ON A CLASS OF SECOND-ORDER IMPULSIVE BOUNDARY VALUE PROBLEM AT RESONANCE

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We consider the following impulsive boundary value problem, $x''(t) = f(t,x,x')$, $t \in J \setminus \{t_1,t_2,\ldots,t_k\}$, $\triangle x(t_i) = I_i(x(t_i),x'(t_i))$, $\triangle x'(t_i) = J_i(x(t_i),x'(t_i))$, $i = 1,2,\ldots,k$, $x(0) = 0$, $x'(1) = \sum_{j=1}^{m-2} \alpha_j x'(\eta_j)$. By using the coincidence degree theory, a general theorem concerning the problem is given. Moreover, we get a concrete existence result which can be applied more conveniently than recent results. Our results extend some work concerning the usual m-point boundary value problem at resonance without impulses.

1. Introduction

In the few past years, boundary value problems for impulsive differential equation have been studied (see [1, 5, 7]). They discussed the existence of solutions for first-order impulsive equations by the use of upper and lower solution methods. Dong [3] researched the periodic boundary value problem for second-order impulsive equations. Liu and Yu [6] considered the boundary value condition $x'(0) = 0$, $x(1) = \sum_{j=1}^{m-2} \alpha_j x(\eta_j)$ with $\sum_{j=1}^{m-2} \alpha_j = 1$ by making use of the coincidence degree which was developed by Gaines and Mawhin [4].

We are concerned with the m-point boundary value problem for the nonlinear impulsive differential equation:

$$\begin{gathered} x''(t) = f\big(t,x(t),x'(t)\big), \quad t \in J', \\ \Delta x(t_i) = I_i\big(x(t_i),\, x'(t_i)\big),\ \Delta x'(t_i) = J_i\big(x(t_i),x'(t_i)\big), \quad i = 1,2,\ldots,k, \end{gathered} \tag{1.1}$$

associated with the boundary value condition

$$x(0) = 0, \quad x'(1) = \sum_{j=1}^{m-2} \alpha_j x'(\eta_j), \tag{1.2}$$

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where $J = [0,1]$, $0 < t_1 < t_2 < \cdots < t_k < 1$, $J' = J \setminus \{t_1, t_2, \ldots, t_k\}$. $x \in R$, $f : J \times R \times R \to R$, $I_i : R \times R \to R$, $J_i : R \times R \to R$ are continuous, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\triangle x(t_i) = x(t_i + 0) - x(t_i)$, $\triangle x'(t_i) = x'(t_i + 0) - x'(t_i)$, $i = 1, 2, \ldots, k$, $\sum_{j=1}^{m-2} \alpha_j = 1$, $\alpha_j > 0$, $j = 1, 2, \ldots, m-2$. A map $x : J \to R$ is said to be solution of (1.1)–(1.2), if it satisfies

(1) $x(t)$ is twice continuously differentiable for $t \in J'$, both $x(t+0)$ and $x(t-0)$ exist at $t = t_i$, and $x(t_i) = x(t_i - 0)$, $i = 1, 2, \ldots, k$;

(2) $x(t)$ satisfies the relations (1.1)–(1.2).

We will use the continuation theorem of coincidence degree [2] to show a general theorem for the existence of solutions to the problem (1.1)–(1.2) and then use it to get concrete existence conditions in Section 3. This paper is motivated by [2, 3, 6, 8, 9].

2. Preliminary lemmas

At first, we recall some notations and present a series of useful lemmas with respect to the problem (1.1)–(1.2) that is important in the proof of our results. Consider an operator equation

$$Lx = Nx, \tag{2.1}$$

where $L : \operatorname{dom} L \cap X \to Z$ is a linear operator, $N : X \to Z$ is a nonlinear operator, X and Z are Banach spaces. If $\dim \operatorname{Ker} L = \dim(Z/\operatorname{Im} L) < \infty$, and $\operatorname{Im} L$ is closed in Z, then L will be called a Fredholm mapping of index *zero*. And at the same time, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q$. It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is reversible. We denote the inverse of this map by K_p.

Let Ω be an open and bounded subset of X. The map N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q) : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $\widehat{J} : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 2.1 (continuation theorem [4]). *Suppose that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. If the following conditions are satisfied:*

(i) *for each $\lambda \in (0,1)$, every solution x of*

$$Lx = \lambda Nx \tag{2.2}$$

is such that $x \notin \partial\Omega$;

(ii) *$QNx \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$, and $\deg(IQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, where $Q : Z \to Z$ is a continuous projector with $\operatorname{Im} L = \operatorname{Ker} Q$, $\widehat{J} : Z/\operatorname{Im} L \to \operatorname{Ker} L$ is an isomorphism. Then the operator equation (2.1) has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.*

In the following, in order to obtain the existence theorem of (1.1)–(1.2) we first introduce the following:

(i) $X = PC^1[J,R] = \{x : J \to R \mid x(t)$ is twice continuously differentiable for $t \in J'$, there exist $x'(t_i + 0)$, $x'(t_i - 0)$ and $x(t_i) = x(t_i - 0)$, $x'(t_i) = x'(t_i - 0)$, $i = 1, 2, \ldots, k$ and (1.2) is satisfied$\}$;

(ii) $Z = PC[J,R] \times R^{2k} = \{y : J \to R \mid y(t)$ is continuous for $t \in J'$, there exist $y(t_i - 0), y(t_i + 0)$, and $y(t_i - 0) = y(t_i)$, $i = 1,2,\dots,k\} \times R^{2k}$.

Let $\|x\|_\infty = \sup_{t\in J} |x(t)|$ for $x \in PC[J,R]$ and $x \in PC^1[J,R]$. $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$. And for every $z = (y,c) \in Z$, denote its norm by

$$\|z\| = \max\left\{\sup_{t\in J} |y(t)|, \|c\|\right\}. \tag{2.3}$$

We can prove that X and Z are Banach spaces. Let $\operatorname{dom} L = \{x : J \to R \mid x(t)$ is twice differentiable for $t \in J'\} \cap X$,

$$\begin{gathered} L : \operatorname{dom} L \longrightarrow Z,\ x \longmapsto (x''(t), \triangle x(t_1),\dots,\triangle x(t_k), \triangle x'(t_1),\dots,\triangle x'(t_k)), \\ N : X \longrightarrow Z,\ x \longmapsto (f(t,x(t), x'(t)), I_1(x(t_1), x'(t_1)),\dots, I_k(x(t_k),x'(t_k)), \\ J_1(x(t_1),x'(t_1)),\dots,J_k(x(t_k),x'(t_k))). \end{gathered} \tag{2.4}$$

Then problem (1.1)–(1.2) can be written as $Lx = Nx$, $x \in \operatorname{dom} L$.

Lemma 2.2. *Assume that L is defined as above and $\sum_{j=1}^{m-2} \alpha_j = 1$. Then L is a Fredholm mapping of index zero. Furthermore, for the problem (1.1)–(1.2),*

$$\operatorname{Ker} L = \{x(t) \in X : x(t) = ct,\ c \in R\}, \tag{2.5}$$

$$\begin{aligned} \operatorname{Im} L = \{&(y,a_1,a_2,\dots,a_k,b_1,\dots,b_k) : x''(t) = y(t), \triangle x(t_i) = a_i, \triangle x'(t_i) = b_i, \\ &i = 1,2,\dots,k, \text{ for some } x(t) \in \operatorname{dom} L\} \\ = \Bigg\{&(y,a_1,a_2,\dots,a_k,b_1,b_2,\dots,b_k) \in PC[0,1] \times R^{2k} : \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 y(s)ds + \sum_{t_i>\eta_j} b_i\right] = 0\Bigg\}. \end{aligned} \tag{2.6}$$

Proof. Firstly, it is easily seen that (2.5) holds. Next we will show that (2.5) holds. Since problem

$$\begin{gathered} x''(t) = y(t), \quad t \in J', \\ \triangle x(t_i) = a_i, \qquad \triangle x'(t_i) = b_i \end{gathered} \tag{2.7}$$

has solution $x(t)$ satisfying (1.2) if and only if

$$\sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 y(s)ds + \sum_{t_i>\eta_j} b_i\right] = 0. \tag{2.8}$$

In fact, if (2.7) has solution $x(t)$ such that (1.2), then from (2.7) we have

$$x(t) = x(0) + x'(0)t + \int_0^t \int_0^s y(\tau)d\tau\, ds + \sum_{t_i<t} b_i(t - t_i) + \sum_{t>t_i} a_i. \tag{2.9}$$

Thus

$$x'(t) = x'(0) + \int_0^t y(s)ds + \sum_{t>t_i} b_i. \tag{2.10}$$

In view of (1.2), we have

$$x(0)=0,\quad x'(\eta_j)=x'(0)+\int_0^{\eta_j}y(s)ds+\sum_{\eta_j>t_i}b_i,$$
$$0=x'(1)-\sum_{j=1}^{m-2}\alpha_j x'(\eta_j)=\sum_{j=1}^{m-2}\alpha_j\left[\int_{\eta_j}^{1}y(s)ds+\sum_{t_i>\eta_j}b_i\right]. \tag{2.11}$$

Hence, (2.8) holds.

On the other hand, if (2.8) holds setting

$$x(t)=ct+\int_0^t\int_0^s y(\tau)d\tau\,ds+\sum_{t_i<t}b_i(t-t_i)+\sum_{t>t_i}a_i, \tag{2.12}$$

where $c\in R$ is an arbitrary constant, then it is clear that $x(t)$ is a solution of (2.7) and satisfies (1.2). Hence, (2.5) holds.

Take the projector $Q:Z\to Z$ as follows:

$$Q(y,a_1,\dots,a_k,b_1,\dots,b_k)=\left(\frac{2}{\sum_{j=1}^{m-2}\alpha_j(1-\eta_j^2)}\sum_{j=1}^{m-2}\alpha_j\left[\int_{\eta_j}^{1}y(\tau)d\tau+\sum_{t_i>\eta_j}b_i\right]\cdot t,0,\dots,0\right) \tag{2.13}$$

and for $(y,a_1,a_2,\dots,a_k,b_1,b_2,\dots,b_k)\in Z$. Let

$$z=(y_1,a_1,a_2,\dots,a_k,b_1,b_2,\dots,b_k)$$
$$=(y,a_1,\dots,a_k,b_1,b_2,\dots,b_k)-Q(y,a_1,a_2,\dots,a_k,b_1,\dots,b_k), \tag{2.14}$$

then $z\in\operatorname{Im}L$. Thus, we have

$$\dim(Z/\operatorname{Im}L)=\dim\operatorname{Im}Q=1=\dim\operatorname{Ker}L, \tag{2.15}$$

moreover by the Ascoli-Arzela theorem, L is a Fredholm mapping of index *zero*. □

3. Main results

By applying Lemma 2.1, a general theorem for the existence of solutions to the problem (1.1)–(1.2) is obtained. And concrete existence conditions for the same problem are also obtained.

For any subset $G\subset R^2$, let

$$\Omega=\{x\in X\mid (x(t),x'(t))\in G,\ (x(t_i^+),x'(t_i^+))\in G,\ \forall t\in J,\ i=1,2,\dots,k\},$$
$$\Omega\cap\operatorname{Ker}L=\{x=ct\mid (ct,c)\in G,\ \forall t\in J\}=G_1,\quad \hat{G}_1=\{c\in R: ct\in G_1\}. \tag{3.1}$$

Theorem 3.1. *Assume that the following conditions are satisfied:*

(1) *let* $G \subset R^2$ *be an open bounded subset such that for every* $\lambda \in (0,1)$*, each possible solution* $x(t)$ *of the auxiliary system*

$$\begin{gathered} x''(t) = \lambda f(t,x,x'), \quad t \in J', \\ \triangle x(t_i) = \lambda I_i(x(t_i),x'(t_i)), \\ \triangle x'(t_i) = \lambda J_i(x(t_i),x'(t_i)), \quad i = 1,2,\dots,k, \\ x(0) = 0, \quad x'(1) = \sum_{j=1}^{m-2} \alpha_j x'(\eta_j) \end{gathered} \tag{3.2}$$

satisfies $x \notin \partial\Omega$*;*

(2) $h(c) \neq 0$*, for* $c \in \partial\widehat{G}_1$*,* $\deg(h,\widehat{G}_1,0) \neq 0$*, where h is defined by*

$$h(c) = \frac{2}{\sum_{j=1}^{m-2} \alpha_j (1-\eta_j^2)} \cdot \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^1 f(\tau,c\tau,c)d\tau + \sum_{t_i > \eta_j} J_i(ct_i,c) \right]. \tag{3.3}$$

Then the BVP (1.1)–(1.2) has at least one solution $x(t)$ *satisfying* $(x(t),x'(t)) \in G$*, for* $t \in J$.

Proof. By Lemma 2.2, we know that L is a Fredholm operator of index *zero*, and the problem (3.2) can be written as $Lx = \lambda Nx$. Set

$$\Omega = \{x \in X : (x(t),x'(t)) \in G, (x(t_i+0),x'(t_i+0)) \in G, \ \forall t \in J, \ i = 1,\dots,k\}. \tag{3.4}$$

Then Ω is open and bounded. To use Lemma 2.1, we show at first N is L-compact on $\overline{\Omega}$. Defining a projector

$$P : X \longrightarrow \operatorname{Ker} L, \quad P(x(t)) = x'(0)t, \tag{3.5}$$

then $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom} L$ can be written in

$$K_p z = K_p(y,a_1,\dots,a_k,b_1,\dots,b_k) = \int_0^t \int_0^s y(\tau)d\tau\, ds + \sum_{t_i<t} b_i(t-t_i) + \sum_{t_i<t} a_i. \tag{3.6}$$

In fact, we have $K_pL = I - P$, thus for any $x \in \operatorname{dom} L$, $K_pLx = x - x'(0)t$, so (3.6) holds. Again from (2.13) and (3.6), we have

$$\begin{aligned}
QNx &= \Bigg(\frac{2}{\sum_{j=1}^{m-2}\alpha_j(1-\eta_j^2)}\cdot\sum_{j=1}^{m-2}\alpha_j\bigg[\int_{\eta_j}^{1} f(\tau,x(\tau),x'(\tau))\,d\tau \\
&\qquad + \sum_{t_i>\eta_j} J_i(x(t_i),x'(t_i))\bigg]\cdot t,0,\dots,0\Bigg), \\
K_p(I-Q)Nx &= \int_0^t\int_0^s f(\tau,x(\tau),x'(\tau))\,d\tau\,ds + \int_0^t \sum_{t_i<s} J_i(x(t_i),x'(t_i))\,ds \\
&\quad + \sum_{t_i<t} I_i(x(t_i),x'(t_i)) - \int_0^t\int_0^s \frac{2}{\sum_{j=1}^{m-2}\alpha_j(1-\eta_j^2)} \\
&\quad \cdot \sum_{j=1}^{m-2}\alpha_j\bigg[\int_{\eta_j}^{1} f(\tau,x(\tau),x'(\tau))\,d\tau + \sum_{t_i>\eta_j} J_i(x(t_i),x'(t_i))\bigg]\cdot s\,ds\,dt.
\end{aligned} \tag{3.7}$$

By using the Ascoli-Arzela theorem, we can prove that $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact, thus N is L-compact on $\overline{\Omega}$.

At last, we will prove that (i), (ii) of Lemma 2.1 are satisfied. Note that $x \in \partial\Omega$, if and only if $(x(t),x'(t)) \in \overline{G}$, for $t \in J$, and either $(x(s),x'(s)) \in \partial G$, for some $s \in J$, or $(x(t_{i_0}+0),x'(t_{i_0}+0)) \in \partial G$, for some $i_0 = \{1,2,\dots,k\}$, then the assumption (i) follows from condition (1).

Let $\hat{J} : \operatorname{Im} Q \to \operatorname{Ker} L : (ct,0,\dots,0) \to ct$ be the isomorphism. Then

$$\hat{J}QNx = \frac{2}{\sum_{j=1}^{m-2}\alpha_j(1-\eta_j^2)}\cdot\sum_{j=1}^{m-2}\alpha_j\bigg[\int_{\eta_j}^{1} f(\tau,x(\tau),x'(\tau))\,d\tau + \sum_{t_i>\eta_j} J_i(x(t_i),x'(t_i))\bigg]\cdot t. \tag{3.8}$$

Take an isomorphism $g : G_1 \to R$, $g(ct) = c$,

$$\begin{gathered}
h(c) = g\big(\hat{J}QNg^{-1}c\big) = \frac{2}{\sum_{j=1}^{m-2}\alpha_j(1-\eta_j^2)}\cdot\sum_{j=1}^{m-2}\alpha_j\bigg[\int_{\eta_j}^{1} f(\tau,c\tau,c)\,d\tau + \sum_{t_i>\eta_j} J_i(ct_i,c)\bigg], \\
g(\Omega\cap\ker L) = \widehat{G}_1,
\end{gathered} \tag{3.9}$$

then

$$\deg\{\hat{J}QN,\Omega\cap\ker L,0\} = \deg\{g\hat{J}QNg^{-1},g(\Omega\cap\ker L),g(0)\} = \deg\{h,\widehat{G}_1,0\} \neq 0 \tag{3.10}$$

in view of (2), $h(c) \neq 0$, for $c \in \partial\widehat{G}_1$, then $\hat{J}QNx \neq 0$, $x \in \partial\Omega \cap \operatorname{Ker} L$, that is, condition (2) yields (ii) of Lemma 2.1, and the proof is finished. □

Remark 3.2. A similar result is given in [3] for the periodic boundary value problems (see Theorem 3.1 therein). However, there is something wrong in its proof. For a subset $G \subset R^{2n}$, define

$$\Omega = \{x \in X : (x(t), x'(t)) \in G \text{ for } t \in [0,T]; (x(t_i+0), x'(t_i+0)) \in G, \text{ for } i = 1,2,\ldots,k\}. \tag{3.11}$$

Then $\Omega \cap \operatorname{Ker} L = \{x \in R^n : (x,0) \in G\}$, which is not the set $G_1 = \{x \in R^n :$ there exists $y \in R$ such that $(x,y) \in G\}$ defined by the author.

For example, take $G = \{(x,y) \in R^2 : x^2 + (y-1)^2 < 4\}$. We have

$$\begin{gathered} \Omega \cap \operatorname{Ker} L = \{x \in R : x^2 + 1 < 4\} = (-\sqrt{3}, \sqrt{3}), \\ G_1 = \{x \in R : x^2 < 4\} = (-2, 2) \end{gathered} \tag{3.12}$$

since $\operatorname{Ker} L = R$. So the proof of Theorem 1 in [3] is not correct. The same problem appears in the proof of Theorem 1 in [6].

Let

$$\begin{aligned} a_1 &= \max_{|y| \le M,\ |x| \le Mt_1} |I_1(x,y)|, \\ a_2 &= \max_{|y| \le M,\ |x| \le Mt_2 + a_1} |I_2(x,y)|, \\ a_3 &= \max_{|y| \le M,\ |x| \le Mt_3 + a_1 + a_2} |I_3(x,y)| \cdots, \\ a_k &= \max_{|y| \le M,\ |x| \le Mt_k + \sum_{l=1}^{k-1} a_l} |I_k(x,y)|. \end{aligned} \tag{3.13}$$

Since $x(t) = \int_0^t x'(s)ds + \sum_{t_i < t} I_i(x(t_i), x'(t_i))$, if $|x'(t)| \le M$, $t \in J$, then

$$|x(t)| \le M + \sum_{i=1}^{k} a_i := \widehat{M}. \tag{3.14}$$

Theorem 3.3. *Let $f : J \times R \times R \to R$ be a continuous function and suppose that there exists a constant $M > 0$ such that*

$$y f(t,x,y) > 0, \quad y J_i(x,y) > 0, \tag{3.15}$$

for $|y| \ge M$, $(t,x) \in J \times [-\widehat{M}, \widehat{M}]$, $i = 1,2,\ldots,k$.

Then BVP(1.1)–(1.2) with $\sum_{j=1}^{m-2} \alpha_j = 1$ has at least one solution $x(t) \in PC^1[0,1]$.

Proof. Let

$$f^*(t,x,y) = \begin{cases} f(t,x,y), & |x| \le \widehat{M}, \\ f(t,\widehat{M},y), & x > \widehat{M}, \\ f(t,-\widehat{M},y), & x < -\widehat{M}, \end{cases}$$
$$J_i^*(x,y) = \begin{cases} J_i(x,y), & |x| \le \widehat{M}, \\ J_i(\widehat{M},y), & x > \widehat{M}, \\ J_i(-\widehat{M},y), & x < -\widehat{M}. \end{cases} \tag{3.16}$$

Suppose $x(t)$ is a solution to BVP (3.2). We show at first that $\|x'\| < M$, when $\lambda \in (0,1)$. Otherwise, there is $t_0 \in [0,1]$ such that $\|x'\| = |x'(t_0)| = \sup_{t\in J}|x'(t)| \ge M$.

Without loss of generality we suppose that $x'(t_0) \ge M$.

If $t_0 \notin \{t_i, i = 1,2,\dots,k\} \cup \{0,1\}$, then one has

$$x'(t_0) = \sup_{t\in J} x'(t) \ge M, \quad x''(t_0) \le 0. \tag{3.17}$$

However, by condition (3.15), $x''(t_0) = \lambda f^*(t_0,x(t_0),x'(t_0)) > 0$, a contradiction.

If $t_0 \in \{t_i, i = 1,2,\dots,k\}$, say $t_0 = t_i$, then $J_i^*(x(t_i),x'(t_i)) > 0$ and hence

$$x'(t_i^+) = x'(t_i) + \lambda J_i^*(x(t_i), x'(t_i)) > x'(t_i) \tag{3.18}$$

which contradicts the assumption $x'(t_i) = \sup_{t\in J} x'(t)$.

If $t_0 = t_i^+ := t_i + \epsilon$, then there is $\sigma \in (0, t_{i+1} - t_i)$, (if $i = k, t_{i+1}$ is replaced by 1), such that $x'(t) > M$, $t \in (t_i, t_i + \sigma)$. Since $x''(t) = \lambda f^*(t,x(t),x'(t))$, $t \in (t_i,t_i+\sigma)$, $x''(t_i^+) = \lambda f^*(t_i,x(t_i^+),x'(t_i^+)) > 0$, then

$$x'(t_i+\sigma) = x'(t_i^+) + \int_{t_i}^{t_i+\sigma} x''(s)ds > x'(t_i^+) \tag{3.19}$$

which contradicts $x'(t_i^+) = \sup_{t\in J} x'(t)$.

If $t_0 = 1$, since $\sum_{j=1}^{m-2}\alpha_j = 1$, $\alpha_j \in (0,1)$, and $x'(1) = \sum_{j=1}^{m-2}\alpha_j x'(\eta_j)$ yield that

$$x'(1) = x'(\eta_1) = \cdots = x'(\eta_{m-2}) = \sup_{t\in J} x'(t) \tag{3.20}$$

which is the case we have discussed in case 1.

If $t_0 = 0$, $x'(0) = \sup_{t\in J}|x(t)| \ge M$, then $x''(0) = \lambda f(0,x(0),x'(0)) > 0$. So there is a $\sigma > 0$ small enough, such that $x''(t) > 0$, $t \in (0,\sigma)$, which yields

$$x'(\sigma) = x'(0) + \int_0^{\sigma} x''(s)ds > x'(0), \tag{3.21}$$

a contraction.

So $\|x'\| < M$ holds for all cases.

Since $x(t) = \int_0^t x'(s)ds + \sum_{t_i<t} I_i(x(t_i),x'(t_i))$, and $|x'(t)| \le M$, $t \in J$, then

$$|x(t)| \le Mt + \sum_{i=1}^{k} a_i \le M + \sum_{i=1}^{k} a_i = \widehat{M} \tag{3.22}$$

and in turn the prior bound is obtained. Let $\Omega = \{x \in X \mid \|x\|_X < \widehat{M}+1\}$. We have $x \notin \partial\Omega$.

By the proof of Theorem 3.1, we know that $h(c) = g\hat{J}QNg^{-1}c$,

$$h(c) = 0 \Longleftrightarrow g\hat{J}QNg^{-1}c = 0 \Longleftrightarrow \hat{J}QNct = 0 \Longleftrightarrow QNct = 0 \Longleftrightarrow Nct \in \operatorname{Im} L, \tag{3.23}$$

one has $x \in \{x = ct : |ct| < \widehat{M}+1,\ t \in J,\ |c| < \widehat{M}+1\} = \{ct : |c| < \widehat{M}+1\}$ and $\widehat{G_1} = (-\widehat{M}-1, \widehat{M}+1)$. When $c = \widehat{M}+1$ or $c = -(\widehat{M}+1)$ by condition (3.15), it holds that

$$\operatorname{sgnc} \cdot \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^{1} f(\tau, c\tau, c)d\tau + \sum_{t_i>\eta_j} J_i(ct_i, c) \right] > 0, \tag{3.24}$$

$c \in \partial\hat{G}_1 = \{-\widehat{M}-1, \widehat{M}+1\}$.

Obviously,

$$\begin{aligned} \operatorname{sgnc} \cdot h(c) &= \operatorname{sgnc} \cdot g\hat{J}QNg^{-1}(ct) \\ &= \operatorname{sgnc} \cdot \frac{2}{\sum_{j=1}^{m-2} \alpha_j (1-\eta_j^2)} \cdot \sum_{j=1}^{m-2} \alpha_j \left[\int_{\eta_j}^{1} f(\tau, c\tau, c)d\tau + \sum_{t_i>\eta_j} J_i(ct_i, c) \right] > 0 \end{aligned} \tag{3.25}$$

for $c \in \partial\hat{G}_1 = \{-\widehat{M}-1, \widehat{M}+1\}$. Then

$$\deg\{g\hat{J}QNg^{-1}, \hat{G}_1, 0\} = \deg\{h, (-\widehat{M}-1, \widehat{M}+1), 0\} = 1. \tag{3.26}$$

Hence, the conditions of Theorem 3.1 are satisfied and the proof of Theorem 3.3 is completed. □

Finally, we present an example to check our result.

Example 3.4. Consider the boundary value problem

$$\begin{gathered} x''(t) = x'(t)[t^2 + \ln(2 + x^2(t)) + x'^2(t)] + 3t\sin(x(t)+1), \quad t \in [0,1],\ t \neq \frac{1}{2}, \\ \triangle x\left(\frac{1}{2}\right) = \sin x\left(\frac{1}{2}\right), \quad \triangle x'\left(\frac{1}{2}\right) = x'\left(\frac{1}{2}\right)\left[1 + x'^2\left(\frac{1}{2}\right)\right] - \left[2 + \cos x\left(\frac{1}{2}\right)\right], \quad t = \frac{1}{2}, \\ x(0) = 0, \quad x'(1) = x'\left(\frac{1}{3}\right), \end{gathered} \tag{3.27}$$

where $f(t,x,x') = x'(t)[t^2 + \ln(2 + x^2(t)) + x'^2(t)] + 3t\sin(x(t)+1)$, $I(x,x') = \sin x(t)$, $J = x'(t)[1 + x'^2(t)] - [2 + \cos x(t)]$.

In this example, we note that $t_k = 1/2$, $k = 1$, $\alpha = 1$, $\eta_j = 1/3$, $j = 1$.
We choose a constant $M > 0$ large enough, let

$$a = \max_{|x'|<M,\ |x|\le Mt_1} \left| I(x,x') \right| = \max_{|x'|<(1/2)M,\ |x|\le M} \left| \sin x\left(\frac{1}{2}\right) \right| = 1. \tag{3.28}$$

Since $x(t) = \int_0^t x'(s)ds + I(x,x') = \int_0^t x'(s)ds + \sin x(1/2)$,

$$\left| x(t) \right| \le M + 1 := \widehat{M}. \tag{3.29}$$

When $|x'| \ge M$, $(t,x) \in J \times [-\widehat{M},\widehat{M}]$, obviously

$$\begin{gathered} x' \cdot f(t,x,x') = x'^2(t)\left[t^2 + \ln\left(2 + x^2(t)\right) + x'^2(t)\right] + x'(t)\left[3t\sin\left(x(t)+1\right)\right] > 0, \\ x' \cdot J = x'^2(t)\left[1 + x'^2(t)\right] - x'(t)\left[2 + \cos x(t)\right] > 0, \end{gathered} \tag{3.30}$$

that is to say, the condition of Theorem 3.3 is satisfied. The BVP (3.27) has at least one solution.

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