LINEARIZATION COEFFICIENTS FOR SHEFFER POLYNOMIAL SETS VIA LOWERING OPERATORS

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The lowering operator σ associated with a polynomial set $\{P_n\}_{n\geq0}$ is an operator not depending on n and satisfying the relation $\sigma P_n = nP_{n-1}$. In this paper, we express explicitly the linearization coefficients for polynomial sets of Sheffer type using the corresponding lowering operators. We obtain some well-known results as particular cases.

1. Introduction

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients. A polynomial sequence $\{P_n\}_{n\geq0}$ in $\mathcal{P}$ is called a *polynomial set* if and only if $\deg P_n = n$ for all nonnegative integers n. Given two polynomial sets $\{S_n\}_{n\geq0}$ and $\{P_n\}_{n\geq0}$, the so-called *connection problem* between them asks to find the coefficients $C_m(n)$ in the expression:

$$S_n(x) = \sum_{m=0}^{n} C_m(n)P_m(x), \tag{1.1}$$

which for $S_n(x) = x^n$ is known as the *inversion problem* for the polynomial set $\{P_n\}_{n\geq0}$. When $S_{i+j}(x) = Q_i(x)R_j(x)$ in (1.1), $\{Q_n\}_n$ and $\{R_n\}_n$ being two polynomial sets, we are faced to the *general linearization problem*

$$Q_i(x)R_j(x) = \sum_{k=0}^{i+j} L_{ij}(k)P_k(x). \tag{1.2}$$

Particular case of this problem is the *standard linearization* or *Clebsch-Gordan-type problem*

$$P_i(x)P_j(x) = \sum_{k=0}^{i+j} L_{ij}(k)P_k(x). \tag{1.3}$$

The computation of the connection and linearization coefficients plays an important role

in many situations of pure and applied mathematics and also in physical and quantum chemical applications [32, 33]. In particular, the study of positivity conditions of $L_{ij}(k)$ has received special attention. This property has many important consequences. It gives rise to a convolution structure associated with the polynomial set $\{P_n\}_{n\geq0}$ [8, 9, 19, 36]. Several sufficient conditions for the sign properties to hold have been derived in [6, 7, 37, 38]. The literature on this topic is extremely vast and a wide variety of methods, based on specific properties of the involved polynomials, have been devised for computing the linearization coefficients $L_{ij}(k)$ either in closed form or by means of recursive relations (usually in k) [10, 26, 27], exploiting for this purpose several of their specific properties: recurrence relation [24], generating function [3, 7, 15, 16, 22], orthogonality weights and Rodrigue's formula [1, 5], inversion formulas [4, 29], and so forth. A combinatorial approach to solve the linearization problems was also given in [21, 28, 39].

A general method, based on lowering operators, was developed by the authors [13, 14] to solve connection problems. The purpose of this work is to show that such a technique can likewise be used to treat linearization problems.

The outline of the paper is as follows. In Section 2, we give a result for a general linearization problem. Then we prove a useful lemma, generalizing the Leibniz formula, to express explicitly the standard linearization coefficients for Sheffer polynomial sets (Theorem 2.6). In Section 3, for practical uses of the main result, we give the standard linearization coefficients for some well-known basic Sheffer polynomial sets. Finally, in Section 4, we apply Theorem 2.6 to orthogonal Sheffer polynomial sets.

2. Linearization coefficients

2.1. A general result. Denote by $\Lambda^{(-1)}$ the space of operators σ acting on analytic functions that reduce the degree of every polynomial by exactly one and $\sigma(1) = 0$.

It was shown, by the first author, that every polynomial set is quasi-monomial [12]. That is to say, there exist a lowering operator σ and a raising operator τ, independent of n, such that

$$\sigma(P_n) = nP_{n-1}, \quad \tau(P_n) = P_{n+1}, \quad n = 1,2,\ldots. \tag{2.1}$$

Definition 2.1. Let $\sigma \in \Lambda^{(-1)}$ and let $\{P_n\}_{n\geq0}$ be a polynomial set. $\{P_n\}_{n\geq0}$ is called a σ-Appell polynomial set if and only if

$$\sigma(P_n) = nP_{n-1}, \quad n = 1,2,\ldots. \tag{2.2}$$

Definition 2.2. Let $\sigma \in \Lambda^{(-1)}$. A polynomial set $\{B_n\}_{n\geq0}$ is called the *sequence of basic polynomials* for σ if and only if
   (i) $\{B_n\}_{n\geq0}$ is a σ-Appell polynomial set;
   (ii) $B_n(0) = \delta_{0,n}$, $n = 0,1,\ldots$.
In [11], it was shown that every $\sigma \in \Lambda^{(-1)}$ has a unique sequence of basic polynomials.

THEOREM 2.3 [11]. *Let $\{P_n\}_{n\geq 0}$ be a polynomial set. Then there exist a unique $\sigma \in \Lambda^{(-1)}$ and a unique power series $A(t) = \sum_{k=0}^{\infty} a_n t^n$, $a_0 \neq 0$, such that $\{P_n\}_{n\geq 0}$ is a σ-Appell polynomial set and*

$$A(\sigma)(B_n) = P_n, \quad n = 0, 1, \ldots, \tag{2.3}$$

where $\{B_n\}_{n\geq 0}$ is the sequence of basic polynomials for σ.

Call $\{P_n\}_{n\geq 0}$ *a σ-Appell polynomial set of transfer power series A.*

A σ-Appell polynomial set of transfer power series A is generated by

$$G(x,t) = A(t)G_0(x,t) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n, \tag{2.4}$$

where $G_0(x,t)$ is a solution of the system

$$\sigma G_0(x,t) = tG_0(x,t), \qquad G_0(x,0) = 1, \tag{2.5}$$

and conversely.

Let $\mathscr{P}'$ be the algebraic dual of $\mathscr{P}$. We denote by $\langle \mathscr{L}, f \rangle$ the effect of the functional $\mathscr{L} \in \mathscr{P}'$ on the polynomial $f \in \mathscr{P}$. Let $\{P_n\}_{n\geq 0}$ be a polynomial set. Its dual sequence $\{\mathbb{P}_n\}_{n\geq 0}$ is defined by

$$\langle \mathbb{P}_n, P_m \rangle = \delta_{n,m}, \quad n, m \geq 0. \tag{2.6}$$

When $\{P_n\}_{n\geq 0}$ is a σ-Appell polynomial set of transfer power series A, an explicit expression of its dual sequence was given in [11] by

$$\langle \mathbb{P}_n, f \rangle = \frac{1}{n!} \sigma^n \widehat{A}(\sigma)(f)(x) \Big|_{x=0}, \quad n = 0, 1, \ldots, f \in \mathscr{P}, \tag{2.7}$$

where $\widehat{A}(t) = 1/A(t)$.

Combining (1.2), (2.6), and (2.7), we state the following general result.

THEOREM 2.4. *Let $\sigma \in \Lambda^{(-1)}$ and $\{P_n\}_{n\geq 0}$ be a σ-Appell polynomial set of transfer power series A. Then the general linearization coefficients in (1.2) are given by*

$$L_{ij}(k) = \frac{1}{k!} \sigma^k \widehat{A}(\sigma)(Q_i R_j)(0), \quad i, j = 0, 1, \ldots, k = 0, 1, \ldots, i+j. \tag{2.8}$$

Next, in this paper, we limit ourselves to standard case for Sheffer polynomial sets case.

2.2. Sheffer polynomials. Recall that a polynomial set $\{P_n\}_{n\geq 0}$ is said to be of Sheffer type A-zero (Sheffer polynomial set, for shorter,) if and only if it has a generating function of the form [25, 30]

$$A(t)\exp(xC(t)) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n, \tag{2.9}$$

where A and C are two formal power series:

$$A(t) = \sum_{k=0}^{\infty} a_k t^k, \quad a_0 \neq 0, \qquad C(t) = \sum_{k=0}^{\infty} c_k t^{k+1}, \quad c_0 \neq 0. \tag{2.10}$$

It was shown in [12] that a Sheffer polynomial set generated by (2.9) is σ-Appell polynomial set of transfer power series A, where $\sigma = C^*(D)$, $D = d/dx$, and C^* is the inverse of C; that is, $C^*(C(t)) = C(C^*(t)) = t$.

If $C(t) = t$, we have $\sigma = D$. That corresponds to Appell polynomial sets [2].

In order to apply (2.8) to Sheffer polynomial sets we need the following.

LEMMA 2.5 (generalized Leibniz formula). *Let $\sigma \in \Lambda^{(-1)}$ and let $\{B_n\}_{n \geq 0}$ be the sequence of basic polynomials for σ. Suppose that σ commutes with the derivative operator D. Let f and g be two formal power series. Then,*

$$\sigma^n(f(z)g(z)) = \sum_{m=n}^{\infty} \sum_{k=0}^{m} c_{k,m}(n) \sigma^k f(z) \sigma^{m-k} g(z), \tag{2.11}$$

where

$$c_{k,m}(n) = \frac{1}{k!(m-k)!} \sigma^n (B_k B_{m-k}) \Big|_{x=0}. \tag{2.12}$$

Proof. Let $z_0 \in \mathbb{C}$. Let us define the translation operators T_{z_0} by $T_{z_0} f(z) = e^{z_0 D} f(z) = f(z + z_0)$. Since σ commutes with T_{z_0}, f and g have the formal power expansions:

$$f(z) = T_{z_0} f(z - z_0) = \sum_{m=0}^{\infty} \frac{\sigma^m f(z_0)}{m!} B_m(z - z_0), \qquad g(z) = \sum_{m=0}^{\infty} \frac{\sigma^m g(z_0)}{m!} B_m(z - z_0), \tag{2.13}$$

by virtue of (2.7). So

$$\sigma^n(f(z)g(z)) = \sum_{m=n}^{\infty} \sum_{k=0}^{m} \frac{\sigma^k f(z_0)}{k!} \frac{\sigma^{m-k} g(z_0)}{(m-k)!} \sigma^n(B_k(z - z_0)B_{m-k}(z - z_0)), \tag{2.14}$$

since $\sigma^n(B_k(z - z_0)B_{m-k}(z - z_0)) = 0$ if $m < n$. Put $z = z_0$ in (2.14), we deduce (2.11) since z_0 is arbitrary. $\qquad \square$

For the particular case $\sigma = D$, the corresponding basic sequence is $B_n(x) = x^n$. Then the coefficients in (2.12) are given by

$$c_{k,m}(n) = \frac{1}{k!(m-k)!} D^n(x^m) \Big|_{x=0} = \frac{n!}{k!(m-k)!} \delta_{n,m}, \tag{2.15}$$

and (2.11) is reduced to the well-known Leibniz formula

$$D^n(f(z)g(z)) = \sum_{k=0}^{n} \binom{n}{k} D^k f(z) D^{n-k} g(z). \tag{2.16}$$

As every Sheffer polynomial set generated by (2.9) may be viewed as a σ-Appell polynomial set of transfer power series A where $\sigma = C^*(D)$, we use this property to state our following main result.

THEOREM 2.6. *The linearization coefficients in (1.3) with $\{P_n\}_{n\geq 0}$ a Sheffer polynomial set generated by (2.9) are given by*

$$L_{ij}(k) = \sum_{m\geq k} \sum_{p=0}^{m} \binom{i}{p}\binom{j}{m-p} l_{p,m-p}(k) \widehat{A}(C^*(D)) (P_{i-p}P_{j-m+p}(x)) \Big|_{x=0}, \qquad (2.17)$$

where $l_{nm}(k)$ are the standard linearization coefficients for the corresponding basic sequence generated by

$$\frac{1}{k!}C^{*k}(C(t)+C(s)) = \sum_{n,m} \frac{l_{nm}(k)}{n!m!} t^n s^m. \qquad (2.18)$$

Proof. $\{P_n\}_{n\geq 0}$ is a σ-Appell polynomial set of transfer power series A, where $\sigma = C^*(D)$. Then by virtue of Theorem 2.4 and (2.11) we derive (2.17).

The basic sequence $\{B_n\}_{n\geq 0}$ is a σ-Appell polynomial set of transfer power series 1. So according to (2.8) and (2.16), we have

$$\begin{aligned}
l_{ij}(k) &= \frac{1}{k!}\sigma^k(B_iB_j)\Big|_{x=0} = \frac{1}{k!}C^{*k}(D)(B_iB_j)\Big|_{x=0} \\
&= \frac{1}{k!}\sum_{n\geq k}\alpha_{n,k}D^k(B_iB_j)\Big|_{x=0}, \qquad \text{where } C^{*k}(t) = \sum_{n\geq k}\alpha_{n,k}t^n, \\
&= \frac{1}{k!}\sum_{n\geq k}\alpha_{n,k}\left(\sum_{p=0}^{n}\binom{n}{p}D^pB_iD^{k-p}B_j\right)\Big|_{x=0} \\
&= \frac{1}{k!}\sum_{n\geq k}\alpha_{n,k}\left(\sum_{p=0}^{n}\binom{n}{p}C(\sigma)^pB_iC(\sigma)^{k-p}B_j\right)\Big|_{x=0} \\
&= \frac{1}{k!}\sum_{n\geq k}\alpha_{n,k}\left(C(\sigma)(B_i)+C(\sigma)(B_j)\right)^n\Big|_{x=0} \\
&= \frac{1}{k!}C^{*k}\left(C(\sigma(B_i))+C(\sigma(B_j))\right)\Big|_{x=0}.
\end{aligned} \qquad (2.19)$$

Put $(1/k!)C^{*k}(C(t)+C(s)) = \sum_{n,m}a_{n,m}(k)t^ns^m$. It follows from (2.19) and Definition 2.2 that

$$\begin{aligned}
l_{i,j}(k) &= \frac{1}{k!}\sum_{n,m}a_{n,m}(k)\sigma^n(B_i)\sigma^m(B_j)\Big|_{x=0} \\
&= \frac{1}{k!}\sum_{n,m}a_{n,m}(k)\frac{i!}{(i-n)!}B_{i-n}(0)\frac{j!}{(j-m)!}B_{j-m}(0)\Big|_{x=0} = i!j!a_{i,j}(k),
\end{aligned} \qquad (2.20)$$

which gives (2.18).   □

Remark 2.7. A similar proof may be used to express general linearization coefficients in (1.2) where the involved three polynomial sets are of Sheffer type.

Next, in Section 3, we use (2.18) to express explicitly the standard linearization coefficients for some well-known basic Sheffer polynomial sets. Then, in Section 4, in order to show the efficiency of the proposed approach, we apply Theorem 2.6 to orthogonal Sheffer polynomial sets to derive some already obtained results in the literature by alternative methods.

3. Linearization coefficients for basic polynomials

3.1. Stirling polynomials. The Stirling polynomial set $\{x^{[n]} = x(x-1)\cdots(x-n+1)\}_{n\geq 0}$ is generated by

$$(1+t)^x = \sum_{n=0}^{\infty} \frac{x^{[n]}}{n!} t^n. \tag{3.1}$$

For this case we have $C(t) = \text{Log}(1+t)$ and $C^*(t) = e^t - 1$.

It follows that $\{x^{[n]}\}_{n\geq 0}$ is a Δ-Appell polynomial set, where $\Delta = e^D - 1$ is the *difference operator*, and

$$\frac{1}{k!}C^{*k}\left(C(t) + C(s)\right) = \frac{1}{k!}(t+s+st)^k = \sum_{n,m} \frac{t^{m+k-n}s^{k-m}}{(k-n)!(n-m)!m!}$$

$$= \sum_{i,j} \frac{t^i s^j}{(k-j)!(k-i)!(i+j-k)!}. \tag{3.2}$$

Then

$$x^{[i]}x^{[j]} = \sum_{k=0}^{\min(i,j)} k! \binom{i}{k}\binom{j}{k} x^{[i+j-k]}. \tag{3.3}$$

According to (3.3), the following relations can be derived

$$\binom{x}{n}\binom{x}{m} = \sum_{k=0}^{\min(n,m)} \frac{(n+m-k)!}{(n-k)!(m-k)!k!}\binom{x}{n+m-k},$$

$$(x)_n(x)_m = \sum_{k=0}^{\min(n,m)} (-1)^k k! \binom{n}{k}\binom{m}{k}(x)_{n+m-k}, \tag{3.4}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$.

Also, from (3.3), one can see that (2.11) contains as a particular case the well-known Jordan formula [20]

$$\Delta^n(fg)(z) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(z) \Delta^{n-k} g(z+k). \tag{3.5}$$

In fact, for the special case $\sigma = \Delta$, the corresponding basic sequence is $B_n(x) = x^{[n]}$. Then (2.12) is reduced to

$$c_{k,m}(n) = \frac{n!}{k!(m-k)!}l_{k,m-k}(n) = \frac{n!}{(n-k)!(n-m)!(m+k-n)!}. \tag{3.6}$$

It follows from (2.11) and (3.6) that

$$\Delta^n(f(z)g(z)) = \sum_{m=0}^{\infty}\sum_{k=0}^{m} c_{k,m}\Delta^k f(z)\Delta^{m-k}g(z) = \sum_{k=0}^{\infty}\frac{n!}{k!}\Delta^k f(z)\left(\sum_{m=0}^{\infty}\frac{l_{k,m}(n)}{m!}\Delta^m g(z)\right)$$

$$= \sum_{k=0}^{\infty}\Delta^k f(z)\left(\sum_{m=0}^{\infty}\frac{n!}{(n-k)!(n-m)!(m+k-n)!}\Delta^m g(z)\right) \tag{3.7}$$

$$= \sum_{k=0}^{\infty}\binom{n}{k}\Delta^k f(z)\Delta^{n-k}\left(\sum_{p=0}^{\infty}\binom{k}{p}\Delta^p g(z)\right),$$

which, in view of the expansion formula [20]

$$g(z+a) = \sum_{p=0}^{\infty}\binom{a}{p}\Delta^p g(z), \tag{3.8}$$

gives (3.5).

3.2. Basic Laguerre polynomials.

The basic Laguerre polynomials $\{L_n(x)\}_{n\geq 0}$ are generated by

$$e^{x(t/(t-1))} = \sum_{n=0}^{\infty}L_n(x)t^n. \tag{3.9}$$

Then $C(t) = C^*(t) = t/(t-1)$. It follows that $\{n!L_n(x)\}_{n\geq 0}$ is a σ-Appell polynomial set, where $\sigma = D/(D-1)$ is the *Laguerre operator*.

For this case we have

$$\frac{1}{k!}C^{*k}(C(t)+C(s)) = \frac{1}{k!}\left(\frac{t+s-2st}{1-st}\right)^k$$

$$= \sum_{i,j}\sum_{n\geq 0}\frac{(k)_n(-2)^{i+j-k-2n}}{(n+k-j)!(n+k-i)!(i+j-k-2n)!}t^i s^j. \tag{3.10}$$

Then

$$l_{ij}(k) = \sum_{n\geq 0}\frac{(k)_n k!(-2)^{i+j-k-2n}}{n!(n+k-i)!(n+k-j)!(i+j-k-2n)!}$$

$$= \frac{(-2)^{i+j-k}k!}{(i+j-k)!(k-i)!(k-j)!}\, {}_3F_2\left(k,\frac{k-i-j}{2},\frac{k-i-j+1}{2};1\atop k-i+1,k-j+1\right). \tag{3.11}$$

3.3. Basic Meixner polynomials. The basic Meixner polynomial set $\{P_n(x)\}_{n\geq 0}$ is generated by

$$e^{x\ln((1-t/a)/(1-t))} = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n. \tag{3.12}$$

For this case we have

$$C(t) = \ln\left(\frac{1-t/a}{1-t}\right), \qquad C^*(t) = \frac{e^t - 1}{e^t - 1/a}. \tag{3.13}$$

It follows that $\{P_n(x)\}_{n\geq 0}$ is a σ-Appell polynomial set, where $\sigma = (e^D - 1)/(e^D - 1/a)$, and

$$\frac{1}{k!}C^{*k}(C(t) + C(s)) = \frac{1}{k!}\left(\frac{t + s - (1 + 1/a)st}{1 - st/a}\right)^k$$

$$= \sum_{i,j}\sum_{n\geq 0} \frac{(k)_n(-1 - 1/a)^{i+j+k-2n}}{n!a^n(n+k-j)!(n+k-i)!(i+j-k-2n)!} t^i s^j. \tag{3.14}$$

Then

$$l_{ij}(k) = \sum_{n\geq 0} \frac{(k)_n i! j! (-1 - 1/a)^{i+j-k-2n}}{n!a^n(n+k-i)!(n+k-j)!(i+j-k-2n)!}$$

$$= \frac{i! j! (-1 - 1/a)^{i+j-k}}{(i+j-k)!(k-i)!(k-j)!} \, {}_3F_2\left(\begin{matrix} k, \dfrac{k-i-j}{2}, \dfrac{k-i-j+1}{2} \\ k-i+1, k-j+1 \end{matrix}; \frac{4a}{(a+1)^2}\right). \tag{3.15}$$

3.4. Basic Meixner-Pollaczek polynomials. The basic Meixner-Pollaczek polynomial set $\{P_n(x)\}_{n\geq 0}$ is generated by

$$e^{x\arctan(t/(1+\delta t))} = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n. \tag{3.16}$$

For this case we have

$$C(t) = \arctan\frac{t}{1+\delta t}, \qquad C^*(t) = \frac{\tan t}{1 - \delta\tan t}. \tag{3.17}$$

It follows that $\{P_n(x)\}_{n\geq 0}$ is a σ-Appell polynomial set, where $\sigma = \tan D/(1 - \delta\tan D)$ and

$$\frac{1}{k!}C^{*k}(C(t) + C(s)) = \frac{1}{k!}\left(\frac{t + s + 2\delta st}{1 - (1 + \delta^2)st}\right)^k$$

$$= \sum_{i,j}\sum_{n\geq 0} \frac{(k)_n(1 + \delta^2)^n(2\delta)^{i+j-k-2n}}{n!(n+k-j)!(n+k-i)!(i+j-k-2n)!} t^i s^j. \tag{3.18}$$

Then

$$
\begin{aligned}
l_{ij}(k) &= \sum_{n\geq 0} \frac{(k)_n (1+\delta^2)^n i! j! (2\delta)^{i+j-k-2n}}{n!(n+k-i)!(n+k-j)!(i+j-k-2n)!} \\
&= \frac{i! j! (2\delta)^{i+j-k}}{(i+j-k)!(k-i)!(k-j)!} \, {}_3F_2\left(k, \frac{k-i-j}{2}, \frac{k-i-j+1}{2}; \frac{1+\delta^2}{\delta^2} \right).
\end{aligned}
\tag{3.19}
$$

4. Orthogonal Sheffer polynomials

Let $\{P_n\}_{n\geq 0}$ be an orthogonal σ-Appell polynomial set of transfer power series A. The linear functional $\mathcal{L}$ for which the orthogonality holds is given by [34]

$$
\langle \mathcal{L}, f \rangle = \hat{A}(\sigma)(f)(0).
\tag{4.1}
$$

We use this relation and Theorem 2.6 to state the following.

COROLLARY 4.1. *The linearization coefficients in (1.3) for $\{P_n\}_{n\geq 0}$ an orthogonal σ-Appell polynomial set of transfer power series A of Sheffer type are given by*

$$
L_{ij}(k) = \sum_{2s\leq i+j-k} \binom{i}{s}\binom{j}{s} l_{i-s,j-s}(k) I_s,
\tag{4.2}
$$

where $I_s = \langle \mathcal{L}, P_s P_s \rangle = \hat{A}(\sigma)(P_s^2)(x)|_{x=0}$ and l_{ij} are the standard linearization coefficients for the corresponding basic sequence.

An immediate consequence of this result is the following.

COROLLARY 4.2. *A sufficient condition to ensure the positivity of the standard linearization coefficients for an orthogonal Sheffer polynomial set is the positivity of those associated with the corresponding basic sequence.*

It follows, from the results obtained in Section 3, that the standard linearization coefficients for Hermite, Charlier, monic Laguerre polynomials $\{(-1)^n n! L_n^\alpha(x)\}_{n\geq 0}$, and Meixner-Pollaczek polynomials are positive.

Let us return now to Corollary 4.1 to mention that this result concerns exactly five classes of Sheffer polynomials according to Meixner characterization [17]. In Table 4.1, we recall these classes with the corresponding lowering operators and transfer power series according to our analysis.

Next, for each case, we use (4.2) to express explicitly the corresponding standard linearization coefficients in terms of hypergeometric series.

4.1. Hermite polynomials.

The Hermite polynomials $\{H_n\}_{n\geq 0}$ are generated by

$$
e^{-t^2} e^{2xt} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n.
\tag{4.3}
$$

Table 4.1. Orthogonal Sheffer polynomial sets.

Polynomial set	Lowering operator	Transfer power series
Hermite $H_n(x)$	$\sigma = \dfrac{D}{2}$	$A(t) = e^{-t^2}$
Charlier $C_n^a(x)$, $a > 0$	$\sigma = \Delta = e^D - 1$	$A(t) = e^{-at}$
Laguerre $L_n^\alpha(x)$, $\alpha > -1$	$\sigma = \dfrac{D}{D-1}$	$A(t) = (1-t)^{-\alpha-1}$
Meixner $M_n(x,\beta,a)$, $\beta > 0$, $0 < a < 1$	$\sigma = \dfrac{\Delta}{(a-1)/a + \Delta}$	$A(t) = (1-t)^{-\beta}$
Meixner-Pollaczek $P_n(x;\delta,\eta)$ $\delta > 0$, $\eta > 0$	$\sigma = \dfrac{\tan D}{1 - \delta \tan D}$	$A(t) = [(1+\delta t)^2 + t^2]^{-\eta/2}$

Then $\sigma = D/2$ and $A(t) = e^{-t^2}$. Since the corresponding basic sequence is $\{(2x)^n\}_{n\geq 0}$, we get $l_{ij}(k) = \delta_{i+j,k}$. The linear functional for which the orthogonality holds is [34]

$$\widehat{A}(\sigma)(f)(0) = \exp(\sigma^2)(f)(0) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} f(x)dx. \tag{4.4}$$

Then we have [17] $I_s = s!2^s$. According to (4.2), we deduce [18]

$$L_{ij}(k) = \begin{cases} \dbinom{i}{s}\dbinom{j}{s} 2^s s! & \text{if } k = i+j-2s, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

4.2. Charlier polynomials. The Charlier polynomials $\{C_n^a\}_{n\geq 0}$ are generated by

$$e^{-at}(1+t)^x = \sum_{n=0}^{\infty} \frac{C_n^a(x)}{n!} t^n. \tag{4.6}$$

Then $\sigma = \Delta$ and $A(t) = e^{-at}$. Since the corresponding basic sequence is the Stirling polynomial set $\{x^{[n]}\}_{n\geq 0}$, we have $l_{ij}(k) = k!\binom{i}{i+j-k}\binom{j}{i+j-k}$ by virtue of (3.3). The linear functional for which the orthogonality holds is [34]

$$\widehat{A}(\sigma)(f)(0) = \exp(a\Delta)(f)(0) = e^{-a} \sum_{j=0}^{\infty} \frac{a^j}{j!} f(j). \tag{4.7}$$

Then we have [17] $I_s = a^s s!$. According to (4.2), we obtain [39]

$$L_{ij}(k) = \sum_{s \geq 0} \frac{i! \, j! \, a^s}{(s+k-i)!(s+k-j)!(s)!(i+j-k-2s)!}$$

$$= \frac{i! \, j!}{(i+j-k)!(k-i)!(k-j)!} \, {}_2F_2 \left(\begin{array}{c} \dfrac{k-i-j}{2}, \dfrac{k-i-j+1}{2} \\ k-i+1, k-j+1 \end{array} ; 4a \right). \tag{4.8}$$

4.3. Laguerre polynomials. The Laguerre polynomials $\{L_n^\alpha\}_{n \geq 0}$ are generated by [17]

$$(1-t)^{-\alpha-1} \exp\left(x\frac{t}{t-1} \right) = \sum_{n=0}^{\infty} L_n^\alpha(x) t^n. \tag{4.9}$$

Then the lowering and transfer operators for $\{n! L_n^\alpha\}_{n \geq 0}$ are, respectively, $\sigma = D/(D-1)$ and $A(\sigma) = (1-\sigma)^{-\alpha-1}$. The linear functional for which the orthogonality holds is [34]

$$\hat{A}(\sigma)(f)(0) = (1-\sigma)^{\alpha+1}(f)(0) = \frac{1}{\Gamma(\alpha+1)} \int_0^{+\infty} f(x) x^\alpha e^{-x} dx, \quad \alpha > -1. \tag{4.10}$$

For this case we have [17, 23] $I_s = s!(\alpha+1)_s$.

According to (3.11) and (4.2), we deduce the linearization coefficients for $\{L_n^\alpha\}_{n \geq 0}$,

$$L_{ij}(k) = \frac{(-2)^{i+j-k}}{k!} \sum_{n,s} \frac{2^{-2(n+s)}(k)_n(\alpha+1)_s}{n! s! ((n+s)+k-i)!((n+s)+k-j)!(i+j-k-2(n+s))!}, \tag{4.11}$$

which, in view of the well-known relationship [31, 35]:

$$\sum_{n,m} \frac{(\rho)_n (\sigma)_m c_{n+m}}{m! n!} x^{n+m} = \sum_n \frac{(\rho+\sigma)_n c_n}{n!} x^n, \quad \{c_n\} \text{ being a sequence of complex numbers,} \tag{4.12}$$

assumes the form [28]

$$L_{ij}(k) = \frac{(-2)^{i+j-k}}{k!} \sum_{p \geq 0} \frac{(\alpha+1+k)_p 2^{-2p}}{p!(p+k-i)!(p+k-j)!(i+j-k-2p)!}$$

$$= \frac{(-2)^{i+j-k}}{k!(i+j-k)!(k-i)!(k-j)!} \, {}_3F_2 \left(\begin{array}{c} k+\alpha+1, \dfrac{k-i-j}{2}, \dfrac{k-i-j+1}{2} \\ k-i+1, k-j+1 \end{array} ; 1 \right). \tag{4.13}$$

4.4. Meixner polynomials. The Meixner polynomial set $\{M_n(x; \beta, a)\}_{n \geq 0}$ is generated by [17]

$$(1-t)^{-\beta} \left(\frac{1-t/a}{1-t} \right)^x = \sum_{n=0}^{\infty} M_n(x; \beta, a) \frac{t^n}{n!}. \tag{4.14}$$

It follows that $\sigma = (e^D - 1)/(e^D - 1/a)$ and $A(t) = (1-t)^{-\beta}$. The linear functional for which the orthogonality holds is [34]

$$\hat{A}(\sigma)(f)(0) = (1-\sigma)^{\beta}(f)(0) = (1-a)^{\beta} \sum_{j=0}^{\infty} \frac{(\beta)_j}{j!} a^j f(j). \tag{4.15}$$

For this case we have [18] $I_s = s!(\beta)_s a^{-s}$. According to (3.15), (4.2), and (4.12) we obtain [9]

$$L_{ij}(k) = i! j! \left(-1 - \frac{1}{a} \right)^{i+j-k} \sum_{p \geq 0} \frac{(\beta+k)_p (1+a)^{-2p} a^p}{p!(p+k-i)!(p+k-j)!(i+j-k-2p)!}$$

$$= \frac{i! j! \left(-1 - 1/a \right)^{i+j-k}}{(i+j-k)!(k-i)!(k-j)!} \,_3F_2 \left(\begin{matrix} k+\beta, \dfrac{k-i-j}{2}, \dfrac{k-i-j+1}{2} \\ k-i+1, k-j+1 \end{matrix} ; \frac{4a}{(a+1)^2} \right). \tag{4.16}$$

So, the linearization coefficients alternate in sign just as in the Laguerre polynomial set case.

4.5. Meixner-Pollaczek polynomials. The Meixner-Pollaczek polynomials are generated by [17]

$$\left[(1+\delta t)^2 + t^2 \right]^{-\eta/2} e^{x \arctan(t/(1+\delta t))} = \sum_{n=0}^{\infty} \frac{P_n(x,\delta,\eta)}{n!} t^n. \tag{4.17}$$

Then we have

$$\sigma = \frac{\tan D}{1 - \delta \tan D}, \qquad A(t) = \left((1+\delta t)^2 + t^2 \right)^{-\eta/2}. \tag{4.18}$$

To obtain the effect of the linear functional $\hat{A}(\sigma)$ on analytic functions, we need the following relation [39]:

$$\frac{1}{\pi \Gamma(\rho)} \int_{-\infty}^{+\infty} e^{-(\pi - 2t)x} \left| \Gamma \left(\frac{\rho + ix}{2} \right) \right|^2 dx = (2 \sin t)^{-\rho}, \quad \rho > 0. \tag{4.19}$$

It follows from (4.18) that

$$\hat{A}(\sigma(f)) = (\cos D - \delta \sin D)^{-\eta}(f) = \left(\frac{\sin(\pi/2 + D + \arctan \delta)}{\sin(\pi/2 + \arctan \delta)} \right)^{-\eta} (f)$$

$$= \frac{\int_{-\infty}^{+\infty} f(x) \omega(x) dx}{\int_{-\infty}^{+\infty} \omega(x) dx}, \tag{4.20}$$

where $\omega(x) = [\Gamma(\eta/2)]^{-2}|\Gamma(\eta + ix/2)|^2 \exp(-x\tan^{-1}\delta)$. So we have [17, 23] $I_s = (\delta^2 + 1)^s s!(\eta)_s$. According to (3.19), (4.2), and (4.12), we obtain [39]

$$
\begin{aligned}
L_{ij}(k) &= i!j!(2\delta)^{i+j-k} \sum_{p\geq 0} \frac{(\eta+k)_p (2\delta)^{-2p}(1+\delta^2)^p}{p!(p+k-i)!(p+k-j)!(i+j-k-2p)!} \\
&= \frac{i!j!(2\delta)^{i+j-k}}{(i+j-k)!(k-i)!(k-j)!} \, _3F_2\left(\begin{matrix} k+\eta, \dfrac{k-i-j}{2}, \dfrac{k-i-j+1}{2} \\ k-i+1, k-j+1 \end{matrix}; \dfrac{1+\delta^2}{\delta^2}\right).
\end{aligned}
\tag{4.21}
$$

Taking $\delta \mapsto 0$ and $\eta = 2\lambda$ in (4.21) and using the well-known relation [35], namely,

$$
\left(\frac{-2k+1}{2}\right)_k = (-1)^k \frac{(2k)!}{2^{2k}k!}, \quad k = 0,1,2,\ldots,
\tag{4.22}
$$

we obtain [3]

$$
p_i^{(\lambda)}(x)p_j^{(\lambda)}(x) = \sum_{k=0}^{\min(i,j)} \frac{\Gamma(i+j-2k+2\lambda+1)(i+j-2k)!}{k!(i-k)!(j-k)!} p_{i+j-2k}^{(\lambda)}(x),
\tag{4.23}
$$

where $p_n^{(\lambda)}(x) = (1/n!)P_n(x;0,2\lambda)$ designates the symmetric Meixner-Pollaczek polynomials.

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