0} T(\kappa'-\kappa)((T(\kappa)x - x)/\kappa - Ax) = 0$, by $\|T(\sigma)\| \leq M$ for each $\sigma \in \Omega_r$. Using the strong continuity of the semigroup $T(\kappa)$, it follows that

$$\lim_{\kappa\to 0}\left[T(\kappa'-\kappa)Ax - T(\kappa')Ax\right] = 0. \tag{3.14}$$

Consequently, $\lim_{\kappa\to 0}((T(\kappa')x - T(\kappa'-\kappa)x)/\kappa - T(\kappa')x) = 0$, and hence the left derivative of $T(\kappa')x$ exists and equals $T(\kappa')Ax$. This completes the proof. □

Remark 3.11. One of the consequences of Theorem 3.10 is that the function $v(\kappa) = T(\kappa)u_0$, $\kappa \in \Omega_r$, for some $u_0 \in D(A)$, is the solution to the homogeneous p-adic differential equation given by

$$\begin{aligned}\frac{d}{d\kappa}u(\kappa) &= Au(\kappa), \quad \kappa \in \Omega_r,\\ u(0) &= u_0,\end{aligned} \tag{3.15}$$

where $A : D(A) \subset \mathbb{E} \mapsto \mathbb{E}$ is the infinitesimal generator of the C_0-semigroup $(T(\kappa))_{\kappa\in\Omega_r}$, and $u : \Omega_r \mapsto D(A)$ is an $\mathbb{E}$-valued function.

Example 3.12. Take $\mathbb{K} = \mathbb{Q}_p$. Let A be the multiplication operator on $\mathbb{E} = C(\mathbb{Z}_p, \mathbb{Q}_p)$ defined by

$$Au = Q(x)u, \quad \forall u \in C(\mathbb{Z}_p, \mathbb{Q}_p), \tag{3.16}$$

where $Q = \sum_{n=0}^{\infty} q_n f_n \in C(\mathbb{Z}_p, \mathbb{Q}_p)$.

Suppose that $\|Q\|_\infty = \sup_{n\in\mathbb{N}} |q_n| < r$ with $r = p^{-1/(p-1)}$ (here, one can take $M = 1$).

In view of Theorem 3.10, the function defined by $v(q) = (\sum_{n\geq 0}((qA)^n/n!))u_0$, $q \in \Omega_r$, for some $u_0 \in \mathbb{E}$, is the solution to the homogeneous p-adic differential equation

$$\frac{d}{d\kappa}u(\kappa) = Q(k)u(\kappa), \quad \kappa \in \Omega_r, \qquad u(0) = u_0 \in \mathbb{Q}_p. \tag{3.17}$$

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