WEAKLY INDUCED MODIFICATIONS OF I-FUZZY TOPOLOGIES

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The aim of this paper is to study weakly induced I-fuzzy topological spaces and weakly induced modifications of I-fuzzy topologies. We give two kinds of weakly induced I-fuzzy topologies for each I-fuzzy topology and prove that I-WIFTOP is a reflective and coreflective full subcategory of I-FTOP. We also discuss some relationships between several categories.

1. Introduction and preliminaries

Since Chang [2] introduced fuzzy theory into topology, many authors have discussed various aspects of fuzzy topology. It is well known that weakly induced and induced topological spaces play an important role in L-topological spaces (see book [8]). According to their value ranges, L-topological spaces form different categories. Clearly, the investigation on their relationships is certainly important and necessary. Lowen was the first author to study the relations between I-topological spaces and classical topological spaces. He introduced two well-known functors—ω and ι. Later, these functors, named Lowen functors, were extended by different authors [7, 12] for various kinds of lattices studying the relations between L-TOP and TOP.

However, in a completely different direction, Höhle [4] created the notion of a topology being viewed as an L-subset of a powerset. Then Kubiak [6] and Šostak [11] independently extended Höhle's notion to L-subsets of L^X. From a logical point of view, Ying [13] introduced fuzzifying topological spaces (Ying's fuzzifying topology is similar to Höhle's topology). In order to discuss the relations between fuzzifying topologies and I-fuzzy topologies, the authors studied Lowen functors in I-fuzzy topological spaces in a Kubiak-Šostak sense and introduced induced I-fuzzy topological spaces in [15]. Zhang and Liu [17] studied weakly induced modifications of L-topologies. The aim of this paper is to study weakly induced I-fuzzy topological spaces and the weakly induced modifications of I-fuzzy topologies.

This paper is organized as follows. In Section 1, we give some preliminary concepts and properties. Two kinds of weakly induced modifications are introduced in Section 2.

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We prove that I-WIFTOP—the category of weakly induced I-fuzzy topological spaces—is a reflective and coreflective full subcategory of I-FTOP. Finally, in Section 3, we discuss the relationship between several important categories.

In this paper, X is a nonempty set and $I = [0,1]$, $I_0 = [0,1)$. The family of all fuzzy sets on X will be denoted by I^X. By 0_X and 1_X, we denote, respectively, the constant fuzzy set on X taking the values 0 and 1. Let $\sigma_r(A) = \{x \mid A(x) > r\}$ for $r \in I$ and $A \in I^X$. $U \in P(X)$, 1_U denotes the characteristic function of U, that is, $1_U(x) = 1$ when $x \in U$ and $1_U(x) = 0$ when $x \notin U$. For the notions about categories, please refer to [1, 5, 9].

Definition 1.1 [4, 13]. A fuzzifying topology on X is a map $\xi : P(X) \to I$ satisfying the following axioms:
(FY1) $\xi(\emptyset) = \xi(X) = 1$;
(FY2) $\xi(U \cap V) \geq \xi(U) \wedge \xi(V)$ for all $U, V \in P(X)$;
(FY3) $\xi(\bigcup_{t\in T} U_t) \geq \bigwedge_{t\in T} \xi(U_t)$ for every family $\{U_t \mid t \in T\} \subseteq P(X)$.

If ξ is a fuzzifying topology on X, the pair (X,ξ) is called a fuzzifying topological space. A fuzzifying continuous map between fuzzifying topological spaces (X,ξ) and (Y,η) is a map $f : X \to Y$ such that $\xi(f^{\leftarrow}(U)) \geq \eta(U)$ for all $U \in P(Y)$. The category of fuzzifying topological spaces and fuzzifying continuous maps is denoted by FYS. Let FYS(X) denote the set of all fuzzifying topologies on X.

Definition 1.2 [6, 11]. An I-fuzzy topology on a set X is defined to be a map $\mathcal{T} : I^X \to I$ satisfying:
(FT1) $\mathcal{T}(1_X) = \mathcal{T}(0_X) = 1$;
(FT2) $\mathcal{T}(A \wedge B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$ for all $A,B \in I^X$;
(FT3) $\mathcal{T}(\bigvee_{t\in T} A_t) \geq \bigwedge_{t\in T} \mathcal{T}(A_t)$ for every family $\{A_t \mid t \in T\} \subseteq I^X$.

If $\mathcal{T}$ is an I-fuzzy topology on X, the pair $(I^X,\mathcal{T})$ is called an I-fuzzy topological space. An I-fuzzy continuous map between I-fuzzy topological spaces $(I^X,\mathcal{T})$ and $(I^Y,\mathcal{S})$ is a map $f : X \to Y$ such that $\mathcal{T}(f_I^{\leftarrow}(B)) \geq \mathcal{S}(B)$ for all $B \in I^Y$, where $f_I^{\leftarrow}(B)(x) = B(f(x))$ (following the notation in [10]). The category of I-fuzzy topological spaces and I-fuzzy continuous maps is denoted by I-FTOP. Let I-FTOP(X) denote the set of all I-fuzzy topologies on X.

Definition 1.3 [13]. Let ξ be a fuzzifying topology on X, $\mathcal{B} : P(X) \to I$, and $\mathcal{B} \leq \mathcal{T}$. $\mathcal{B}$ is called a base of ξ if $\mathcal{B}$ satisfies the following condition:

$$\forall U \in P(X), \quad \xi(U) = \bigvee_{\bigcup_{\lambda\in\wedge} V_\lambda = U} \bigwedge_{\lambda\in\wedge} \mathcal{B}(V_\lambda), \tag{1.1}$$

where the expression $\bigvee_{\bigcup_{\lambda\in\wedge} V_\lambda = U} \bigwedge_{\lambda\in\wedge} \mathcal{B}(V_\lambda)$ will be denoted by $\mathcal{B}^{(\sqcup)}(U)$, that is, $\xi = \mathcal{B}^{(\sqcup)}$.

A map $\phi : P(X) \to I$ is called a subbase of ξ if $\phi^{(\sqcap)} : P(X) \to I$ defined by $\phi^{(\sqcap)}(U) = \bigvee_{(\sqcap)_{\lambda\in J} V_\lambda = U} \bigwedge_{\lambda\in J} \phi(V_\lambda)$ for all $U \in P(X)$ is a base, where $(\sqcap)$ stands for "finite intersection." $\phi : P(X) \to I$ is a subbase of one fuzzifying topology if and only if $\phi^{(\sqcup)}(X) = 1$.

Definition 1.4 [14]. Let $\{(X_t,\xi_t)\}_{t\in T}$ be a family of fuzzifying topological spaces and let $P_t : \prod_{t\in T} X_t \to X_t$ be the projection. Then the fuzzifying topology whose subbase is defined by

$$\forall W \in P\left(\prod_{t\in T} X_t\right), \quad \phi(W) = \bigvee_{t\in T} \bigvee_{P_t^{\leftarrow}(U)=W} \xi_t(U) \tag{1.2}$$

is called the product topology of $\{\xi_t \mid t \in T\}$, denoted by $\prod_{t\in T}\xi_t$. $(\prod_{t\in T} X_t,\ \prod_{t\in T}\xi_t)$ is called the product space of $\{(X_t,\xi_t)\}_{t\in T}$.

Fang and Yue [3] extended the above definitions and results to I-fuzzy topological spaces. For more explicitly, we list them as follows.

(1) Let $\mathcal{T}$ be an I-fuzzy topology on X, $\mathcal{B} : I^X \to I$ s.t. $\mathcal{B} \le \mathcal{T}$ (in a pointwise sense). Then $\mathcal{B}$ is called a base of $\mathcal{T}$ if $\mathcal{B}$ satisfies the following condition:

$$\forall A \in I^X, \quad \mathcal{T}(A) = \bigvee_{\bigvee_{\lambda\in\wedge} B_\lambda = A} \bigwedge_{\lambda\in\wedge} \mathcal{B}(B_\lambda), \tag{1.3}$$

where the expression $\bigvee_{\bigvee_{\lambda\in\wedge} B_\lambda=A} \bigwedge_{\lambda\in\wedge} \mathcal{B}(B_\lambda)$ will be denoted by $\mathcal{B}^{(\sqcup)}(A)$.

(2) Let $\phi : I^X \to I$ be a map. Then ϕ is called a subbase of $\mathcal{T}$ if $\phi^{(\sqcap)} : I^X \to I$ is a base, where $\phi^{(\sqcap)}(A) = \bigvee_{(\sqcap)_{\lambda\in J} B_\lambda = A} \bigwedge_{\lambda\in J} \phi(B_\lambda)$ for all $A \in I^X$ with $(\sqcap)$ standing for "finite intersection." A map $\phi : I^X \to I$ is a subbase if and only if $\phi^{(\sqcup)}(1_X) = 1$.

(3) Let $\{(I^{X_t}, \mathcal{T}_t)\}_{t\in T}$ be a family of I-fuzzy topological spaces and let $P_t : \prod_{t\in T} X_t \to X_t$ be the projection. Then the I-fuzzy topology whose subbase is defined by

$$\forall A \in I^{\prod_{t\in T} X_t}, \quad \phi(A) = \bigvee_{t\in T} \bigvee_{(P_t)_I^{\leftarrow}(B)=A} \mathcal{T}_t(B) \tag{1.4}$$

is called the product topology of $\{\mathcal{T}_t \mid t \in T\}$, denoted by $\prod_{t\in T}\mathcal{T}_t$. $(I^{\prod_{t\in T} X_t},\ \prod_{t\in T}\mathcal{T}_t)$ is called the product space of $\{(I^{X_t}, \mathcal{T}_t)\}_{t\in T}$.

Definition 1.5. Let $\{(I^{X_t}, \mathcal{T}_t)\}_{t\in T}$ be a family of I-fuzzy topological spaces, let different $X_t's$ be disjoint and $X = \bigcup_{t\in T} X_t$, and let $\mathcal{T} : I^X \to I$ be defined as follows:

$$\forall A \in I^X, \quad \mathcal{T}(A) = \bigwedge_{t\in T} \mathcal{T}_t(A \mid X_t). \tag{1.5}$$

Then it is easy to verify that $\mathcal{T}$ is an I-fuzzy topology on X, and $\mathcal{T}$ is called the sum topology of $\{\mathcal{T}_t\}_{t\in T}$, denoted by $\bigoplus_{t\in T}\mathcal{T}_t$.

Definition 1.6. Let $(I^X, \mathcal{T})$ be an I-fuzzy topological space and let $f : X \to Y$ be a surjective map. Define the I-fuzzy quotient topology $\mathcal{T}/f_I^{\to}$ of $\mathcal{T}$ with respect to f by

$$\forall A \in I^Y, \quad \mathcal{T}/f_I^{\to}(A) = \mathcal{T}\left(f_I^{\leftarrow}(A)\right). \tag{1.6}$$

It is easy to verify that $\mathcal{T}/f_I^{\to}$ is an I-fuzzy topology on Y. $(I^Y, \mathcal{T}/f_I^{\to})$ is called the I-fuzzy quotient space of $(I^X, \mathcal{T})$ with respect to f and $f_I^{\to}$ is called an I-fuzzy quotient map.

Definition 1.7 [9]. Let $(I^X, \mathcal{T})$ be an I-fuzzy topological space and $Y \subseteq X$. $(I^Y, \mathcal{T} \mid Y)$ is called the subspace of $(I^X, \mathcal{T})$, where $\mathcal{T} \mid Y : I^Y \to I$ is defined by $\mathcal{T} \mid Y(B) = \bigvee\{\mathcal{T}(A) \mid A \in I^X, A \mid Y = B\}$ for all $B \in I^Y$.

Lemma 1.8 [5]. *I-FTOP(X) is a complete lattice.*

Using the similar argument in [5], it is easy to show that FYS(X) is also a complete lattice.

Lemma 1.9 [15]. *Let $\{\xi_t\}_{t\in T} \subseteq FYS(X)$. Then $\phi : P(X) \to I$ defined by $\phi(U) = \bigvee_{t\in T} \xi_t(U)$ is the subbase of $\bigvee_{t\in T} \xi_t$, that is, $\bigvee_{t\in T} \xi_t = (\phi^{(\sqcap)})^{(\sqcup)}$.*

2. Weakly induced modifications of I-fuzzy topologies

The purpose of this section is to study weakly induced I-fuzzy topological spaces and the weakly induced modifications of I-fuzzy topologies.

Definition 2.1 [15]. Let $(I^X, \mathcal{T})$ be an I-fuzzy topological space on X. If $\mathcal{T}(A) \leq \bigwedge_{r\in I_0} \mathcal{T}(1_{\sigma_r(A)})$ for all $A \in I^X$, then $(I^X, \mathcal{T})$ is called a weakly induced I-fuzzy topological space. Let I-WIFTOP denote the category of weakly induced I-fuzzy topological spaces.

Example 2.2. Let ξ be a fuzzifying topology on X. Define $\mathcal{T}_\xi : I^X \to I$ as follows:

$$\mathcal{T}_\xi(A) = \begin{cases} \xi(U) & \text{if } A \text{ is a characteristic function, that is, } A = 1_U, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

It is easy to check that $\mathcal{T}_\xi$ is an I-fuzzy topology on X and it is weakly induced. Specially, $\mathcal{T}$ is weakly induced, where

$$\mathcal{T}(A) = \begin{cases} 1, & A = 0_X, 1_X, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Example 2.3. Let $\mathcal{T} : I^X \to I$ be defined by $\mathcal{T}(A) = 1$ for all $A \in L^X$. Then $\mathcal{T}$ is a weakly induced I-fuzzy topology on X.

In the following discussion, we will give the right adjoint functor and left adjoint functor of the inclusion functor $i : I$-WIFTOP $\to I$-FTOP, and show that I-WIFTOP is a reflective and coreflective full subcategory of I-FTOP.

Lemma 2.4. *Let $(I^X, \mathcal{T})$ be an I-fuzzy topological space and let $\mathcal{T}_* : I^X \to I$ be defined by*

$$\forall A \in I^X, \quad \mathcal{T}_*(A) = \bigwedge_{r\in I_0} \mathcal{T}(1_{\sigma_r(A)}) \wedge \mathcal{T}(A). \tag{2.3}$$

Then $\mathcal{T}_*$ is the biggest weakly induced I-fuzzy topology smaller than $\mathcal{T}$. Hence, if $\mathcal{T}$ is weakly induced, then $\mathcal{T} = \mathcal{T}_*$.

Proof. It is routine to prove that $\mathcal{T}_*$ is an I-fuzzy topology on X. The following computation can show that $\mathcal{T}_*$ is weakly induced:

$$\begin{aligned}\bigwedge_{r\in I_0}\mathcal{T}_*\left(1_{\sigma_r(A)}\right) &= \bigwedge_{r\in I_0}\bigwedge_{s\in I_0}\mathcal{T}\left(1_{\sigma_s(1_{\sigma_r(A)})}\right)\wedge\mathcal{T}\left(1_{\sigma_r(A)}\right) = \bigwedge_{r\in I_0}\mathcal{T}\left(1_{\sigma_r(A)}\right)\\ &\geq \bigwedge_{r\in I_0}\mathcal{T}\left(1_{\sigma_r(A)}\right)\wedge\mathcal{T}(A) = \mathcal{T}_*(A).\end{aligned} \tag{2.4}$$

Let $\mathcal{S}$ be any weakly induced I-fuzzy topology on X satisfying $\mathcal{S}\leq\mathcal{T}$. We need to prove that $\mathcal{S}\leq\mathcal{T}_*$. Since $\mathcal{S}$ is weakly induced, we have $\mathcal{S}(A)\leq\bigwedge_{r\in I_0}\mathcal{S}(1_{\sigma_r(A)})$ for all $A\in I^X$. Hence we get that

$$\mathcal{S}(A)\leq\mathcal{T}(A)\wedge\bigwedge_{r\in I_0}\mathcal{S}\left(1_{\sigma_r(A)}\right)\leq\mathcal{T}(A)\wedge\bigwedge_{r\in I_0}\mathcal{T}\left(1_{\sigma_r(A)}\right)=\mathcal{T}_*(A), \tag{2.5}$$

thus the conclusion. □

Lemma 2.5. *Let $(I^Y,\mathcal{T})$ be weakly induced and let $(I^X,\mathcal{S})$ be an I-fuzzy topological space. Then $f_I^{\rightarrow}:(I^X,\mathcal{S})\to(I^Y,\mathcal{T})$ is I-fuzzy continuous if and only if $f_I^{\rightarrow}:(I^X,\mathcal{S}_*)\to(I^Y,\mathcal{T}_*)=(I^Y,\mathcal{T})$ is I-fuzzy continuous.*

Proof. The sufficiency is obvious and it needs to show the necessity. Let $f_I^{\rightarrow}:(I^X,\mathcal{S})\to(I^Y,\mathcal{T})$ be I-fuzzy continuous, that is, $\mathcal{T}(B)\leq\mathcal{S}(f_I^{\leftarrow}(B))$ for all $B\in I^Y$. Since $\mathcal{T}$ is weakly induced, we have $\mathcal{T}(B)\leq\bigwedge_{r\in I_0}\mathcal{T}(1_{\sigma_r(B)})$. Hence

$$\mathcal{T}(B)\leq\mathcal{S}\left(f_I^{\leftarrow}(B)\right)\wedge\bigwedge_{r\in I_0}\mathcal{T}\left(1_{\sigma_r(B)}\right)\leq\mathcal{S}\left(f_I^{\leftarrow}(B)\right)\wedge\bigwedge_{r\in I_0}\mathcal{S}\left(1_{f^{\leftarrow}(\sigma_r(B))}\right)=\mathcal{S}_*\left(f_I^{\leftarrow}(B)\right). \tag{2.6}$$

Therefore, $f_I^{\rightarrow}:(I^X,\mathcal{S}_*)\to(I^Y,\mathcal{T})$ is I-fuzzy continuous. □

Remark 2.6. From Lemma 2.5, we also can get that $f_I^{\rightarrow}:(I^X,\mathcal{S}_*)\to(I^Y,\mathcal{T}_*)$ is I-fuzzy continuous if $f_I^{\rightarrow}:(I^X,\mathcal{S})\to(I^Y,\mathcal{T})$ is I-fuzzy continuous. Hence we know that $(\cdot)_*$ is a functor from I-FTOP to I-WIFTOP. Furthermore, we have the following theorem.

Theorem 2.7. *$(\cdot)_*$ is the left adjoint of i.*

Lemma 2.8. *Let $(I^X,\mathcal{T})$ be an I-fuzzy topological space and let $\phi:I^X\to I$ be defined by*

$$\phi^{\mathcal{T}}(A)=\begin{cases}\bigvee_{r\in I_0}\bigvee\{\mathcal{T}(B)\mid\sigma_r(B)=U\} & \text{if } A \text{ is a characteristic function, that is, } A=1_U,\\ \mathcal{T}(A) & \text{otherwise.}\end{cases} \tag{2.7}$$

Then $\phi^{\mathcal{T}}$ is a subbase of one I-fuzzy topology, and denote this I-fuzzy topology by $\mathrm{wi}(\mathcal{T})$. $\mathrm{wi}(\mathcal{T})$ is called the weakly induced modification of $\mathcal{T}$.

Proof. It is trivial to verify that $\phi^{\mathcal{T}}$ is a subbase of one I-fuzzy topology. □

Theorem 2.9. *Let $(I^X,\mathcal{T})$ be an I-fuzzy topological space. Then $\mathrm{wi}(\mathcal{T})$ is the smallest weakly induced I-fuzzy topology bigger than $\mathcal{T}$. Hence, if $\mathcal{T}$ is weakly induced, then $\mathcal{T}=\mathrm{wi}(\mathcal{T})$.*

Proof. We need to prove that $\mathrm{wi}(\mathcal{T})(A) \le \bigwedge_{r\in I_0} \mathrm{wi}(\mathcal{T})(1_{\sigma_r(A)})$, that is, $\mathrm{wi}(\mathcal{T})(A) \le \mathrm{wi}(\mathcal{T})(1_{\sigma_r(A)})$ for all $r \in I_0$. In fact, noting that

$$\mathrm{wi}(\mathcal{T})(A) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\lambda\beta} = B_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi^{\mathcal{T}}(C_{\lambda\beta}),$$
$$\mathrm{wi}(\mathcal{T})(1_{\sigma_r(A)}) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = 1_{\sigma_r(A)}} \bigwedge_{\lambda\in\Lambda} \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\lambda\beta} = B_\lambda} \bigwedge_{\beta\in\Lambda_\lambda} \phi^{\mathcal{T}}(C_{\lambda\beta}), \tag{2.8}$$

we have $\mathrm{wi}(\mathcal{T})(A) \le \mathrm{wi}(\mathcal{T})(1_{\sigma_r(A)})$ according to $\phi^{\mathcal{T}}(C_{\lambda\beta}) \le \phi^{\mathcal{T}}(1_{\sigma_r(C_{\lambda\beta})})$, as desired.

We now prove that $\mathrm{wi}(\mathcal{T})$ is the smallest weakly induced I-fuzzy topology bigger than $\mathcal{T}$. Let $\mathcal{T}^*$ be any weakly induced I-fuzzy topology on X bigger than $\mathcal{T}$. We need to prove that $\mathrm{wi}(\mathcal{T}) \le \mathcal{T}^*$. It suffices to show that $\phi^{\mathcal{T}}(A) \le \mathcal{T}^*(A)$ for all $A \in I^X$. Then it suffices to show that $\phi^{\mathcal{T}}(1_U) \le \mathcal{T}^*(1_U)$ for all $U \subseteq X$, and this can be obtained by the following computation:

$$\begin{aligned}\phi^{\mathcal{T}}(1_U) &= \bigvee_{r\in I_0} \bigvee \{\mathcal{T}(B) \mid \sigma_r(B) = U\} \le \bigvee_{r\in I_0} \bigvee \{\mathcal{T}^*(B) \mid \sigma_r(B) = U\} \\ &\le \bigvee_{r\in I_0} \bigvee \left\{ \bigwedge_{s\in I_0} \mathcal{T}^*(1_{\sigma_s(B)}) \mid \sigma_r(B) = U \right\} \le \mathcal{T}^*(1_U),\end{aligned} \tag{2.9}$$

thus the conclusion. □

Lemma 2.10. *Let $(I^Y, \mathcal{T})$ be weakly induced and let $(I^X, \mathcal{S})$ be an I-fuzzy topological space. Then $f_I^{\rightarrow} : (I^Y, \mathcal{T}) \to (I^X, \mathcal{S})$ is I-fuzzy continuous if and only if $f_I^{\rightarrow} : (I^Y, \mathcal{T}) \to (I^X, \mathrm{wi}(\mathcal{S}))$ is I-fuzzy continuous.*

Proof. The sufficiency is obvious. We need to prove the necessity. It suffices to show that $\phi^{\mathcal{S}}(A) \le \mathcal{T}(f_I^{\leftarrow}(A))$ for all $A = 1_U \in I^X$. Since $f_I^{\rightarrow} : (I^Y, \mathcal{T}) \to (I^X, \mathcal{S})$ is I-fuzzy continuous, we have

$$\phi^{\mathcal{S}}(1_U) = \bigvee_{r\in I_0} \bigvee \{\mathcal{S}(B) \mid \sigma_r(B) = U\} \le \bigvee_{r\in I_0} \bigvee \{\mathcal{T}(f_I^{\leftarrow}(B)) \mid \sigma_r(B) = U\} \le \mathcal{T}(f_I^{\leftarrow}(1_U)), \tag{2.10}$$

thus the conclusion. □

Remark 2.11. From Lemma 2.10 above, we also can get that $f_I^{\rightarrow} : (I^X, \mathrm{wi}(\mathcal{S})) \to (I^Y, \mathrm{wi}(\mathcal{T}))$ is I-fuzzy continuous if $f_I^{\rightarrow} : (I^X, \mathcal{S}) \to (I^Y, \mathcal{T})$ is I-fuzzy continuous. Hence wi is another functor from I-FTOP to I-WIFTOP. Furthermore, we have the following theorem.

Theorem 2.12. wi *is the right adjoint of i.*

From Theorems 2.7 and 2.12, we have the main theorem in this paper as follows.

Theorem 2.13. *I-WIFTOP is a reflective and coreflective full subcategory of I-FTOP.*

By the properties of right adjoint, we have the following corollaries.

Corollary 2.14. *Let $(I^X, \mathcal{T})$ be an I-fuzzy topological space and $Y \subseteq X$. Then* $\mathrm{wi}(\mathcal{T} \mid Y) = \mathrm{wi}(\mathcal{T}) \mid Y$.

Corollary 2.15. *Let $\{(I^{X_t}, \mathcal{T}_t)\}_{t\in T}$ be a family of I-fuzzy topological spaces and $X = \prod_{t\in T} X_t$. Then* $\mathrm{wi}(\prod_{t\in T}\mathcal{T}_t) = \prod_{t\in T}\mathrm{wi}(\mathcal{T}_t)$.

Theorem 2.16. *Let $\{(I^{X_t}, \mathcal{T}_t)\}_{t\in T}$ be a family of I-fuzzy topological spaces and let different X_t's be disjoint. Then* $\mathrm{wi}(\bigoplus_{t\in T}\mathcal{T}_t) = \bigoplus_{t\in T}\mathrm{wi}(\mathcal{T}_t)$.

Proof. First, we have

$$\begin{aligned}\bigoplus_{t\in T}\mathrm{wi}(\mathcal{T}_t)(A) &= \bigwedge_{t\in T}\mathrm{wi}(\mathcal{T}_t)(A\mid X_t) \le \bigwedge_{t\in T}\bigwedge_{r\in I_0}\mathrm{wi}(\mathcal{T}_t)\left(1_{\sigma_r(A\mid X_t)}\right)\\ &= \bigwedge_{t\in T}\bigwedge_{r\in I_0}\mathrm{wi}(\mathcal{T}_t)\left(1_{\sigma_r(A)}\mid X_t\right) = \bigwedge_{r\in I_0}\bigwedge_{t\in T}\mathrm{wi}(\mathcal{T}_t)\left(1_{\sigma_r(A)}\mid X_t\right)\\ &= \bigwedge_{r\in I_0}\bigoplus_{t\in T}\mathrm{wi}(\mathcal{T}_t)\left(1_{\sigma_r(A)}\right).\end{aligned} \tag{2.11}$$

Hence, $\bigoplus_{t\in T}\mathrm{wi}(\mathcal{T}_t)$ is weakly induced. Therefore, $\mathrm{wi}(\bigoplus_{t\in T}\mathcal{T}_t) \le \bigoplus_{t\in T}\mathrm{wi}(\mathcal{T}_t)$.

Conversely, let $\lambda < \bigoplus_{t\in T}\mathrm{wi}(\mathcal{T}_t)(A)$, that is,

$$\lambda < \bigoplus_{t\in T}\mathrm{wi}(\mathcal{T}_t)(A) = \bigwedge_{t\in T}\mathrm{wi}(\mathcal{T}_t)(A\mid X_t) = \bigwedge_{t\in T}\ \bigvee_{\bigvee_{\lambda\in\Lambda^t} D_\lambda^t = A\mid X_t}\ \bigwedge_{\lambda\in\Lambda^t}\ \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda^t} E_{\lambda\beta}^t = D_\lambda^t}\ \bigwedge_{\beta\in\Lambda_\lambda^t}\phi^{\mathcal{T}_t}(E_{\lambda\beta}^t). \tag{2.12}$$

Then, for all $t \in T$, there exists $\{D_\lambda^t\}_{\lambda\in\Lambda^t} \subseteq I^{X_t}$ such that

(i) $\bigvee_{\lambda\in\Lambda^t} D_\lambda^t = A \mid X_t$;

(ii) for each $\lambda \in \Lambda^t$, there exists $\{E_{\lambda\beta}^t\}_{\beta\in\Lambda_\lambda^t} \subseteq I^{X_t}$ such that $(\sqcap)_{\beta\in\Lambda_\lambda^t} E_{\lambda\beta}^t = D_\lambda^t$;

(iii) for each $\beta \in \Lambda_\lambda^t$, we have $\lambda \le \phi^{\mathcal{T}_t}(E_{\lambda\beta}^t)$.

Let $(D_\lambda^t)^* \in I^X$ and $(E_{\lambda\beta}^t)^* \in I^X$ be defined as follows:

$$\begin{gathered}\left(D_\lambda^t\right)^*(x) = \begin{cases} D_\lambda^t(x), & x\in X_t,\\ 0, & x\notin X_t,\end{cases}\\ \left(E_{\lambda\beta}^t\right)^*(x) = \begin{cases} E_{\lambda\beta}^t(x), & x\in X_t,\\ 0, & x\notin X_t.\end{cases}\end{gathered} \tag{2.13}$$

Then we have

$$\bigvee_{t\in T}\bigvee_{\lambda\in\Lambda^t}\left(D_\lambda^t\right)^* = A, \qquad (\sqcap)_{\beta\in\Lambda_\lambda^t}\left(E_{\lambda\beta}^t\right)^* = \left(D_\lambda^t\right)^*, \qquad \phi^{\mathcal{T}_t}(E_{\lambda\beta}^t) = \phi^{\oplus_{t\in t}\mathcal{T}_t}\left(\left(E_{\lambda\beta}^t\right)^*\right). \tag{2.14}$$

Therefore, $\lambda \le \phi^{\oplus_{t\in t}\mathcal{T}_t}((E_{\lambda\beta}^t)^*)$ due to $\lambda \le \phi^{\mathcal{T}_t}(E_{\lambda\beta}^t)$. Note that

$$\mathrm{wi}\left(\bigoplus_{t\in T}\mathcal{T}_t\right)(A) = \bigvee_{\bigvee_{\lambda\in\Lambda} B_\lambda = A}\ \bigwedge_{\lambda\in\Lambda}\ \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda} C_{\lambda\beta} = B_\lambda}\ \bigwedge_{\beta\in\Lambda_\lambda}\phi^{\oplus_{t\in T}\mathcal{T}_t}(C_{\lambda\beta}). \tag{2.15}$$

We have $\lambda \leq \mathrm{wi}(\bigoplus_{t\in T}\mathcal{T}_t)(A)$. Then $\bigoplus_{t\in T}\mathrm{wi}(\mathcal{T}_t)(A) \leq \mathrm{wi}(\bigoplus_{t\in T}\mathcal{T}_t)(A)$. This completes the proof. □

The readers can easily prove the following theorem.

Theorem 2.17. *Let $(I^X,\mathcal{T})$ be an I-fuzzy topological space and let $(I^Y,\mathcal{T}/f_I^{\to})$ be the I-fuzzy quotient space of $(I^X,\mathcal{T})$ with respect to $f : X \to Y$. If $(I^X,\mathcal{T})$ is weakly induced, then $(I^Y,\mathcal{T}/f_I^{\to})$ is weakly induced.*

3. On the relationships between several categories

In Section 2, we study weakly induced modifications of I-fuzzy topologies. Since weakly induced and induced topological spaces play an important role in L-topology, in this section, we will study induced I-fuzzy topologies and the relationships between the categories FYS, I-WIFTOP, I-SFTOP, I-IFTOP, and I-FTOP, where I-IFTOP and I-SFTOP denote the categories of induced I-fuzzy topological spaces and stratified I-fuzzy topological spaces, respectively. In the following discussion, we always assume that I-TOP denotes the category of stratified Chang-Goguen topological spaces. We know that TOP can be regarded as a full (moreover, simultaneously reflective and coreflective) subcategory of I-TOP by Lowen functors. Zhang [16] proved that TOP is a reflective and coreflective full subcategory of FYS and FYS is a reflective and coreflective full subcategory of I-TOP. From [15], we know that FYS is isomorphic to I-IFTOP. We will prove that I-IFTOP is a reflective and coreflective full subcategory of I-SFTOP and I-IFTOP is a coreflective full subcategory of I-WIFTOP.

Let $(I^X,\mathcal{T})$ be an I-fuzzy topological space and let $[\mathcal{T}] : P(X) \to I$ be defined by $[\mathcal{T}](U) = \mathcal{T}(1_U)$ for all $U \in P(X)$. Then it is easy to verify that $[\mathcal{T}]$ is a fuzzifying topology on X.

Definition 3.1 [15]. Let $(I^X,\mathcal{T})$ be an I-fuzzy topological space. $[\mathcal{T}]$ is called the background topology of $\mathcal{T}$ and $(X,[\mathcal{T}])$ is called the background space of $(I^X,\mathcal{T})$.

From the definition above, we get a functor $[\cdot]$ from I-FTOP to FYS. It is easy to verify the following two theorems.

Theorem 3.2. *If $f_I^{\to} : (I^X,\mathcal{T}_1) \to (I^Y,\mathcal{T}_2)$ is I-fuzzy continuous, then $f : (X,[\mathcal{T}_1]) \to (Y,[\mathcal{T}_2])$ is a fuzzifying continuous.*

Theorem 3.3. *Let $\{(I^{X_t},\mathcal{T}_t)\}_{t\in T}$ be a family of I-fuzzy topological spaces and let different X_t's be disjoint. Then $[\bigoplus_{t\in T}\mathcal{T}_t] = \bigoplus_{t\in T}[\mathcal{T}_t]$.*

Definition 3.4 [15]. Let $(I^X,\mathcal{T})$ be an I-fuzzy topological space on X. If $\mathcal{T}(A) = \bigwedge_{r\in I_0}\mathcal{T}(1_{\sigma_r(A)})$ for all $A \in I^X$, then $(I^X,\mathcal{T})$ is called an induced I-fuzzy topological space. If $\mathcal{T}(\bar{\lambda}) = 1$ for all $\lambda \in I$, where $\bar{\lambda}$ is the constant function from X to I with value λ, then $(X,\mathcal{T})$ is called a stratified I-fuzzy topological space.

Lemma 3.5 [15]. *Let $\mathcal{T}$ be an I-fuzzy topology on X and let $\phi_{\mathcal{T}} : P(X) \to I$ be defined by $\phi_{\mathcal{T}}(U) = \bigvee_{r\in I}\bigvee\{\mathcal{T}(B) \mid B \in I^X, \sigma_r(B) = U\}$ for $U \in P(X)$. Then $\phi_{\mathcal{T}}$ is the subbase of one fuzzifying topology, and let this fuzzifying topology be denoted by $\iota(\mathcal{T})$.*

Definition 3.6 [15]. Let $\mathcal{T}$ be an I-fuzzy topology on X. $\iota(\mathcal{T})$ is called a generated fuzzifying topology by $\mathcal{T}$.

We get another functor ι from I-FTOP to FYS.

Lemma 3.7 [15]. *Let (X,ξ) be a fuzzifying topological space and define $\omega(\xi): I^X \to I$ as follows: $\omega(\xi)(A) = \bigwedge_{r\in I_0} \xi(\sigma_r(A))$ for all $A \in I^X$. Then $\omega(\xi)$ is an I-fuzzy topology on X.*

From Lemma 3.7, we know that ω is a functor from FYS to I-FTOP.

Lemma 3.8 [15]. (1) *For every $\xi \in FYS(X)$, $\iota(\omega(\xi)) = \xi$.*

(2) *For every $\mathcal{T} \in L\text{-}FTOP(X)$, $\omega(\iota(\mathcal{T})) \geq \mathcal{T}$. If $\mathcal{T} = \omega(\xi)$, then $\omega(\iota(\mathcal{T})) = \mathcal{T}$.*

Corollary 3.9 [15]. *Both $\omega: FYS(X) \to \omega(FYS(X))$ and $\iota: \omega(FYS(X)) \to FYS(X)$ are complete lattice isomorphisms.*

Corollary 3.10. *FYS is isomorphic to I-IFTOP.*

Now we begin to study the relations between the categories FYS, I-WIFTOP, I-SFTOP, I-IFTOP, and I-FTOP. Firstly, we give the left adjoint and the right adjoint of the inclusion functor i from I-IFTOP to I-FTOP.

Lemma 3.11. *Let $(I^X,\mathcal{S})$ be a stratified I-fuzzy topological space and let $(I^Y,\mathcal{T})$ be an induced I-fuzzy topological space. Then $f_I^{\rightarrow}: (I^X,\mathcal{S}) \to (I^Y,\mathcal{T})$ is I-fuzzy continuous if and only if $f_I^{\rightarrow}: (I^X,\omega([\mathcal{S}])) \to (I^Y,\omega([\mathcal{T}])) = (I^Y,\mathcal{T})$ is I-fuzzy continuous.*

Proof. Since $(I^X,\mathcal{S})$ is stratified, we have

$$\begin{aligned} \omega([\mathcal{S}])(A) &= \bigwedge_{r\in I_0} \mathcal{S}(1_{\sigma_r(A)}) = \bigwedge_{r\in I_0} \mathcal{S}(\bar{r}) \wedge \mathcal{S}(1_{\sigma_r(A)}) \\ &\leq \bigwedge_{r\in I_0} \mathcal{S}(\bar{r}1_{\sigma_r(A)}) \leq \mathcal{S}\left(\bigvee_{r\in I_0} \bar{r}1_{\sigma_r(A)}\right) = \mathcal{S}(A). \end{aligned} \tag{3.1}$$

Hence we get that $f_I^{\rightarrow}: (I^X,\mathcal{S}) \to (I^Y,\mathcal{T})$ is I-fuzzy continuous if $f_I^{\rightarrow}: (I^X,\omega([\mathcal{S}])) \to (I^Y,\omega([\mathcal{T}])) = (I^Y,\mathcal{T})$ is I-fuzzy continuous. Conversely, let $f_I^{\rightarrow}: (I^X,\mathcal{S}) \to (I^Y,\mathcal{T})$ be I-fuzzy continuous, that is, $\mathcal{T}(B) \leq \mathcal{S}(f_I^{\leftarrow}(B))$ for all $B \in I^Y$. Since $\mathcal{T}$ is induced, we have $\mathcal{T}(B) = \bigwedge_{r\in I_0} \mathcal{T}(1_{\sigma_r(B)})$. Hence

$$\mathcal{T}(B) = \bigwedge_{r\in I_0} \mathcal{T}(1_{\sigma_r(B)}) \leq \bigwedge_{r\in I_0} \mathcal{S}(f_I^{\leftarrow}(1_{\sigma_r(B)})) = \bigwedge_{r\in I_0} \mathcal{S}(1_{\sigma_r(f_I^{\leftarrow}(B))}) = \omega([\mathcal{S}])(f_I^{\leftarrow}(B)). \tag{3.2}$$

Therefore $f_I^{\rightarrow}: (I^X,\omega([\mathcal{S}])) \to (I^Y,\mathcal{T})$ is I-fuzzy continuous. □

Lemma 3.12. *Let $(I^X,\mathcal{T})$ be an I-fuzzy topological space and let $(I^Y,\mathcal{S})$ be an induced I-fuzzy topological space. Then $f_I^{\rightarrow}: (I^Y,\mathcal{S}) \to (I^X,\mathcal{T})$ is I-fuzzy continuous if and only if $f_I^{\rightarrow}: (I^Y,\mathcal{S}) \to (I^X,\omega\circ\iota(\mathcal{T}))$ is I-fuzzy continuous.*

Proof. The sufficiency is obvious. We need to prove the necessity. In fact, we have

$$\begin{aligned}\omega(\iota(\mathcal{T}))(A) &= \bigwedge_{r\in I_0}\ \bigvee_{\bigcup_{\lambda\in\Lambda}V_\lambda=\sigma_r(A)}\ \bigwedge_{\lambda\in\Lambda}\ \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda}W_{\lambda\beta}=V_\lambda}\ \bigwedge_{\beta\in\Lambda_\lambda}\ \bigvee_{\mu\in I_0}\bigvee\{\mathcal{T}(D)\mid\sigma_\mu(D)=W_{\lambda\beta}\}\\ &\leq \bigwedge_{r\in I_0}\ \bigvee_{\bigcup_{\lambda\in\Lambda}V_\lambda=\sigma_r(A)}\ \bigwedge_{\lambda\in\Lambda}\ \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda}W_{\lambda\beta}=V_\lambda}\ \bigwedge_{\beta\in\Lambda_\lambda}\ \bigvee_{\mu\in I_0}\bigvee\{\mathcal{S}(f_I^{\leftarrow}(D))\mid\sigma_\mu(D)=W_{\lambda\beta}\}\\ &\leq \bigwedge_{r\in I_0}\ \bigvee_{\bigcup_{\lambda\in\Lambda}V_\lambda=\sigma_r(A)}\ \bigwedge_{\lambda\in\Lambda}\ \bigvee_{(\sqcap)_{\beta\in\Lambda_\lambda}W_{\lambda\beta}=V_\lambda}\ \bigwedge_{\beta\in\Lambda_\lambda}\mathcal{S}(1_{f^{\leftarrow}(W_{\lambda\beta})})\leq\mathcal{S}(f_I^{\leftarrow}(A)),\end{aligned}\tag{3.3}$$

thus the conclusion. □

From Lemmas 3.11 and 3.12, we have the following theorems.

Theorem 3.13. (1) $\omega\circ\iota$ *is the right adjoint of the inclusion functor* $i: I$*-IFTOP* $\to I$*-FTOP.*
(2) $\omega\circ[\cdot]$ *is the left adjoint of the inclusion functor* $i: I$*-IFTOP* $\to I$*-SFTOP.*

Theorem 3.14. *I-IFTOP is a reflective and coreflective full subcategory of I-SFTOP and I-IFTOP is a coreflective full subcategory of I-WIFTOP. Hence, I-IFTOP is also a coreflective full subcategory of I-FTOP.*

Corollary 3.15. *FYS is a reflective and coreflective full subcategory of I-SFTOP. Hence TOP is a reflective and coreflective full subcategory of I-SFTOP.*

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