# THE EQUIVALENCE BETWEEN THE MANN AND ISHIKAWA ITERATIONS DEALING WITH GENERALIZED CONTRACTIONS

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We show that the Mann and Ishikawa iterations are equivalently used to approximate fixed points of generalized contractions.



## 1. Introduction

Let $X$ be a real Banach space, $A$ a nonempty convex subset of $X$, $T$ a selfmap of $A$, and let $x_0 = u_0 \in A$. The Mann iteration (see [2]) is defined by

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n. \tag{1.1}$$

The Ishikawa iteration is defined (see [1]) by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$.

These methods were applied, in [3], to a class of functions $T$ satisfying the inequality

$$\|Tx - Ty\| \le Q(M(x,y)), \tag{1.3}$$

where $Q$ is a real-valued function satisfying

(a) $0 < Q(s) < s$ for each $s > 0$ and $Q(0) = 0$,

(b) $Q$ is nondecreasing on $(0,\infty)$,

(c) $g(s) := s/(s - Q(s))$ is nonincreasing on $(0,\infty)$,

$$M(x,y) := \max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|y-Tx\|\}. \tag{1.4}$$

In [4], the following conjecture was given: "if the Mann iteration converges, then so does the Ishikawa iteration." In a series of papers [4–8], the authors have given a positive answer to this conjecture, showing, for example, the equivalence between Mann and

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Ishikawa iterations for strongly and uniformly pseudocontractive maps. In this note, we show that the convergence of Mann iteration is equivalent to the convergence of Ishikawa iteration, used to approximate fixed points of a map which satisfies condition (1.3). Such a map is independent of the class of strongly pseudocontractive maps. The class of generalized contractions satisfying (1.3) generalizes the class of quasi-contractions, see, for example, [9]. Thus, our result generalizes the main [9, Theorem 1], which states the equivalence between Mann and Ishikawa iterations when applied to quasi-contractions.

Lemma 1.1 [10]. *Let $\{a_n\}$ be a nonnegative sequence which satisfies the following inequality:*

$$a_{n+1} \le (1-\lambda_n)a_n + \sigma_n, \tag{1.5}$$

*where $\lambda_n \in (0,1)$ for all $n \ge n_0$, $\sum_{n=1}^{\infty}\lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$. Then $\lim_{n\to\infty} a_n = 0$.*

The following result is a lemma from [3, page 351].

Lemma 1.2 [3]. *Let $A$ be a nonempty closed convex subset of a Banach space $X$, and $T$ a self-map of $A$ satisfying (1.3). Let $\{\alpha_n\}$ satisfy the conditions $\alpha_n > 0$ for all $n \ge 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{Tx_n\}$, $\{Ty_n\}$, and $\{Tu_n\}$ are bounded.*

## 2. Main result

Theorem 2.1. *Let $A$ be a nonempty closed convex subset of a Banach space $X$, and $T$ a self-map of $A$ satisfying (1.3). Let $\{\alpha_n\}$ satisfy the conditions $\alpha_n > 0$ for all $n \ge 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Denote by $x^*$ the unique fixed point of $T$. Then for $u_0 = x_0 \in A$, the following are equivalent:*

(i) *the Mann iteration (1.1) converges to $x^*$;*
(ii) *the Ishikawa iteration (1.2) converges to $x^*$.*

*Proof.* Lemma 1.2 assures that both Mann and Ishikawa iterations are bounded and hence, in order to prove the equivalence between (1.1) and (1.2), we need to prove that

$$\lim_{n\to\infty} ||x_n - u_n|| = 0. \tag{2.1}$$

Set

$$\begin{aligned} r_n = \max\{ &\sup(||x_n - Ty_j|| : j \ge n) \cup \sup(||u_n - Tu_j|| : j \ge n) \\ &\cup \sup(||x_n - Tu_j|| : j \ge n) \cup \sup(||u_n - Ty_j|| : j \ge n)\}. \end{aligned} \tag{2.2}$$

Then the following are true:

$$\begin{aligned} ||x_n - Ty_j|| &\le (1-\alpha_{n-1})||x_{n-1} - Ty_j|| + \alpha_{n-1}||Ty_{n-1} - Ty_j|| \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}Q(M(y_{n-1}, y_j)) \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}Q(r_{n-1}), \\ ||u_n - Tu_j|| &\le (1-\alpha_{n-1})||u_{n-1} - Tu_j|| + \alpha_{n-1}||Tu_{n-1} - Tu_j|| \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}Q(M(u_{n-1}, u_j)) \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}Q(r_{n-1}), \end{aligned} \tag{2.3}$$

moreover,

$$\begin{aligned}||x_n - Tu_j|| &\le (1-\alpha_{n-1})||x_{n-1} - Tu_j|| + \alpha_{n-1}||Ty_{n-1} - Tu_j|| \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}Q(M(y_{n-1}, u_j)) \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}Q(r_{n-1}),\end{aligned} \tag{2.4}$$

also,

$$\begin{aligned}||u_n - Ty_j|| &\le (1-\alpha_{n-1})||u_{n-1} - Ty_j|| + \alpha_{n-1}||Tu_{n-1} - Ty_j|| \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}Q(M(u_{n-1}, y_j)) \\ &\le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}Q(r_{n-1}).\end{aligned} \tag{2.5}$$

Eventually, one gets the following evaluation:

$$r_n \le (1-\alpha_{n-1})r_{n-1} + \alpha_{n-1}Q(r_{n-1}) \Longleftrightarrow \alpha_{n-1}g(r_{n-1}) \le r_{n-1} - r_n, \tag{2.6}$$

which implies that $\{r_n\}$ is nonincreasing in $n$ and positive. Hence, there exists $\lim_{n\to\infty} r_n$, denoted by $r \ge 0$. Suppose $r > 0$. From (2.6), we obtain

$$\alpha_{n-1}g(r) \le \alpha_{n-1}g(r_{n-1}) \le r_{n-1} - r_n \Longleftrightarrow g(r)\sum_{k=0}^{n}\alpha_k \le \sum_{k=0}^{n}(r_k - r_{k-1}) = r_0 - r_{n+1}. \tag{2.7}$$

The right-hand side is bounded and the left-hand side is unbounded. Thus, $r = 0$. Hence,

$$\begin{aligned}&\lim_{n\to\infty}||x_n - Tu_n|| = 0, \qquad \lim_{n\to\infty}||u_n - Ty_n|| = 0, \\ &\lim_{n\to\infty}||x_n - Ty_n|| = 0, \qquad \lim_{n\to\infty}||u_n - Tu_n|| = 0.\end{aligned} \tag{2.8}$$

Suppose now that the Mann iteration converges, then one has

$$\begin{aligned}||x_{n+1} - u_{n+1}|| &\le (1-\alpha_n)||x_n - u_n|| + \alpha_n||Ty_n - Tu_n|| \\ &\le (1-\alpha_n)||x_n - u_n|| + \alpha_n(||Ty_n - x_n|| + ||x_n - Tu_n||).\end{aligned} \tag{2.9}$$

Using (2.8) and (2.9) and Lemma 1.1, with

$$\begin{aligned}\lambda_n &:= ||x_n - u_n||, \\ \sigma_n &:= \alpha_n(||Ty_n - x_n|| + ||x_n - Tu_n||), \\ \sigma_n &= o(\alpha_n),\end{aligned} \tag{2.10}$$

we have $\lim_{n\to\infty}\lambda_n = 0$, that is, (2.1) holds. The relation

$$\|x_n - x^*\| \le \|x_n - u_n\| + \|x^* - u_n\| \longrightarrow 0 \tag{2.11}$$

leads to the conclusion that Ishikawa iteration converges too. Suppose now that the Ishikawa iteration converges, then one has

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\|Ty_n - Tu_n\| \\ &\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n(\|Ty_n - u_n\| + \|u_n - Tu_n\|).\end{aligned} \tag{2.12}$$

Using (2.8) and (2.12) and Lemma 1.1, with

$$\begin{gathered}\lambda_n := \|x_n - u_n\|, \\ \sigma_n := \alpha_n(\|Ty_n - u_n\| + \|u_n - Tu_n\|), \\ \sigma_n = o(\alpha_n),\end{gathered} \tag{2.13}$$

we have $\lim_{n\to\infty}\lambda_n = 0$, that is, (2.1) holds. The relation

$$\|u_n - x^*\| \le \|x_n - u_n\| + \|x_n - x^*\| \longrightarrow 0 \tag{2.14}$$

leads to the conclusion that Mann iteration converges too. □

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