PRODUCTS OF DERIVATIONS WHICH ACT AS LIE DERIVATIONS ON COMMUTATORS OF RIGHT IDEALS

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Let R be a prime ring of characteristic different from 2, I a nonzero right ideal of R, d and δ nonzero derivations of R, and $s_4(x_1,x_2,x_3,x_4)$ the standard identity of degree 4. If the composition $(d\delta)$ is a Lie derivation of $[I,I]$ into R, then either $s_4(I,I,I,I)I = 0$ or $\delta(I)I = d(I)I = 0$.

Throughout this note, R will be always a prime ring of characteristic different from 2 with center $Z(R)$, extended centroid C, and two-sided Martindale quotient ring Q. Let $f : R \to R$ be additive mapping of R into itself. It is said to be a derivation of R if $f(xy) = f(x)y + xf(y)$, for all $x, y \in R$. Let $S \subseteq R$ be any subset of R. If for any $x, y \in S$, $f([x, y]) = [f(x), y] + [x, f(y)]$, then the mapping f is called a Lie derivation on S. Obviously any derivation of R is a Lie derivation on any arbitrary subset S of R.

A typical example of a Lie derivation is an additive mapping which is the sum of a derivation and a central map sending commutators to zero.

The well-known Posner first theorem states that if δ and d are two nonzero derivations of R, then the composition $(d\delta)$ cannot be a nonzero derivation of R [12, Theorem 1]. An analog of Posner's result for Lie derivations was proved by Lanski [8]. More precisely, Lanski showed that if δ and d are two nonzero derivations of R and L is a Lie ideal of R, then $(d\delta)$ cannot be a Lie derivation of L into R unless $\mathrm{char}(R) = 2$ and either R satisfies $s_4(x_1,\dots,x_4)$, the standard identity of degree 4, or $d = \alpha\delta$, for $\alpha \in C$.

This note is motived by the previous cited results. Our main theorem gives a generalization of Lanski's result to the case when $(d\delta)$ is a Lie derivation of the subset $[I,I]$ into R, where I is a nonzero right ideal of R and the characteristic of R is different from 2. The statement of our result is the following.

Theorem 1. *Let R be a prime ring of characteristic different from 2, I a nonzero right ideal of R, d and δ nonzero derivations of R, and $s_4(x_1,\dots,x_4)$ the standard identity of degree 4. If the composition $(d\delta)$ is a Lie derivation of $[I,I]$ into R, then either $s_4(I,I,I,I)I = 0$ or $\delta(I)I = d(I)I = 0$.*

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Remark 2. Notice that for all $u,v \in [I,I]$, we obviously have that

$$(d\delta)([u,v]) = [(d\delta)(u),v] + [u,(d\delta)(v)] + [\delta(u),d(v)] + [d(u),\delta(v)]. \tag{1}$$

Hence, since we suppose that $(d\delta)$ is a Lie derivation on $[I,I]$, we will always assume as a main hypothesis that $[\delta(u),d(v)] + [d(u),\delta(v)] = 0$, for any $u,v \in [I,I]$.

Remark 3. The assumption $S_4(I,I,I,I)I \neq 0$ is essential to the main result. For example, consider $R = M_3(F)$, for F a field of characteristic 3, and let e_{ij} be the usual matrix unit in R. Let $I = (e_{11}+e_{22})R$, δ the inner derivation induced by the element e_{13}, d the inner derivation induced by the element e_{12}, that is, $\delta(x) = [e_{13},x] = e_{13}x - xe_{13}$, and $d(x) = [e_{12},x] = e_{12}x - xe_{12}$, for all $x \in R$. In this case, notice that $S_4(x_1,x_2,x_3,x_4)x_5$ is an identity for I, moreover

$$\begin{aligned}
&[\delta([(e_{11}+e_{22})x_1,(e_{11}+e_{22})x_2]),d([(e_{11}+e_{22})y_1,(e_{11}+e_{22})y_2])] \\
&\qquad + [d([(e_{11}+e_{22})x_1,(e_{11}+e_{22})x_2]),\delta([(e_{11}+e_{22})y_1,(e_{11}+e_{22})y_2])] \\
&= (d([(e_{11}+e_{22})x_1,(e_{11}+e_{22})x_2])[(e_{11}+e_{22})y_1,(e_{11}+e_{22})y_2])e_{13} \\
&\qquad - (d([(e_{11}+e_{22})y_1,(e_{11}+e_{22})y_2])[(e_{11}+e_{22})x_1,(e_{11}+e_{22})x_2])e_{13} = 0
\end{aligned} \tag{2}$$

for any $x_1,x_2,y_1,y_2 \in R$, but clearly $d(I)I = [e_{12},I]I \neq 0$.

In the particular case $I = R$ and both d, δ are inner derivations, induced, respectively, by some elements $a,b \in R$, our theorem has the following flavor.

Lemma 4. *Let R be a prime ring of characteristic different from 2, $a,b \in R$ such that $[[a,v],[b,u]] + [[b,v],[a,u]] = 0$, for all $v,u \in [R,R]$. Then either a is a central element of R or b is a central one.*

The proof is a clear special case of [8, Theorem 6].

We first fix some notations and recall some useful facts.

Remark 5. Denote by $T = Q *_C C\{X\}$ the free product over C of the C-algebra Q and the free C-algebra $C\{X\}$, with X a countable set consisting of noncommuting indeterminates $\{x_1,\ldots,x_n\}$. The elements of T are called generalized polynomials with coefficients in Q. I, IR, and IQ satisfy the same generalized polynomial identities with coefficients in Q. For more details about these objects, we refer the reader to [1, 2, 4].

Remark 6. Any derivation of R can be uniquely extended to a derivation of Q, and so any derivation of R can be defined on the whole of Q [2, Proposition 2.5.1]. Moreover Q is a prime ring as well as R and the extended centroid C of R coincides with the center of Q [2, Proposition 2.1.7, Remark 2.3.1].

Remark 7. Let $f(x_1,\ldots,x_n,d(x_1),\ldots,d(x_n))$ be a differential identity of R. One of the following holds (see [7]):

(1) either d is an inner derivation in Q, in the sense that there exists $q \in Q$ such that $d(x) = [q,x]$, for all $x \in Q$ and Q satisfies the generalized polynomial identity $f(x_1,\ldots,x_n,[q,x_1],\ldots,[q,x_n])$;

(2) or R satisfies the generalized polynomial identity

$$f(x_1,\dots,x_n,y_1,\dots,y_n). \tag{3}$$

Moreover I, IR, and IQ satisfy the same differential identities with coefficients in Q (see [9]).

Finally, as a consequence of [11, Theorem 2], we have the following.

Remark 8. Let R be a prime ring and $\sum_{i=1}^{m} a_iXb_i + \sum_{j=1}^{n} c_jXd_j = 0$, for all $X \in R$, where $a_i,b_i,c_j,d_j \in RC$. If $\{a_1,\dots,a_m\}$ are linearly C-independent, then each b_i is C-dependent on $d_1,\dots,d_n$. Analogously, if $\{b_1,\dots,b_m\}$ are linearly C-independent, then each a_i is C-dependent on $c_1,\dots,c_n$.

For the remainder of the note we will assume that the hypothesis of the theorem holds but that the conclusion is false.

Thus, we will always suppose that there exist $c_1,c_2,c_3,c_4,c_5,c_6,c_7,c_8,c_9 \in I$ such that $s_4(c_1,c_2,c_3,c_4)c_5 \neq 0$, and either $\delta(c_6)c_7 \neq 0$ or $d(c_8)c_9 \neq 0$.

We begin with the following.

Lemma 9. *Let δ and d both be Q-inner derivations such that either $\delta(I)I \neq 0$ or $d(I)I \neq 0$. Then R is a ring satisfying a nontrivial generalized polynomial identity.*

Proof. By Remark 2, we assume that $[\delta(u),d(v)] + [d(u),\delta(v)] = 0$, for any $u,v \in [I,I]$. Let $a,b \in Q$ such that $\delta(x) = [a,x]$ and $d(x) = [b,x]$, for all $x \in R$.

Without loss of generality, we may assume in this context that $\delta(I)I \neq 0$. Notice that if $\{y,ay\}$ are linearly C-dependent for all $y \in I$, then there exists $\alpha \in C$, such that $(a-\alpha)I = 0$ (see [10, Lemma 3]). If we replace a by $a-\alpha$, since they induce the same inner derivation, it follows that $\delta(I)I = [a-\alpha,I]I = 0$, a contradiction. Thus there exists $x \in I$ such that $\{x,ax\}$ are linearly C-independent.

Let $x \in I$ such that $\{x,ax\}$ are linearly C-independent and r_1, r_2, r_3, r_4 are any elements of R. Then

$$[[a,[xr_1,xr_2]],[b,[xr_3,xr_4]]] + [[b,[xr_1,xr_2]],[a,[xr_3,xr_4]]] = 0. \tag{4}$$

Denote

$$\begin{aligned}
F_1 &= (r_1xr_2 - r_2xr_1)b[xr_3,xr_4] - (r_1xr_2 - r_2xr_1)[xr_3,xr_4]b \\
&\quad - (r_3xr_4 - r_4xr_3)b[xr_1,xr_2] + (r_3xr_4 - r_4xr_3)[xr_1,xr_2]b, \\
F_2 &= -(r_3xr_4 - r_4xr_3)a[xr_1,xr_2] + (r_3xr_4 - r_4xr_3)[xr_1,xr_2]a \\
&\quad + (r_1xr_2 - r_2xr_1)a[xr_3,xr_4] - (r_1xr_2 - r_2xr_1)[xr_3,xr_4]a, \\
F_3 &= (r_3xr_4 - r_4xr_3)ba[xr_1,xr_2] - (r_1xr_2 - r_2xr_1)ba[xr_3,xr_4] \\
&\quad + (r_1xr_2 - r_2xr_1)a[xr_3,xr_4]b - (r_3xr_4 - r_4xr_3)b[xr_1,xr_2]a \\
&\quad - (r_1xr_2 - r_2xr_1)ba[xr_3,xr_4] + (r_1xr_2 - r_2xr_1)b[xr_3,xr_4]a \\
&\quad + (r_3xr_4 - r_4xr_3)ab[xr_1,xr_2] - (r_3xr_4 - r_4xr_3)a[xr_1,xr_2]b.
\end{aligned} \tag{5}$$

Hence (4) is $axF_1 + bxF_2 + xF_3 = 0$. If $\{ax, bx, x\}$ are linearly C-independent, then (4) is a nontrivial generalized polynomial identity for R, since $F_1 \neq 0$ in T, using $b \notin C$. On the other hand, if there exist $\alpha_1, \alpha_2 \in C$ such that $bx = \alpha_1 x + \alpha_2 ax$, it follows that R satisfies

$$axF_1 + \alpha_1 xF_2 + \alpha_2 axF_2 + xF_3 = 0, \tag{6}$$

that is, again a nontrivial GPI, because $\{x, ax\}$ are linearly C-independent, by the choice of x and since $F_1 + F_2 \neq 0$ in T, using $a, b \notin C$.

The same argument shows that if $d(I)I \neq 0$, then there exists $x \in I$ such that $\{x, bx\}$ are linearly C-independent and R satisfies in any case a nontrivial GPI. □

At this point, we need a result that will be useful in the continuation of the note.

Remark 10. Let $R = M_n(F)$ be the ring of $n \times n$ matrices over the field F, denote by e_{ij} the usual matrix unit with 1 in the (i, j)-entry and zero elsewhere. Since there exists a set of matrix units that contains the idempotent generator of a given minimal right ideal, we observe that any minimal right ideal is part of a direct sum of minimal right ideals adding to R. In light of this and applying [6, Proposition 5, page 52], we may assume that any minimal right ideal of R is a direct sum of minimal right ideals, each of the form $e_{ii}R$.

Lemma 11. *Let $R = M_n(F)$ be the ring of $n \times n$ matrices over the field F of characteristic different from 2 and $n \geq 2$. Let d be a nonzero inner derivation of R, and I a nonzero right ideal of R. If a is a nonzero element of I such that $(d([x_1, x_2])[x_3, x_4] - d([x_3, x_4])[x_1, x_2])a = 0$, for all $x_1, x_2, x_3, x_4 \in I$, then either $s_4(I, I, I, I)I = 0$ or d is induced by an element $b \in R$ such that $(b - \beta)I = 0$, for a suitable $\beta \in Z(R)$.*

Proof. Let b be an element of R which induces the derivation d, that is, $d(x) = [b, x]$, for all $x \in R$. As above, let e_{ij} be the usual matrix unit with 1 in the (i, j)-entry and zero elsewhere and write $a = \sum a_{ij}e_{ij}$, $b = \sum b_{ij}e_{ij}$, with a_{ij} and b_{ij} elements of F.

We know that I has a number of uniquely determinated simple components: they are minimal right ideals of R and I is their direct sum. In light of Remark 10, we may write $I = eR$ for some $e = \sum_{i=1}^{t} e_{ii}$ and $t \in \{1, 2, \ldots, n\}$. Since $s_4(I, I, I, I)I = 0$ in case $t \leq 2$, we may suppose that $t \geq 3$.

First of all, we want to prove that $b_{rs} = 0$ for all $s \leq t$ and $r \neq s$. To do this, suppose by contradiction that there exist $i \neq j$ such that $b_{ij} \neq 0$ ($j \leq t$). Without loss of generality, we replace b by $b_{ij}^{-1}(b - b_{jj}I_n)$, where I_n is the identity matrix in $M_n(F)$ so that we assume $b_{ij} = 1$ and $b_{jj} = 0$. Moreover $a = ex$ for a suitable $x \in R$.

Let now $k \leq t$, $k \neq i, j$, $[x_1, x_2] = e_{ki}$, $[x_3, x_4] = e_{ji}$. In this case, we have

$$0 = ([b, e_{ki}]e_{ji}a - [b, e_{ji}]e_{ki})a \tag{7}$$

and left multiplying by e_{kk},

$$e_{ki}be_{ji}a = 0, \tag{8}$$

that is, since $b_{ij} = 1$, $e_{ii}a = 0$.

On the other hand, if we choose $[x_1,x_2]=e_{ki}$ and $[x_3,x_4]=e_{jk}$, we have

$$0=([b,e_{ki}]e_{jk}-[b,e_{jk}]e_{ki})a=[b,e_{ki}]e_{jk}a=-b_{ij}e_{kk}a. \tag{9}$$

Therefore $e_{rr}a=0$ for all $r\neq j$, that is, $a=e_{jj}a$. Finally, consider $[x_1,x_2]=e_{ki}$ and $[x_3,x_4]=e_{kk}-e_{jj}$. Then

$$0=([b,e_{ki}](e_{kk}-e_{jj})-[b,e_{kk}-e_{jj}]e_{ki})a=e_{ki}be_{jj}a, \tag{10}$$

that is, $e_{jj}a=0$. This implies that $ea=0$, so that $a=0$, a contradiction.

This argument says that if $a\neq 0$, then $b_{ij}=0$ for all $i\neq j$, $j\leq t$.

Suppose that $(b-\beta)I\neq 0$, for $\beta\in F$. In this case, there exist $1\leq r$, $s\leq t$, with $r\neq s$, such that $b_{rr}\neq b_{ss}$.

Let f be the F-automorphism of R defined by $f(x)=(1-e_{rs})x(1+e_{rs})$. Thus we have that $f(x)\in I$, for all $x\in I$ and

$$([f(b),[x_1,x_2]][x_3,x_4]-[f(b),[x_3,x_4]][x_1,x_2])f(a)=0 \tag{11}$$

for all $x_1,x_2,x_3,x_4\in I$. If $a\neq 0$, then $f(a)\neq 0$, and as above, the (r,s)-entry of $f(b)$ is zero. On the other hand,

$$f(b)=(1-e_{rs})b(1+e_{rs})=b+b_{rr}e_{rs}-b_{ss}e_{rs}, \tag{12}$$

that is, $b_{rr}=b_{ss}$, a contradiction. This means that there exists $\beta\in F$ such that $(b-\beta)I=0$. Denote $b-\beta=p$. Since b and p induce the same inner derivation d, we have that $([p,[x_1,x_2]][x_3,x_4]-[p,[x_3,x_4]][x_1,x_2])a=0$ with $pI=0$. □

Lemma 12. *Let R be a prime ring of characteristic different from 2, d a nonzero inner derivation of R, I a nonzero right ideal of R. If a is a nonzero element of I such that $(d([x_1,x_2])[x_3,x_4]-d([x_3,x_4])[x_1,x_2])a=0$, for all $x_1,x_2,x_3,x_4\in I$, then either $s_4(I,I,I,I)I=0$ or d is induced by an element $b\in R$ such that $(b-\beta)I=0$, for a suitable $\beta\in Z(R)$.*

Proof. As a reduction of Lemma 9, we have that if R is not a GPI ring, then we are done. Thus consider the only case when R satisfies a nontrivial generalized polynomial identity.

Thus the Martindale quotient ring Q of R is a primitive ring with nonzero socle $H=\mathrm{Soc}(Q)$. H is a simple ring with minimal right ideals. Let D be the associated division ring of H, by [11] D is a simple central algebra finite-dimensional over $C=Z(Q)$. Thus $H\otimes_C F$ is a simple ring with minimal right ideals, with F an algebraic closure of C. Let b be an element of R which induces the derivation d. Moreover $([b,[x_1,x_2]][x_3,x_4]-[b,[x_3,x_4]][x_1,x_2])a=0$, for all $x_1,x_2,x_3,x_4\in IH\otimes_C F$ (see, e.g., [4, Theorem 2]). Notice that if C is finite, we choose $F=C$.

Now we claim that for any $c\in IH$, there exists $\beta\in C$ with $(b-\beta)c=0$. If not, then for some $c\in IH$, $(b-\beta)c\neq 0$ for all $\beta\in C$, so in particular $bc\neq 0$. Since H is regular [5], there exists $g^2=g\in IH$, such that $c\in gIH$, and $e^2=e\in H\otimes_C F$, such that

$$g,bg,gb,a,c,bc,cb\in e(H\otimes_C F)e\cong M_n(F),\quad n\geq 3. \tag{13}$$

Let $x_1,x_2,x_3,x_4 \in ge(H \otimes_C F)e$ and $a = eae \neq 0$, then

$$0 = e([b,[x_1,x_2]][x_3,x_4] - [b,[x_3,x_4]][x_1,x_2])eae. \tag{14}$$

Applying Lemma 11, we have that $e(b-\lambda)ec = 0$ for $\lambda \in C$, so $(b-\beta)c = 0$, contradicting the choice of c.

As in the proof of Lemma 9, by [10, Lemma 3], we conclude that there exists $\beta \in C$ such that $(b-\beta)I = 0$. □

LEMMA 13. *If δ and d are both inner derivations, then the theorem holds.*

Proof. By Remark 2, we assume that $[\delta(u),d(v)] + [d(u),\delta(v)] = 0$, for any $u,v \in [I,I]$. Let $a,b \in Q$ such that $\delta(x) = [a,x]$ and $d(x) = [b,x]$, for all $x \in R$. Since in light of Lemma 9, R satisfies a nontrivial GPI, then without loss of generality, R is simple and equal to its own socle and $IR = I$. In fact, Q has nonzero socle H with nonzero right ideal $J = IH$ [11]. Note that H is simple, $J = JH$, and J satisfies the same basic conditions as I. Now just replace R by H, I by J, and we are done.

Recall that $s_4(c_1,c_2,c_3,c_4)c_5 \neq 0$ and either $\delta(c_6)c_7 \neq 0$ or $d(c_8)c_9 \neq 0$. By the regularity of R, there exists an element $e^2 = e \in IR$ such that $eR = c_1R + c_2R + c_3R + c_4R + c_5R + c_6R + c_7R + c_8R + c_9R$ and $ec_i = c_i$, for $i = 1,\dots,9$. We note that $s_4(eRe,eRe,eRe,eRe) \neq 0$ (and $\dim_C(eRe) \geq 9$).

Let $x,y,z \in R$, so

$$[[a,[e,ex(1-e)]],[b,[ey,ez]]] + [[b,[e,ex(1-e)]],[a,[ey,ez]]] = 0. \tag{15}$$

Denote $A = (1-e)ae$, $B = (1-e)be$. Assume that $A = 0$ but $B \neq 0$. Consider first the case when $\{1-e,(1-e)a\}$ are linearly C-independent. Equation (15), multiplied on the left by $(1-e)$, says that

$$-(1-e)b[ey,ez]aex(1-e) + (1-e)b[ey,ez]ex(1-e)a = 0. \tag{16}$$

By Remark 8 and since $\{1-e,(1-e)a\}$ are linearly C-independent, it follows that there exists $\lambda_1 \in C$ such that $-(1-e)b[ey,ez]ae = \lambda_1(1-e)b[ey,ez]e$.

Therefore

$$(1-e)b[ey,ez]ex\lambda_1(1-e) + (1-e)b[ey,ez]ex(1-e)a = 0, \tag{17}$$

which implies that $(1-e)b[ey,ez]e = 0$, again since $\{1-e,(1-e)a\}$ are linearly C-independent. If we denote $T = eR$, $(1-e)b[T,T]T = 0$ forces $(1-e)bT[T,T]T = 0$, so either $(1-e)bT = 0$ or $[T,T]T = 0$. Thus we have that either $B = (1-e)be = 0$ or $[x_1,x_2]x_3$ is an identity for eR. In this last case, a fortiori $s_4(x_1,x_2,x_3,x_4)x_5$ is an identity for eR. In both cases, we have a contradiction, since we suppose that $B \neq 0$ and $s_4(c_1,c_2,c_3,c_4)c_5 \neq 0$.

Suppose now that $(1-e)a = \lambda(1-e)$, for some $\lambda \in C$. Equation (16) says that

$$-(1-e)b[ey,ez]aex(1-e) + \lambda(1-e)b[ey,ez]ex(1-e) = 0, \tag{18}$$

and so

$$-(1-e)b[ey,ez]ae+\lambda(1-e)b[ey,ez]e=0, \tag{19}$$

that is, for $a'=\lambda e-ae$,

$$(1-e)b[ey,ez]a'=0. \tag{20}$$

Denote $U=[ey,ez]a'$. Since $(1-e)be[Ux_1,ex_2]a'=0$, for all $x_1,x_2\in R$, it follows that $(1-e)bex_2Ux_1a'=0$, and so either $a'=0$ or $U=0$. Again denote $T=eR$. If $U=0$, we have $[T,T]a'=0$, so that $[T,T]Ta'=0$, which implies either $a'=0$ or $[T,T]T=0$. Since $[eR,eR]e\neq 0$, we have $ae=\lambda e$ in any case.

All the previous arguments say that $(a-\lambda)e=0$. Replacing a by $a-\lambda=a''$, since they induce the same inner derivation, we may assume that for all $x,y,z,t\in R$,

$$[[a'',[ex,ey]],[b,[ez,et]]]+[[b,[ex,ey]],[a'',[ez,et]]]=0. \tag{21}$$

Left multiplying (21) by $(1-e)$, we have

$$(1-e)be[[ez,et],[ex,ey]]a''=0, \tag{22}$$

in particular

$$0=(1-e)be[[ez,et],[ex,ey(1-e)]]a''=(1-e)be[ez,et]exey(1-e)a'', \tag{23}$$

and by the previous same argument, $(1-e)a''=0$, that is, $a''=ea''$. In light of this, by (22),

$$(eR(1-e)be)[[eze,ete],[exe,eye]](ea''Re)=0. \tag{24}$$

Let G be the subgroup of eRe generated by the polynomial $[[eze,ete],[exe,eye]]$. It is easy to see that G is a noncentral Lie ideal of eRe. In this condition, it is well known that $[eRe,eRe]\subseteq G$, and so $eR(1-e)be[eRe,eRe]ea''Re=0$.

Consider now the simple Artinian ring eRe, then we have that

$$eR(1-e)be[ex_1e,ex_2e](ea''Re)=0 \quad \forall x_1,x_2\in R. \tag{25}$$

Let $U=[ex_1e,ex_2e](ea''Re)$, so $eR(1-e)beU=0$. Since

$$(eR(1-e)be)[Uex_1e,ex_2e](ea''Re)=0, \tag{26}$$

then

$$(eR(1-e)be)x_2Uex_1(ea''Re)=0. \tag{27}$$

It follows that if $(1-e)be\neq 0$, then $a''=0$, that is, $a=\lambda\in C$, a contradiction. Thus the conclusion is that if $A=(1-e)ae=0$, then $B=(1-e)be=0$.

Similarly, $A=(1-e)ae=0$ follows from $B=(1-e)be=0$.

Now we assume that neither $A = 0$ nor $B = 0$, and proceed to get contradictions, proving that $A = B = 0$.

Left multiplying (15) by $(1-e)$ and right multiplying by e, we get

$$\begin{aligned}(1-e)aex(1-e)b[ey,ez]e+(1-e)b[ey,ez]ex(1-e)ae \\ +(1-e)bex(1-e)a[ey,ez]e+(1-e)a[ey,ez]ex(1-e)be=0.\end{aligned} \tag{28}$$

If we denote $A' = A[ye,ze]$, $B' = B[ye,ze]$, it follows that

$$AxB' + B'xA + BxA' + A'xB = 0. \tag{29}$$

Consider now the case when A,B are linearly C-independent.

In light of Remark 8 and (29), it follows that there exist α_1, α_2, α_3, α_4 in C such that $B' = \alpha_1 A + \alpha_2 B$, $A' = \alpha_3 A + \alpha_4 B$. So we rewrite (29) as follows:

$$2\alpha_1 AxA + 2\alpha_4 BxB + (\alpha_2+\alpha_3)AxB + (\alpha_2+\alpha_3)BxA = 0, \tag{30}$$

that is,

$$Ax(2\alpha_1 A + (\alpha_2+\alpha_3)B) + Bx(2\alpha_4 B + (\alpha_2+\alpha_3)A) = 0. \tag{31}$$

Since A, B are C-independent, by (31) and again Remark 8, it follows that $2\alpha_1 A + (\alpha_2 + \alpha_3)B = 0$ and $2\alpha_4 B + (\alpha_2+\alpha_3)A = 0$, so the independence of A and B forces $\alpha_1 = \alpha_4 = \alpha_2 + \alpha_3 = 0$.

Therefore we have that $B[eRe,eRe] \subseteq CB$. Notice that $B[eRe,eRe] \neq 0$. In fact, if $B[eRe,eRe] = 0$, since $[eRe,eRe] \neq (0)$ is a noncentral Lie ideal of the simple Artinian ring eRe, the contradiction $B = 0$ is immediate.

Let $u,v \in [eRe,eRe]$. Hence there exist ω_1, ω_2, $0 \neq \omega \in C$ such that

$$B[u,v] = \omega B \neq 0, \qquad Bu = \omega_1 B, \qquad Bv = \omega_2 B, \tag{32}$$

and by calculation we get the contradiction

$$0 \neq \omega B = B[u,v] = 0. \tag{33}$$

Hence we may assume that A and B are linearly C-dependent, say $A = \alpha B$, for $0 \neq \alpha \in C$, so also $A' = \alpha B'$. Equation (29) is now $2\alpha BxB' + 2\alpha B'xB = 0$, and it follows that B and B' must be linearly C-dependent, so that $BxB = 0$ and $B = B' = 0$.

Therefore in any case, we have that if $s_4(eR,eR,eR,eR)e \neq 0$, then $(1-e)be = (1-e)ae = 0$.

Let $J = eR$, $\overline{J} = J/J \cap l_R(J)$; $\overline{J}$ is a prime C-algebra. Since $d(J) \subseteq J$ and $\delta(J) \subseteq J$, d and δ induce on $\overline{J}$ the following two derivations:

$$\begin{aligned}\overline{d} : \overline{J} \longrightarrow \overline{J} \quad &\text{such that } \overline{d}(\overline{x}) = \overline{d(x)}, \\ \overline{\delta} : \overline{J} \longrightarrow \overline{J} \quad &\text{such that } \overline{\delta}(\overline{x}) = \overline{\delta(x)}.\end{aligned} \tag{34}$$

Therefore, we have

$$0 = [\overline{\delta([r_1,r_2])}, \overline{d[r_3,r_4]}] + [\overline{d([r_1,r_2])}, \overline{\delta[r_3,r_4]}] \tag{35}$$

for all $\overline{r_1},\overline{r_2},\overline{r_3},\overline{r_4} \in \overline{J}$. By Lemma 4, we have that one of the following holds:

$$\overline{\delta} = \overline{0}, \qquad \overline{d} = \overline{0}, \qquad \overline{J} \text{ is commutative.} \tag{36}$$

Since $s_4(J,J,J,J)J \neq 0$, the last case cannot occur. On the other hand, now we prove that also the other cases lead us to contradictions.

Suppose that the first case occurs, that is, $\delta(J)J = 0$. By the lemma in [3], there exists an element $q = a - \alpha \in Q$, with $\alpha \in C$, such that $(a-\alpha)J = 0$. Moreover a and q induce the same inner derivation δ, so that we have

$$([b,[x_1,x_2]][x_3,x_4] - [b,[x_3,x_4]][x_1,x_2])q = 0 \quad \forall x_1,x_2,x_3,x_4 \in J. \tag{37}$$

In particular, for any $r \in R$, choose $[x_1,x_2] = [e, er(1-e)] = er(1-e)$. From (37), it follows that

$$[b,[x_3,x_4]]eR(1-e)q = 0. \tag{38}$$

If $(1-e)q = 0$, we have $q = eq \in J$. Under this condition, by Lemma 12, it follows from (37) that either $q = 0$, which implies the contradiction $a \in C$ and $\delta = 0$, or $(b-\beta)J = 0$ for a suitable $\beta \in C$, that is, $d(eR)eR = 0$. So consider the case when $[b,[x_3,x_4]]e = 0$ for all $x_3,x_4 \in J$, and remember that $be = ebe$. This implies that $[ebe,[y_1,y_2]] = 0$ for all $y_1,y_2 \in eRe$, that is, either eRe is a commutative central simple algebra or $ebe \in Ce$. In the first case, we have the contradiction $0 = s_4(ec_1,ec_2,ec_3,ec_4)ec_5 = s_4(c_1,c_2,c_3,c_4)c_5 \neq 0$. In the second one, we get again $d(eR)eR = 0$. Therefore we conclude that in any case, $\delta(eR)eR = d(eR)eR = 0$, which is again a contradiction because of $\delta(c_6)c_7 \neq 0$ or $d(c_8)c_9 \neq 0$.

Obviously by a similar argument and (36), we are also finished when $d(J)J = 0$. □

For the proof of the main theorem, we need the following results.

LEMMA 14. *Let R be a prime ring of characteristic different from 2 and I a nonzero right ideal of R. If for any $x_1,x_2,x_3,x_4 \in I$, $[[x_1,x_2],[x_3,x_4]] = 0$, then $[I,I]I = 0$.*

Proof. First note that if $[y,[I,I]] = 0$, for some $y \in R$, then, for any $s,t \in I$, we have $0 = [y,[st,t]] = [s,t][y,t]$. In particular, for any $x \in IR$, $0 = [sx,t][y,t] = [s,t]x[y,t]$, that is $[s,t]IR[y,t] = 0$. By the primeness of R, we have that either $[s,t]I = 0$, that is, $[I,I]I = 0$, or $[y,I] = 0$. In this last case, $0 = [y,IR] = I[y,R]$ forcing $y \in Z(R)$.

Therefore, if we assume that $[I,I]I \neq 0$, the assumption $[[I,I],[I,I]] = 0$ forces $0 \neq [I,I] \subseteq Z(R)$. Let $s,t \in I$ be such that $[s,t]I \neq 0$ and $[s,t] \in Z(R)$. Then $2s[s,t] = [s^2,t] \in Z(R)$, so $[s,t] \neq 0$ forces $s \in Z(R)$ and we have the contradiction $[s,I] = 0$. □

LEMMA 15. *Let R be a noncommutative prime ring of characteristic different from 2, q a noncentral element of R, and I a nonzero right ideal of R. If for any $x_1,x_2,x_3,x_4 \in I$, $[[q,[x_1,x_2]],[x_3,x_4]] = 0$, then $[I,I]I = 0$.*

Proof. Suppose that $[I,I]I \neq 0$. As in Lemma 14, first we recall that the condition $[y,[I,I]]=0$ forces $y \in Z(R)$. This means that $[q,[I,I]] \subseteq Z(R)$, since $[[q,[I,I]],[I,I]]=0$. Moreover we may assume that $[q,[I,I]] \neq 0$, if not, then we are finished by Lemma 14.

Note that from $[q,[I,I]] \subseteq Z(R)$, it follows that $[q,[[I,I],I]] \subseteq Z(R)$. Expanding this yields $[[I,I],[q,I]] \subseteq Z(R)$. Since for all $x \in I$, we have $[[I,I],[q,xq]] \subseteq Z(R)$, then $[[I,I],[q,I]q] \subseteq Z(R)$. Hence

$$0=[[[I,I],[q,I]q],q]=[q,[q,I]][q,[I,I]]. \tag{39}$$

Since the second factor is nonzero and central, we have $[q,[q,I]]=0$, which implies that for all $x,y \in I$,

$$0=[q,[q,xy]]=[q,[q,x]y+x[q,y]]=2[q,x][q,y]. \tag{40}$$

This means that $[q,I][q,I]=0$ and a fortiori $[q,[I,I]][q,[I,I]]=0$ giving the contradiction $[q,[I,I]]=0$. □

We are ready to prove the following main result.

Theorem 16. *Let R be a prime ring of characteristic different from 2, I a nonzero right ideal of R, d and δ nonzero derivations of R, $s_4(x_1,\dots,x_4)$ the standard identity of degree 4. If the composition $(d\delta)$ is a Lie derivation of $[I,I]$ into R, then either $s_4(I,I,I,I)I=0$ or $\delta(I)I=d(I)I=0$.*

Proof. By Remark 2, we assume that $[\delta(u),d(v)]+[d(u),\delta(v)]=0$, for any $u,v \in [I,I]$. Suppose by contradiction that there exist c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9 in I such that $s_4(c_1,c_2,c_3,c_4)c_5 \neq 0$ and either $\delta(c_6)c_7 \neq 0$ or $d(c_8)c_9 \neq 0$.

First suppose that δ and d are C-independent modulo D_{int}.

Let $t_1,t_2,t_3,t_4 \in I$, by assumption, R satisfies

$$\begin{aligned}
&[[\delta(t_1x_1),t_2x_2]+[t_1x_1,\delta(t_2x_2)],[d(t_3x_3),t_4x_4]+[t_3x_3,d(t_4x_4)]]\\
&\quad+[[d(t_1x_1),t_2x_2]+[t_1x_1,d(t_2x_2)],[\delta(t_3x_3),t_4x_4]+[t_3x_3,\delta(t_4x_4)]]\\
&=[[\delta(t_1)x_1+t_1\delta(x_1),t_2x_2]+[t_1x_1,\delta(t_2)x_2+t_2\delta(x_2)],\\
&\quad[d(t_3)x_3+t_3d(x_3),t_4x_4]+[t_3x_3,d(t_4)x_4+t_4d(x_4)]]\\
&\quad+[[d(t_1)x_1+t_1d(x_1),t_2x_2]+[t_1x_1,d(t_2)x_2+t_2d(x_2)],\\
&\quad[\delta(t_3)x_3+t_3\delta(x_3),t_4x_4]+[t_3x_3,\delta(t_4)x_4+t_4\delta(x_4)]]=0.
\end{aligned} \tag{41}$$

By Kharchenko's theorem [7], R satisfies the generalized polynomial identity

$$\begin{aligned}
&[[\delta(t_1)x_1+t_1y_1,t_2x_2]+[t_1x_1,\delta(t_2)x_2+t_2y_2],[d(t_3)x_3+t_3z_3,t_4x_4]+[t_3x_3,d(t_4)x_4+t_4z_4]]\\
&+[[d(t_1)x_1+t_1z_1,t_2x_2]+[t_1x_1,d(t_2)x_2+t_2z_2],[\delta(t_3)x_3+t_3y_3,t_4x_4]+[t_3x_3,\delta(t_4)x_4+t_4y_4]]
\end{aligned} \tag{42}$$

in particular R satisfies $[[t_1y_1,t_2x_2],[t_3x_3,t_4z_4]]$, so Q satisfies this as well, and for all $y_1 = x_2 = x_3 = z_4 = 1 \in Q$, it follows that $[[I,I],[I,I]] = 0$. Thus by Lemma 14, we conclude that $[I,I]I = 0$, that is, $s_4(I,I,I,I)I = 0$, which contradicts $s_4(c_1,c_2,c_3,c_4)c_5 \neq 0$.

Now let δ and d be C-dependent modulo D_{int}. There exist $\gamma_1,\gamma_2 \in C$, such that $\gamma_1\delta + \gamma_2 d \in D_{\text{int}}$, and by Lemma 13, it is clear that at most one of the two derivations can be inner.

Without loss of generality, we may assume that $\gamma_1 \neq 0$, so that $\delta = \alpha d + ad(q)$, for $\alpha \in C$ and $ad(q)$ the inner derivation induced by the element $q \in Q$.

If d is inner, then also δ is inner, and we have that d is an outer derivation of R. Let $t_1,t_2,t_3,t_4 \in I$, R satisfies

$$\begin{aligned}
&\alpha[[d(t_1x_1),t_2x_2]+[t_1x_1,d(t_2x_2)],[d(t_3x_3),t_4x_4]+[t_3x_3,d(t_4x_4)]]\\
&\quad+[[q,[t_1x_1,t_2x_2]],[d(t_3x_3),t_4x_4]+[t_3x_3,d(t_4x_4)]]\\
&\quad+\alpha[[d(t_1x_1),t_2x_2]+[t_1x_1,d(t_2x_2)],[d(t_3x_3),t_4x_4]+[t_3x_3,d(t_4x_4)]]\\
&\quad+[[d(t_1x_1),t_2x_2]+[t_1x_1,d(t_2x_2)],[q,[t_3x_3,t_4x_4]]]\\
&=\alpha[[d(t_1)x_1+t_1d(x_1),t_2x_2]+[t_1x_1,d(t_2)x_2+t_2d(x_2)],\\
&\qquad[d(t_3)x_3+t_3d(x_3),t_4x_4]+[t_3x_3,d(t_4)x_4+t_4d(x_4)]]\\
&\quad+[[q,[t_1x_1,t_2x_2]],[d(t_3)x_3+t_3d(x_3),t_4x_4]+[t_3x_3,d(t_4)x_4+t_4d(x_4)]]\\
&\quad+\alpha[[d(t_1)x_1+t_1d(x_1),t_2x_2]+[t_1x_1,d(t_2)x_2+t_2d(x_2)],\\
&\qquad[d(t_3)x_3+t_3d(x_3),t_4x_4]+[t_3x_3,d(t_4)x_4+t_4d(x_4)]]\\
&\quad+[[d(t_1)x_1+t_1d(x_1),t_2x_2]+[t_1x_1,d(t_2)x_2+t_2d(x_2)],[q,[t_3x_3,t_4x_4]]],
\end{aligned} \tag{43}$$

and so the Kharchenko theorem shows that R satisfies

$$\begin{aligned}
&\alpha[[d(t_1)x_1+t_1y_1,t_2x_2]+[t_1x_1,d(t_2)x_2+t_2y_2],[d(t_3)x_3+t_3y_3,t_4x_4]+[t_3x_3,d(t_4)x_4+t_4y_4]]\\
&+[[q,[t_1x_1,t_2x_2]],[d(t_3)x_3+t_3y_3,t_4x_4]+[t_3x_3,d(t_4)x_4+t_4y_4]]\\
&+\alpha[[d(t_1)x_1+t_1y_1,t_2x_2]+[t_1x_1,d(t_2)x_2+t_2y_2],[d(t_3)x_3+t_3y_3,t_4x_4]+[t_3x_3,d(t_4)x_4+t_4y_4]]\\
&+[[d(t_1)x_1+t_1y_1,t_2x_2]+[t_1x_1,d(t_2)x_2+t_2y_2],[q,[t_3x_3,t_4x_4]]].
\end{aligned} \tag{44}$$

In case $\alpha \neq 0$, for $x_1 = x_4 = 0$ in (44), we have that R satisfies

$$2\alpha[[t_1y_1,t_2x_2],[t_3x_3,t_4y_4]], \tag{45}$$

so Q satisfies this as well and for all $y_1 = x_2 = x_3 = y_4 = 1 \in Q$, it follows that $2\alpha[[I,I],[I,I]] = 0$. Hence, if $\alpha \neq 0$, by Lemma 14, we have the contradiction $[I,I]I = 0$.

Now let $\alpha = 0$. In this case for $x_4 = 0$ in (44), we have that R satisfies

$$[[q,[t_1x_1,t_2x_2]],[t_3x_3,t_4y_4]]. \tag{46}$$

As above Q satisifes this and, taking $x_1, x_2, x_3, y_4 = 1$, it follows that

$$[[q,[I,I]],[I,I]] = 0. \tag{47}$$

Then, by Lemma 15, we conclude again with the contradiction $[I,I]I = 0$.

Similarly, when $y_2 \neq 0$, then $d = \beta\delta + ad(q)$, for some $\beta \in C$, and mimicking the argument above gives another contradiction. □

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