MODULAR RELATIONS AND EXPLICIT VALUES OF RAMANUJAN-SELBERG CONTINUED FRACTIONS

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By employing a method of parameterizations for Ramanujan's theta-functions, we find several modular relations and explicit values of the Ramanujan-Selberg continued fractions.

1. Introduction

For $q := e^{2\pi i z}$, $\operatorname{Im}(z) > 0$, define Ramanujan's theta-functions as

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \vartheta_3(0, 2z), \tag{1.1}$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = 2^{-1} q^{-1/8} \vartheta_2(0, z), \tag{1.2}$$

$$f(-q) := (q;q)_\infty = q^{-1/24} \eta(z), \tag{1.3}$$

where ϑ_2 and ϑ_3 are classical theta-functions [8, page 464], η denotes the Dedekind eta-function, and $(a;q)_\infty$ is defined by

$$(a;q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k). \tag{1.4}$$

Now, Ramanujan-Selberg continued fraction $S_1(q)$ is defined by

$$S_1(q) := \frac{q^{1/8}\psi(q)}{\phi(q)} = \frac{q^{1/8}}{1} \, \frac{q}{+1+q} \, \frac{q^2}{+1+q^2} \, \frac{q^3}{+1+q^3} {}_{+\cdots}, \quad |q| < 1. \tag{1.5}$$

This continued fraction was recorded by Ramanujan at the beginning of Chapter 19 of his second notebook [1, page 221]. The equality in (1.5) was proved by Ramanathan [3].

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Closely related to $S_1(q)$ is the continued fraction $H(q)$ [7, page 82] defined by

$$H(q) := \frac{f(-q)}{q^{1/8}f(-q^4)} = q^{1/8} - \frac{q^{7/8}}{1-q}\,{}_{+}\,\frac{q^2}{1+q^2}\,{}_{-}\,\frac{q^3}{1-q^3}\,{}_{+}\,\frac{q^4}{1+q^4}\,{}_{-\cdots}, \quad |q|<1. \tag{1.6}$$

By [1, page 115, Entry 8(xii)] and (1.6), we find that

$$H(q) = \frac{\phi(-q^2)}{q^{1/8}\psi(q)}. \tag{1.7}$$

Also, employing (1.2) and [1, page 37, equation (22.4)], we have

$$H(q) = \frac{(q;q^2)_\infty}{q^{1/8}(-q^2;q^2)_\infty}. \tag{1.8}$$

Again, for $|q|<1$, define

$$N(q) := 1 + \frac{q}{1}\,{}_{+}\,\frac{q+q^2}{1}\,{}_{+}\,\frac{q^3}{1}\,{}_{+}\,\frac{q^2+q^4}{1}\,{}_{+\cdots}. \tag{1.9}$$

In his notebook [4, page 290], Ramanujan asserted that

$$N(q) = \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}. \tag{1.10}$$

This formula was first proved in print by Selberg [6].

In his lost notebook, Ramanujan [5, page 44] also stated that if $|q|<1$ and

$$L(q) = \frac{1+q}{1}\,{}_{+}\,\frac{q^2}{1}\,{}_{+}\,\frac{q+q^3}{1}\,{}_{+}\,\frac{q^4}{1}\,{}_{+\cdots}, \tag{1.11}$$

then

$$L(q) = \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}. \tag{1.12}$$

From (1.5) and (1.9)–(1.12), we easily see that

$$S_1(q) = \frac{q^{1/8}}{N(q)} = \frac{q^{1/8}}{L(q)} = \frac{q^{1/8}(-q^2;q^2)_\infty}{(-q;q^2)_\infty}. \tag{1.13}$$

By setting

$$T(q) := \frac{q^{1/8}}{1}\,{}_{+}\,\frac{-q}{1}\,{}_{+}\,\frac{-q+q^2}{1}\,{}_{+}\,\frac{-q^3}{1}\,{}_{+\cdots}, \tag{1.14}$$

we also note that

$$T(q) = \frac{q^{1/8}}{N(-q)} = \frac{q^{1/8}}{L(-q)} = \frac{q^{1/8}(-q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{1.15}$$

In this paper, we find several modular relations connecting the above continued fractions in different arguments. We present these in Sections 3–5.

We observe that Vasuki and Shivashankara [7] had found explicit values of $H(e^{-\pi\sqrt{n}})$ for $n = 3, 1/3, 5, 1/5, 7, 1/7, 13$, and $1/13$ by using eta-function identities and transformation formulas. In Sections 6 and 7, we also find several new explicit values of $H(e^{-\pi\sqrt{n}})$ by using the parameter J_n, defined by

$$J_n = \frac{f(-q)}{\sqrt{2}q^{1/8}f(-q^4)}, \quad q := e^{-\pi\sqrt{n}}, \tag{1.16}$$

where n is any positive real number. We note that the parameter J_n is equivalent to the parameter $r_{4,n}$, which is a particular case of the parameter $r_{k,n}$, introduced by Yi [10, page 4, equation (1.11)] (see also [9, page 11, equation (2.1.1)]), and defined by

$$r_{k,n} := \frac{f(-q)}{k^{1/4}q^{(k-1)/24}f(-q^k)}, \quad q = e^{-2\pi\sqrt{n/k}}, \tag{1.17}$$

where n and k are positive real numbers.

We note that Zhang [11, page 11, Theorems 2.1 and 2.2] established general formulas for explicit evaluations of $S_1(e^{-\pi\sqrt{n}})$ and $T(e^{-\pi\sqrt{n}})$ in terms of Ramanujan's singular moduli. In fact, he proved that

$$S_1(q) = \frac{\alpha_n^{1/8}}{\sqrt{2}}, \tag{1.18}$$

$$T(q) = \frac{1}{\sqrt{2}}\left(\frac{\alpha_n}{1-\alpha_n}\right)^{1/8}, \tag{1.19}$$

where $q = e^{-\pi\sqrt{n}}$ and the singular modulus α_n is that unique positive number between 0 and 1 satisfying

$$\sqrt{n} = \frac{{}_2F_1(1/2, 1/2; 1; 1-\alpha_n)}{{}_2F_1(1/2, 1/2; 1; \alpha_n)}. \tag{1.20}$$

In Section 8, we establish general formulas for explicit evaluations of $S_1(e^{-\pi\sqrt{n}})$ and $S_1(e^{-\pi/\sqrt{n}})$ in terms of the parameter $r_{k,n}$. We also give some particular examples.

Since Ramanujan's modular equations are central in our evaluations, we now give the definition of a modular equation as given by Ramanujan. Let $K, K' := K(k'), L$, and $L' := L(l')$ denote the complete elliptic integral of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, l, and $l' := \sqrt{1-l^2}$, respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.21}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is implied by (1.21).

If we set

$$q = \exp\left(-\pi\frac{K'}{K}\right), \qquad q' = \exp\left(-\pi\frac{L'}{L}\right), \tag{1.22}$$

we see that (1.21) is equivalent to the relation $q^n = q'$. Thus, a modular equation can be viewed as an identity involving theta-functions at the arguments q and q^n. Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α. The multiplier m connecting α and β is defined by

$$m = \frac{K}{L}. \tag{1.23}$$

If $q = \exp(-\pi K'/K)$, one of the most fundamental relations in the theory of elliptic functions is given by the formula [1, pages 101-102]

$$\phi^2(q) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, k^2\right) = \frac{2}{\pi}K(k). \tag{1.24}$$

So the multiplier m of degree n defined in (1.23) can also be written as

$$m = \frac{z_1}{z_n}, \tag{1.25}$$

where $z_r = \phi^2(q^r)$.

Again, if we put $q = \exp(-\pi K'/K)$, $z = z_1$, and $x = \alpha$ in [1, Entries 10(i), 11(i), 12(i), 12(ii), and 12(iv), pages 122–124], then we have the representations

$$\phi(q) = \sqrt{z_1}, \tag{1.26}$$

$$\psi(q) = \sqrt{\frac{z_1}{2}}\left(\frac{\alpha}{q}\right)^{1/8}, \tag{1.27}$$

$$f(q) = \sqrt{z_1}2^{-1/6}\left(\frac{\alpha(1-\alpha)}{q}\right)^{1/24}, \tag{1.28}$$

$$f(-q) = \sqrt{z_1}2^{-1/6}\left(\frac{(1-\alpha)^4\alpha}{q}\right)^{1/24}, \tag{1.29}$$

$$f(-q^4) = \sqrt{z_1}2^{-2/3}\left(\frac{(1-\alpha)\alpha^4}{q^4}\right)^{1/24}, \tag{1.30}$$

respectively. It is to be noted that if we replace q by q^n, then z_1 and α will be replaced by z_n and β, respectively, where β has degree n over α.

In the next section, we give the values of $r_{k,n}$, eta-function identities and modular equations, which will be used in our subsequent sections.

2. Some values of $r_{k,n}$ and modular equations

In the following theorem, we record the values of $r_{k,n}$ from [9].

Theorem 2.1 (Yi [9]). *One has*

$$\begin{gathered}
r_{2,1}=1, \qquad r_{2,2}=2^{1/8}, \qquad r_{2,3}=\left(1+\sqrt{2}\right)^{1/6}, \qquad r_{2,4}=2^{1/8}\left(1+\sqrt{2}\right)^{1/8}, \\
r_{2,5}=\sqrt{\frac{1+\sqrt{5}}{2}}, \qquad r_{2,6}=2^{1/24}\left(\sqrt{3}+1\right)^{1/4}, \qquad r_{2,8}=2^{3/16}\left(1+\sqrt{2}\right)^{1/4}, \\
r_{2,9}=\left(\sqrt{2}+\sqrt{3}\right)^{1/3}, \qquad r_{2,10}=\left(\frac{1}{2}\left(1+\sqrt{5}\right)\left(\sqrt{\sqrt{5}+1}+\sqrt{2}\right)\right)^{1/4}, \\
r_{2,12}=\left(1+\sqrt{2}\right)^{5/24}\left(2\left(1+\sqrt{2}+\sqrt{6}\right)\right)^{1/8}, \qquad r_{2,16}=2^{1/8}\left(1+\sqrt{2}\right)^{1/4}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8}, \\
r_{2,18}=\frac{\left(1+\sqrt{3}\right)^{1/3}\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}{2^{11/24}} \qquad r_{2,20}=\frac{\left(1+\sqrt{5}\right)^{5/8}\left(2+3\sqrt{2}+\sqrt{5}\right)^{1/8}}{\sqrt{2}}, \\
r_{2,32}=2^{3/16}\left(1+\sqrt{2}\right)^{1/4}\left(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4}\right)^{1/8}, \\
r_{2,36}=\frac{\left\{2\left(1+35\sqrt{2}-28\sqrt{3}\right)\right\}^{1/8}}{\left(\sqrt{3}-\sqrt{2}\right)^{2/3}} \qquad r_{2,50}=\frac{2^{5/8}}{5^{1/4}-1}, \\
r_{2,72}=\frac{\left(\sqrt{2}+\sqrt{3}\right)^{1/3}\left(-\sqrt{2}+4+2\sqrt{3}+3^{3/4}\left(\sqrt{3}+1\right)\right)^{1/3}}{2^{13/48}\left(\sqrt{2}-1\right)^{5/12}}, \\
r_{2,3/2}=\frac{\left(1+\sqrt{3}\right)^{1/4}}{2^{7/24}}, \qquad r_{2,5/2}=\frac{\left(\sqrt{\sqrt{5}+1}+\sqrt{2}\right)^{1/4}}{2^{1/4}}, \\
r_{2,7/2}=\frac{\left(3+\sqrt{7}\right)^{1/4}}{2^{3/8}}, \qquad r_{2,9/2}=\frac{\left(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4}\right)^{1/3}}{2^{13/24}}, \qquad r_{2,25/2}=\frac{5^{1/4}+1}{2^{5/8}}.
\end{gathered} \tag{2.1}$$

Note that is we have recorded the corrected version of $r_{2,72}$ that is given by Yi [9].

From [9, page 12, Theorem 2.1.2(i)–(iii)], we also note that $r_{k,1}=1$, $r_{k,1/n}=1/r_{k,n}$, and $r_{k,n}=r_{n,k}$, where k and n are any positive real numbers.

In the next three theorems, we state three eta-function identities of Yi [9].

Theorem 2.2 (Yi [9, page 36, Theorem 3.5.1]). *If*

$$P=\frac{f(-q)}{q^{1/8}f\left(-q^{4}\right)}, \qquad Q=\frac{f\left(-q^{2}\right)}{q^{1/4}f\left(-q^{8}\right)}, \tag{2.2}$$

then

$$(PQ)^{4}+\frac{4}{PQ}^{4}=\left(\frac{Q}{P}\right)^{12}-16\left(\frac{Q}{P}\right)^{4}-16\left(\frac{P}{Q}\right)^{4}. \tag{2.3}$$

Theorem 2.3 (Yi [9, page 37, Theorem 3.5.2]). *If*

$$P=\frac{f(-q)}{q^{1/8}f\left(-q^{4}\right)}, \qquad Q=\frac{f\left(-q^{3}\right)}{q^{3/8}f\left(-q^{12}\right)}, \tag{2.4}$$

then

$$PQ+\frac{4}{PQ}=\left(\frac{Q}{P}\right)^2+\left(\frac{P}{Q}\right)^2. \tag{2.5}$$

Theorem 2.4 (Yi [9, page 38, Theorem 3.5.3]). *If*

$$P=\frac{f(-q)}{q^{1/8}f(-q^4)},\qquad Q=\frac{f(-q^5)}{q^{5/8}f(-q^{20})}, \tag{2.6}$$

then

$$(PQ)^2+\left(\frac{4}{PQ}\right)^2=\left(\frac{Q}{P}\right)^3-5\left(\frac{Q}{P}+\frac{P}{Q}\right)+\left(\frac{P}{Q}\right)^3. \tag{2.7}$$

The remaining theorems of this section are devoted to stating some modular equations of Ramanujan.

Theorem 2.5 (Berndt [1, page 230, Entry 5(ii)]). *If β has degree 3 over α, then*

$$(\alpha\beta)^{1/4}+((1-\alpha)(1-\beta))^{1/4}=1. \tag{2.8}$$

Theorem 2.6 (Berndt [1, page 282, Entry 13(xv)]). *If β has degree 5 over α, then*

$$\left(Q-\frac{1}{Q}\right)^3+8\left(Q-\frac{1}{Q}\right)=4\left(P-\frac{1}{P}\right), \tag{2.9}$$

where $P=(\alpha\beta)^{1/4}$ and $Q=(\beta/\alpha)^{1/8}$.

Theorem 2.7 (Berndt [1, page 314, Entry 19(i)]). *If β has degree 7 over α, then*

$$(\alpha\beta)^{1/8}+((1-\alpha)(1-\beta))^{1/8}=1. \tag{2.10}$$

Theorem 2.8 (Berndt [1, page 363, Entry 7(i)]). *If β has degree 11 over α, then*

$$(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}+2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}=1. \tag{2.11}$$

Theorem 2.9 (Berndt [2, page 387, Entry 62]). *Let P, Q, and R be defined by*

$$\begin{gathered}P=1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)},\\ Q=64\left(\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}-\sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right),\\ R=32\sqrt{\alpha\beta(1-\alpha)(1-\beta)},\end{gathered} \tag{2.12}$$

respectively. Then, if β has degree 13 over α,

$$\sqrt{P}(P^3+8R)-\sqrt{R}(11P^2+Q)=0. \tag{2.13}$$

Theorem 2.10 (Berndt [2, page 385, Entry 53]). *Let*

$$P = 1 + (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8},$$
$$Q = 4\Big((\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\Big), \tag{2.14}$$
$$R = 4\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}.$$

Then, if β has degree 15 over α,

$$P(P^2 - Q) + R = 0. \tag{2.15}$$

Theorem 2.11 (Berndt [2, page 387, Entry 62]). *Let P, Q, and R be defined as in Theorem 2.9, then if β has degree 17 over α,*

$$P^3 - R^{1/3}(10P^2 + Q) + 13R^{2/3}P + 12R = 0. \tag{2.16}$$

Theorem 2.12 (Berndt [2, page 386, Entry 58]). *Let*

$$P = 1 - (\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4},$$
$$Q = 16\Big((\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}\Big), \tag{2.17}$$
$$R = 16\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}.$$

Then, if β has degree 19 over α,

$$P^5 - 7P^2R - QR = 0. \tag{2.18}$$

Theorem 2.13 (Berndt [1, page 411, Entry 15(i)]). *If β has degree 23 over α, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1. \tag{2.19}$$

Theorem 2.14 (Berndt [2, page 385, Entry 54]). *Let P, Q, and R be defined in as Theorem 2.10. If β has degree 31 over α, then*

$$P^2 - Q = \sqrt{PR}. \tag{2.20}$$

3. Relations between $H(q)$ and $H(q^n)$

In this section, we state and prove some relations between $H(q)$ and $H(q^n)$.

Theorem 3.1. *One has*

(i) $\alpha = 16/(16 + H^8(q))$,

(ii) $\beta = 16/(16 + H^8(q^n))$,

where β has degree n over α.

Proof. We apply (1.29) and (1.30) in the definition of $H(q)$ in (1.6) to complete the proof. □

Theorem 3.2. *One has*

(i) $\alpha = -16/(H^8(-q))$,

(ii) $\beta = -16/(H^8(-q^n))$,

where β has degree n over α.

Proof. We replace q by $-q$ in the definition of $H(q)$ and then employ (1.28) and (1.30) to arrive at the desired result. □

Remark 3.3. By Theorem 3.1 and for any given modular equation of degree n, we can obtain a relation between $H(q)$ and $H(q^n)$. In the following theorem, we illustrate this with $n = 3, 5$, and 7 in (iii), (iv), and (v), respectively.

Theorem 3.4. *Let $a = H(q)$, $b = H(-q)$, $c = H(q^2)$, $u = H(q^3)$, $v = H(q^5)$, and $w = H(q^7)$. Then one has*

(i) $a^8 + b^8 + 16 = 0$,

(ii) $256a^8 + 16a^{16} + 16a^8c^8 + a^{16}c^8 - c^{16} = 0$,

(iii) $a^4 - 4au - a^3u^3 + u^4 = 0$,

(iv) $a^6 - 16av - 5a^4v^2 - 5a^2v^4 - a^5v^5 + v^6 = 0$,

(v) $a^8 - 64aw - 112a^2w^2 - 112a^3w^3 - 70a^4w^4 - 28a^5w^5 - 7a^6w^6 + a^7w^7 + w^8 = 0$.

Proof. From Theorem 3.1(i) and Theorem 3.2, we easily arrive at (i). To prove (iii)–(v), we employ Theorem 3.1 in Theorems 2.5, 2.6, and 2.7, respectively. We note that (ii)–(iv) can also be proved by employing Theorems 2.2–2.4. □

4. Relations between $S_1(q)$ and $S_1(q^n)$

Theorem 4.1. *One has*

(i) $\alpha = 16S_1^8(q)$,

(ii) $\beta = 16S_1^8(q^n)$,

(iii) $\alpha = 16T^8(q)/(1 + 16T^8(q))$,

where β has degree n over α.

Proof. To prove (i) and (ii), we employ (1.26) and (1.27) in the definition of $S_1(q)$ in (1.5). Proof of (iii) follows easily from (1.19). □

Remark 4.2. For any given modular equation of degree n, we can easily obtain the relations connecting $S_1(q)$ and $S_1(q^n)$ by using Theorem 4.1. We give some examples in the following theorem.

Theorem 4.3. *Let $U = S_1(q)$, $V = S_1(q^3)$, $W = S_1(q^5)$, and $X = S_1(q^7)$. Then, one has*

(i) $U^4 - UV + 4U^3V^3 - V^4 = 0$,

(ii) $U^6 - UW + 5U^4W^2 - 5U^2W^4 + 16U^5W^5 - W^6 = 0$,

(iii) $U^8 + X^8 - UX + 7U^2X^2 - 28U^3X^3 + 70U^4X^4 - 112U^5X^5 + 112U^6X^6 - 64U^7X^7 = 0$.

Proof. Employing Theorem 4.1 in Theorems 2.5–2.7, we readily deduce (i)–(iii), respectively. □

5. Relations connecting $H(\pm q)$, $S_1(q)$ and $T(q)$

THEOREM 5.1. *Let $u = H(q)$, $x = H(-q)$, $v = S_1(q)$, and $y = T(q)$. One has*
(i) $u^8 v^8 + 16v^8 - 1 = 0$,
(ii) $x^8 u^8 + 1 = 0$,
(iii) $u = 1/y$,
(iv) $x^8 y^8 + 16y^8 + 1 = 0$.

Proof. (i) follows from Theorem 3.1(i) and Theorem 4.1(i). To prove (ii), we use Theorem 3.2(i) and Theorem 4.1(i). To prove (iii), we employ Theorem 3.1(i) and Theorem 4.1(iii). Finally, employing Theorem 3.2(i) and Theorem 4.1(iii), we easily arrive at (iv). □

6. Theorems on J_n and explicit values

In this section, we establish some general theorems for the explicit evaluations of J_n and find some of its explicit values.

First we recall the following transformation formula for $f(-q)$ from [1, page 43, Entry 27(iii)]. Let $\alpha\beta = \pi^2$, then

$$e^{-\alpha/12}\sqrt[4]{\alpha} f\left(-e^{-2\alpha}\right) = e^{-\beta/12}\sqrt[4]{\beta} f\left(-e^{-2\beta}\right). \tag{6.1}$$

THEOREM 6.1. *If J_n is defined as in (1.16), then one has*

$$J_1 = 1, \qquad J_{1/n} = \frac{1}{J_n}. \tag{6.2}$$

The proof follows directly from (6.1) and the definition of J_n.

THEOREM 6.2. *One has*
(i) $16((J_nJ_{4n})^4 + 1/(J_nJ_{4n})^4) = (J_n/J_{4n})^{12} - 16(J_{4n}/J_n)^4 - 16(J_n/J_{4n})^4$,
(ii) $2(J_nJ_{9n} + 1/J_nJ_{9n}) = (J_{9n}/J_n)^2 + (J_n/J_{9n})^2$,
(iii) $4((J_nJ_{25n})^2 + 1/(J_nJ_{25n})^2) = (J_{25n}/J_n)^3 - 5(J_{25n}/J_n) - 5(J_n/J_{25n}) + (J_n/J_{25n})^3$,
(iv) $(1 + J_nJ_{49n})^8 - (1 + J_n^8)(1 + J_{49n}^8) = 0$.

Proof. Employing the definition J_n in Theorems 2.2–2.4, and 2.7, we complete the proof of (i)–(iv), respectively. □

THEOREM 6.3. *One has*
(i) $J_2 = 2^{1/8}(1 + \sqrt{2})^{1/8}$,
(ii) $J_3 = (2 + \sqrt{3})^{1/4}$,
(iii) $J_4 = 2^{5/16}(1 + \sqrt{2})^{1/4}$,
(iv) $J_5 = (1/\sqrt{2})(1 + \sqrt{5} + \sqrt{2(1 + \sqrt{5})})^{1/2}$,
(v) $J_7 = (8 + 3\sqrt{7})^{1/4}$,
(vi) $J_9 = 1/2 + 3^{1/4}/\sqrt{2} + \sqrt{3}/2$,
(vii) $J_{25} = (1/2)(3 + \sqrt[4]{5} + \sqrt{5} + \sqrt[4]{5^3})$,
(viii) $J_{49} = (1/4)(\sqrt{4 + \sqrt{7} + \sqrt{21 + 8\sqrt{7}}} + \sqrt{\sqrt{7} + \sqrt{21 + 8\sqrt{7}}})^2$,
(ix) $J_8 = 2^{1/4}(1 + \sqrt{2})^{3/8}(4 + \sqrt{2 + 10\sqrt{2}})^{1/8}$.

Proof. First we set $n = 1/2, 1/3, 1, 1/5, 1/7, 1, 1,$ and 1 in Theorem 6.2(i), Theorem 6.2(ii), Theorem 6.2(i), Theorem 6.2(iii), Theorem 6.2(iv), Theorem 6.2(ii), Theorem 6.2(iii), and Theorem 6.2(iv), respectively, and then simplify by using Theorem 6.1. Solving the resulting polynomial equations, we readily arrive at (i)–(viii).

Setting $n = 2$ in Theorem 8.3(i), employing the value of J_2 in (i), and solving the resulting equation, we deduce (ix). □

Remark 6.4. From Theorem 6.1 and the above theorem, the values of J_n for $n = 1/2, 1/3, 1/4, 1/5, 1/7, 1/9, 1/25, 1/49$, and $1/8$ also follow immediately.

Theorem 6.5. *One has*

(i) $J_6 = r_{4,6} = (1+\sqrt{2})^{3/8}(2(1+\sqrt{2}+\sqrt{6}))^{1/8}$,

(ii) $J_{10} = (1+\sqrt{5})^{9/8}(2+3\sqrt{2}+\sqrt{5})^{1/8}/2$,

(iii) $J_{16} = 2^{3/8}(1+\sqrt{2})^{1/2}(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4})^{1/8}$,

(iv) $J_{18} = 2^{1/8}(\sqrt{3}+\sqrt{2})(1+35\sqrt{2}-28\sqrt{3})^{1/8}$,

(v) $J_{36} = (\sqrt{3}+1)^{2/3}(-\sqrt{2}+4+2\sqrt{3}+3^{3/4}(\sqrt{3}+1))^{1/3}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3}/2^{35/48}(\sqrt{3}-\sqrt{2})^{1/3}(\sqrt{2}-1)^{5/12}$.

Proof. First we recall from [9, page 14, Corollary 2.1.5(i)] that

$$r_{k^2,n} = r_{k,nk}r_{k,n/k}. \tag{6.3}$$

Setting $k = 2$ and $n = 6$ in (6.3), we obtain

$$r_{4,6} = r_{2,12}\cdot r_{2,3}. \tag{6.4}$$

Now, from Section 2, we recall that

$$r_{2,3} = \left(1+\sqrt{2}\right)^{1/6}, \qquad r_{2,12} = \left(1+\sqrt{2}\right)^{5/24}\left(2\left(1+\sqrt{2}+\sqrt{6}\right)\right)^{1/8}. \tag{6.5}$$

Substituting these in (6.4), we complete the proof of (i).

The proofs of (ii)–(v) can be given in a similar fashion. □

Remark 6.6. By using Theorem 6.1 and the above theorem, we can easily evaluate $J_{1/n}$ for $n = 6, 10, 16, 18$, and 36.

Theorem 6.7. *One has*

(i) $J_{11} = ((1+\sqrt{1-4a^{12}})/2a^6)^{1/4}$, *where* $a = -(2^{1/3}/3)+(1/6)(38-6\sqrt{33})^{1/3}+(19+3\sqrt{33})^{1/3}/(3\cdot 2^{2/3})$,

(ii) $J_{13} = (18+5\sqrt{13}+6\sqrt{18+5\sqrt{13}})^{1/4}$,

(iii) $J_{15} = ((16+\sqrt{3}(7+\sqrt{5}))/(7-3\sqrt{5}))^{1/4}$,

(iv) $J_{17} = ((2+\sqrt{4-4(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}})^2})/(40+10\sqrt{17}-4\sqrt{206+50\sqrt{17}}))^{1/4}$,

(v) $J_{19} = ((1+\sqrt{1-4k^4})/2k^2)^{1/4}$, *where* $k = (1/24)(-20+(2944-384\sqrt{57})^{1/3}+4(46+6\sqrt{57})^{1/3})$,

(vi) $J_{23} = ((1+\sqrt{1-4n^{24}})/2n^{12})^{1/4}$, *where* $n = -1/(3\cdot 2^{1/3})+(1/6)(50-6\sqrt{69})^{1/3}+(25+3\sqrt{69})^{1/3}/(3\cdot 2^{2/3})$,

(vii) $J_{31} = ((1+\sqrt{1-4d^8})/2d^4)^{1/4}$, *where* $d = 1/2+(1/6)(-27+3\sqrt{93}/2)^{1/3}-1/(2^{2/3}(-27+3\sqrt{93})^{1/3})$.

Proof of (i). Using the definition of J_n in Theorem 3.1, we find that

$$\alpha = \frac{1}{1+J_n^8}, \qquad \beta = \frac{1}{1+J_{121n}^8}, \tag{6.6}$$

where β has degree 11 over α.

Setting $n = 1/11$ in (6.6) and simplifying by using Theorem 6.1, we find that

$$\alpha = J_{11}^8\beta, \qquad \beta = \frac{1}{1+J_{11}^8}, \qquad 1-\alpha = \beta, \qquad \alpha\beta = J_{11}^8\beta^2. \tag{6.7}$$

Substituting (6.7) in Theorem 2.8 and simplifying, we obtain

$$2\left(J_{11}^4\beta\right)^{1/2} + 2^{4/3}\left(J_{11}^4\beta\right)^{1/3} - 1 = 0. \tag{6.8}$$

Solving the above polynomial equation for real positive $a := (J_{11}^4\beta)^{1/6}$, we obtain

$$a = -\frac{2^{1/3}}{3} + \frac{1}{6}\left(38 - 6\sqrt{33}\right)^{1/3} + \frac{\left(19+3\sqrt{33}\right)^{1/3}}{3\cdot 2^{2/3}}. \tag{6.9}$$

Then, from (6.7) and (6.9), we arrive at

$$a^6 J_{11}^8 - J_{11}^4 + a^6 = 0. \tag{6.10}$$

Solving (6.10) for J_{11}, we complete the proof of (i).

Similarly, we can prove (ii)–(vii) by using the definition of J_n in Theorem 3.1, setting $n = 1/13, 1/15, 1/17, 1/19, 1/23$, and $1/31$, in turn, and then appealing to Theorems 2.9–2.14, respectively. □

Remark 6.8. By Theorem 6.1 and the above theorem, the values of $J_{1/n}$ for $n = 11, 13, 15, 17, 19, 23$, and 31 can also be found easily.

7. Explicit values of $H(q)$

In this section, we establish a general formula for the explicit evaluation of $H(e^{-\pi\sqrt{n}})$ and find some explicit values by using the particular values of J_n evaluated in the above section.

Theorem 7.1. *One has*

$$H\left(e^{-\pi\sqrt{n}}\right) = \sqrt{2}J_n. \tag{7.1}$$

Proof. The proof follows directly from the definitions of $H(q)$ and J_n. □

Theorem 7.2. *One has*

(i) $H(e^{-\pi}) = \sqrt{2}$,
(ii) $H(e^{-\pi\sqrt{2}}) = 2^{5/8}(1+\sqrt{2})^{1/8}$,
(iii) $H(e^{-\pi\sqrt{3}}) = \sqrt{2}(2+\sqrt{3})^{1/4}$,
(iv) $H(e^{-2\pi}) = 2^{13/16}(1+\sqrt{2})^{1/4}$,
(v) $H(e^{-\pi\sqrt{5}}) = (1+\sqrt{5}+\sqrt{2}\sqrt{1+\sqrt{5}})^{1/2}$,

(vi) $H(e^{-\pi\sqrt{7}}) = \sqrt{2}(8+3\sqrt{7})^{1/4}$,
(vii) $H(e^{-3\pi}) = (1+\sqrt{2}\sqrt[4]{3}+\sqrt{3})/\sqrt{2}$,
(viii) $H(e^{-5\pi}) = (3+\sqrt[4]{5}+\sqrt{5}+\sqrt[4]{5^3})/\sqrt{2}$,
(ix) $H(e^{-7\pi}) = (1/(2\sqrt{2}))(\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}+\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}})^2$,
(x) $H(e^{-2\sqrt{2}\pi}) = 2^{3/4}(1+\sqrt{2})^{3/8}(4+\sqrt{2+10\sqrt{2}})^{1/8}$.

Proof. Employing the value that $J_1 = 1$ in Theorem 7.1, we arrive at (i). To prove (ii)–(x), we employ the values of J_n from Theorem 6.3 in Theorem 7.1. □

Remark 7.3. From Theorems 6.1 and 7.1, it is obvious that

$$H(e^{-\pi/\sqrt{n}}) = \sqrt{2}J_{1/n} = \frac{\sqrt{2}}{J_n}. \tag{7.2}$$

So by employing the values of J_n from Theorem 6.3 in (7.2), we can easily evaluate $H(e^{-\pi/\sqrt{n}})$ for $n = 2,3,4,5,7,9,25,49$, and 8. For examples

$$\begin{gathered} H(e^{-\pi/2}) = 2^{3/16}\left(\sqrt{2}-1\right)^{1/4}, \qquad H(e^{-\pi/\sqrt{5}}) = \left(1+\sqrt{5}-\sqrt{2}\sqrt{1+\sqrt{5}}\right)^{1/2}, \\ H(e^{-\pi/7}) = \frac{1}{2\sqrt{2}}\left(\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}-\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}}\right)^2. \end{gathered} \tag{7.3}$$

Theorem 7.4. *One has*
(i) $H(e^{-\pi\sqrt{6}}) = \sqrt{2}(1+\sqrt{2})^{3/8}(2(1+\sqrt{2}+\sqrt{6}))^{1/8}$,
(ii) $H(e^{-\pi\sqrt{10}}) = ((1+\sqrt{5})^{9/8}(2+3\sqrt{2}+\sqrt{5})^{1/8})/\sqrt{2}$,
(iii) $H(e^{-4\pi}) = 2^{7/8}(\sqrt{2}+1)^{1/2}(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4})^{1/8}$,
(iv) $H(e^{-3\sqrt{2}\pi}) = 2^{5/8}(\sqrt{3}+\sqrt{2})(1+35\sqrt{2}-28\sqrt{3})^{1/8}$,
(v) $H(e^{-6\pi}) = (\sqrt{3}+1)^{2/3}(-\sqrt{2}+4+2\sqrt{3}+3^{3/4}(\sqrt{3}+1))^{1/3}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3}/(2^{11/48}(\sqrt{3}-\sqrt{2})^{1/3}(\sqrt{2}-1)^{5/12})$.

Proof. We use the values of J_n from Theorem 6.5 in Theorem 7.1 to complete the proof. □

The values of $H(e^{-\pi/\sqrt{n}})$ for $n = 6,10,18$, and 36 also follow from Theorem 6.5 and (7.2).

Theorem 7.5. *One has*
(i) $H(e^{-\pi\sqrt{11}}) = \sqrt{2}((1+\sqrt{1-4a^{12}})/2a^6)^{1/4}$, *where* $a = -2^{1/3}/3+(1/6)(38-6\sqrt{33})^{1/3}+(19+3\sqrt{33})^{1/3}/(3\cdot 2^{2/3})$,
(ii) $H(e^{-\pi\sqrt{13}}) = \sqrt{2}(18+5\sqrt{13}+6\sqrt{18+5\sqrt{13}})^{1/4}$,
(iii) $H(e^{-\pi\sqrt{15}}) = \sqrt{2}((16+\sqrt{3(54+14\sqrt{5})})/(7-3\sqrt{5}))^{1/4}$,
(iv) $H(e^{-\pi\sqrt{17}}) = (\sqrt{2})((2+\sqrt{4-4(20+5\sqrt{17}-2\sqrt{206+50\sqrt{17}})^2})/(40+10\sqrt{17}-4\sqrt{206+50\sqrt{17}}))^{1/4}$,
(v) $H(e^{-\pi\sqrt{19}}) = \sqrt{2}((1+\sqrt{1-4k^4})/2k^2)^{1/4}$, *where* $k = (1/24)(-20+(2944-384\sqrt{57})^{1/3}+4(46+6\sqrt{57})^{1/3})$,
(vi) $H(e^{-\pi\sqrt{23}}) = \sqrt{2}((1+\sqrt{1-4n^{12}})/2n^6)^{1/4}$, *where* $n = -1/(3\cdot 2^{1/3})+(1/6)(50-6\sqrt{69})^{1/3}+(25+3\sqrt{69})^{1/3}/(3\cdot 2^{2/3})$,

(vii) $H(e^{-\pi\sqrt{31}})=\sqrt{2}((1+\sqrt{1+4d^8})/2d^4)^{1/4}$, *where* $d=1/2+(1/6)((-27+3\sqrt{93})/2)^{1/3}-1/(2^{2/3}(-27+3\sqrt{93})^{1/3})$.

The proof of the theorem follows directly from Theorems 6.7 and 7.1.

Remark 7.6. Values of $H(e^{-\pi/\sqrt{n}})$ for $n=11,13,15,17,19,23$, and 31 also follow readily from Theorem 6.7 and (7.2).

8. Theorems on $S_1(q)$ and explicit values

The Weber-Ramanujan class invariants G_n and g_n are defined by

$$G_n := 2^{-1/4}q^{-1/24}(-q;q^2)_\infty, \qquad g_n := 2^{-1/4}q^{-1/24}(q;q^2)_\infty, \tag{8.1}$$

where $q := e^{-\pi\sqrt{n}}$.

The two class invariants satisfy the properties (see [2, page 187, Entry 2.1], [9, page 18, Corollary 2.2.4(i), (ii)])

$$g_{4n}=2^{1/4}g_nG_n, \qquad g_n^{-1}=g_{4/n}, \qquad G_{1/n}=G_n. \tag{8.2}$$

We also note from [9, page 13, Lemma 2.1.3(i)] and [9, page 18, Theorem 2.2.3] that

$$r_{k,n/m}=r_{mk,n}r_{nk,m}^{-1}, \tag{8.3}$$

$$g_n=r_{2,n/2}, \qquad G_n=\frac{r_{2,2n}}{2^{1/4}r_{2,n/2}}, \tag{8.4}$$

respectively, where $r_{k,n}$ is as defined in (1.17) and k and n are positive real numbers.

Now, we state and prove two general formulas for the explicit evaluations of $S_1(q)$ and then calculate some specific values.

Theorem 8.1. *One has*

$$S_1(e^{-\pi\sqrt{n}})=\frac{1}{2^{3/4}G_n^2g_n}=\frac{r_{2,n/2}}{2^{1/4}r_{2,2n}^2}=\frac{r_{4,n}}{2^{1/4}r_{2,2n}^3}, \tag{8.5}$$

where G_n and g_n are Ramanujan's class invariants as defined in (8.1).

Proof. By [1, page 39, Entry 24(iii)], we have

$$\psi(q)=\frac{f^2(-q^2)}{f(-q)}, \qquad \phi(q)=\frac{f^2(q)}{f(-q^2)}. \tag{8.6}$$

Substituting (8.6) in (1.5), we obtain

$$S_1(q)=\frac{f^2(-q^2)}{2^{-1/2}q^{-1/12}f^2(q)}\times\frac{f(-q^2)}{2^{-1/4}q^{-1/24}f(-q)}. \tag{8.7}$$

From [1, page 39, Entry 24(iii)], we also note that

$$\chi(q)=\frac{f(q)}{f(-q^2)}. \tag{8.8}$$

Now, setting $q := e^{-\pi\sqrt{n}}$ and then applying (8.1), (8.7), and (8.8), we complete the proof of the first equality. Employing (8.4) to the first equality, we arrive at the second equality.

To prove the third equality, we employ (8.3) to the second equality. □

Corollary 8.2. *One has*

(i) $S_1(e^{-\pi}) = 2^{-5/8}$,
(ii) $S_1(e^{-\pi\sqrt{2}}) = 2^{-1/2}(1+\sqrt{2})^{-1/2}$,
(iii) $S_1(e^{-\pi\sqrt{3}}) = 2^{-17/24}(1+\sqrt{3})^{-1/4}$,
(iv) $S_1(e^{-2\pi}) = 2^{-3/8}(1+\sqrt{2})^{-1/4}$,
(v) $S_1(e^{-\pi\sqrt{5}}) = (1+\sqrt{5})^{-1/2}(\sqrt{\sqrt{5}+1}+\sqrt{2})^{-1/4}$,
(vi) $S_1(e^{-\pi\sqrt{6}}) = 2^{-1/2}(1+\sqrt{2})^{-1/4}(1+\sqrt{2}+\sqrt{6})^{-1/4}$,
(vii) $S_1(e^{-\pi\sqrt{7}}) = 2^{-7/8}(3+\sqrt{7})^{-1/4}$,
(viii) $S_1(e^{-2\pi\sqrt{2}}) = 2^{-3/8}(1+\sqrt{2})^{-3/8}(4+\sqrt{2+10\sqrt{2}})^{-1/4}$,
(ix) $S_1(e^{-3\pi}) = 2^{1/8}/(1+\sqrt{3})^{2/3}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3}$,
(x) $S_1(e^{-\pi\sqrt{10}}) = 2^{1/4}(1+\sqrt{5})^{-3/4}(2+3\sqrt{2}+\sqrt{5})^{-1/4}$,
(xi) $S_1(e^{-4\pi}) = 2^{-7/16}(1+\sqrt{2})^{-1/4}(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4})^{-1/4}$,
(xii) $S_1(e^{-3\pi\sqrt{2}}) = 2^{-1/2}(\sqrt{3}+\sqrt{2})^{-1}(1+35\sqrt{2}-28\sqrt{3})^{-1/4}$,
(xiii) $S_1(e^{-5\pi}) = 2^{-17/8}(\sqrt{5}-1)(5^{1/4}-1)$,
(xiv) $S_1(e^{-6\pi}) = ((\sqrt{2}-1)^{5/6}(1+\sqrt{3})^{1/3}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3})/(2^{1/6}(\sqrt{2}+\sqrt{3})^{2/3}(4-\sqrt{2}+2\sqrt{3}+3^{3/4}(\sqrt{3}+1))^{2/3})$.

Proof. The parts (i)–(vi) and (viii)–(xiv) easily follow from Theorem 8.1 with the help of the values of $r_{k,n}$ in Section 2. To prove (vii), we use the values of G_7 and $g_7 = r_{2,7/2}$ from [2] and Section 2, respectively. □

Theorem 8.3. *One has*

$$S_1(e^{-\pi/\sqrt{n}}) = \frac{g_n}{2^{1/2}G_n} = \frac{r_{2,n/2}^2}{2^{1/4}r_{2,2n}} = \frac{r_{4,n}^2}{2^{1/4}r_{2,2n}^3}. \tag{8.9}$$

Proof. Replacing n by $1/n$ in Theorem 8.1 and then simplifying by using (8.2), we arrive at the first equality. To prove the second equality, we employ (8.4) to the first. Using (8.3) to the second equality, we finish the proof of the third one. □

Corollary 8.4. *One has*

(i) $S_1(e^{-\pi/\sqrt{2}}) = 2^{-3/8}(\sqrt{2}-1)^{1/8}$,
(ii) $S_1(e^{-\pi/\sqrt{3}}) = 2^{-7/8}(\sqrt{3}+1)^{1/4}$,
(iii) $S_1(e^{-\pi/2}) = 2^{-3/16}(\sqrt{2}-1)^{1/4}$,
(iv) $S_1(e^{-\pi/2\sqrt{2}}) = 2^{-1/8}(4+\sqrt{2+10\sqrt{2}})^{-1/8}$,
(v) $S_1(e^{-\pi/5}) = 2^{-17/8}(\sqrt{5}-1)(5^{1/4}+1)$,
(vi) $S_1(e^{-\pi/3}) = (1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{1/3}/(2^{7/8}(1+\sqrt{3})^{1/3})$,
(vii) $S_1(e^{-\pi/\sqrt{6}}) = 2^{-3/8}((\sqrt{2}+1)/(1+\sqrt{2}+\sqrt{6}))^{1/8}$,
(viii) $S_1(e^{-\pi/4}) = (2^{3/16}(\sqrt{2}+1)^{1/4})/(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4})^{1/8}$,
(ix) $S_1(e^{-\pi/3\sqrt{2}}) = 2^{-3/8}(1+35\sqrt{2}-28\sqrt{3})^{-1/8}$,
(x) $S_1(e^{-\pi/6}) = ((1+\sqrt{3})^{2/3}(\sqrt{2}-1)^{5/12}(1+\sqrt{3}+\sqrt{2}\cdot 3^{3/4})^{2/3})/(2^{15/16}(\sqrt{2}+\sqrt{3})^{1/3}(4-\sqrt{2}+2\sqrt{3}+3^{3/4}(1+\sqrt{3}))^{1/3})$,
(xi) $S_1(e^{-\pi/\sqrt{10}}) = ((\sqrt{5}+1)^{3/8})/(2^{3/4}(2+3\sqrt{2}+\sqrt{5})^{1/8})$.

Proof. With the help of Theorem 8.3 and the values of $r_{k,n}$ listed in Section 2, we readily complete the proof. □

Remark 8.5. From the last equalities of Theorems 8.1 and 8.3, we have the transformation formula for $S_1(q)$:

$$S_1\left(e^{-\pi/\sqrt{n}}\right) = r_{4,n} S_1\left(e^{-\pi\sqrt{n}}\right). \tag{8.10}$$

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