COMMON FIXED POINT THEOREMS IN MENGER SPACES

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We proved two common fixed point theorems for four self-mappings and two set-valued mappings with ϕ-contractive condition in a Menger space.

1. Introduction and preliminaries

Probabilistic metric space was first introduced by Menger [6]. Later, there are many authors who have some detailed discussions and applications of a probabilistic metric space, for example, we may see Schweizer and Sklar [8]. Besides, there are many results about fixed point theorems in a probabilistic metric space with contractive types having appeared; we may see the papers [1–3, 9–12].

In this paper, we will prove two common fixed point theorems for four self-mappings and two set-valued mappings with ϕ-contractive condition in a Menger space, which generalize some results of Dedeić and Sarapa [4, 5], and Sehgal and Bharucha-Reid [9].

A mapping $F:\mathbb{R}\to\mathbb{R}^+$ is said to be a distribution if it is nondecreasing left continuous with $\inf\{F(t):t\in\mathbb{R}\}=0$ and $\sup\{F(t):t\in\mathbb{R}\}=1$.

We will denote by $\mathscr{L}$ the set of all distribution functions while G will always denote the specific distribution function defined by

$$G(t)=\begin{cases}0, & t\le 0,\\ 1, & t>0.\end{cases} \tag{1.1}$$

A probabilistic metric space (PM-space) [7] is an ordered pair $(X,\mathscr{F})$ consisting of a nonempty set X and a mapping $\mathscr{F}$ from $X\times X$ into the collections of all distribution functions on $\mathbb{R}$. For $x,y\in X$, we denote the distribution function $\mathscr{F}(x,y)$ by $F_{x,y}$ and $F_{x,y}(u)$ represents the value of $\mathscr{F}(x,y)$ at $u\in\mathbb{R}$. The functions $F_{x,y}$ are assumed to satisfy the following conditions:

(1) $F_{x,y}(u)=1$ for all $u>0$ if and only if $x=y$,

(2) $F_{x,y}(0)=0$ for all x,y in X,

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(3) $F_{x,y}(u) = F_{y,x}(u)$ for all x, y in X, and

(4) if $F_{x,y}(u) = 1$ and $F_{y,z}(v) = 1$, then $F_{x,z}(u+v) = 1$ for all x, y, z in X and $u, v > 0$.

A mapping $t : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if

(1) $t(a,1) = a$, $t(0,0) = 0$,

(2) $t(a,b) = t(b,a)$,

(3) $t(c,d) \geq t(a,b)$ for $c \geq a$, $d \geq b$, and

(4) $t(t(a,b),c) = t(a,t(b,c))$.

A Menger space is a triplet $(X,\mathcal{F},t)$, where $(X,\mathcal{F})$ is a PM-space, t is a T-norm, and the generalized triangle inequality

$$F_{x,y}(u+v) \geq t\big(F_{x,y}(u), F_{y,z}(v)\big) \tag{1.2}$$

holds for all x, y, z in X and $u, v > 0$.

The concept of neighborhoods in a Menger space was introduced by Schweizer and Sklar [8].

Let $(X,\mathcal{F},t)$ be a Menger space. If $x \in X$, $\varepsilon > 0$, and $\lambda \in (0,1)$, then an (ε,λ)-neighborhood of x, called $U_x(\varepsilon,\lambda)$, is defined by

$$U_x(\varepsilon,\lambda) = \big\{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\big\}. \tag{1.3}$$

An (ε,λ)-topology in X is the topology induced by the family $\{U_x(\varepsilon,\lambda) : x \in X,\ \varepsilon > 0,\ \lambda \in (0,1)\}$ of neighborhood.

Remark 1.1. If t is continuous, then Menger space $(X,\mathcal{F},t)$ is a Hausdorff space in the (ε,λ)-topology. (see [8]).

Let $(X,\mathcal{F},t)$ be a complete Menger space and $A \subset X$. Then A is called a bounded set if

$$\lim_{u \to \infty} \inf_{x,y \in A} F_{x,y}(u) = 1. \tag{1.4}$$

Throughout this paper, $B(X)$ will denote the family of nonempty bounded subsets of a complete Menger space X.

For all $A, B \in B(X)$ and for all $u > 0$, we define

$$\begin{aligned} {}_{\delta}F_{A,B}(u) &= \inf\{F_{x,y}(u) : x \in A,\ y \in B\}, \\ {}_{D}F_{A,B}(u) &= \sup\{F_{x,y}(u) : x \in A,\ y \in B\}, \\ {}_{H}F_{A,B}(u) &= \inf\left\{\sup_{a\in A}\inf_{b \in B} F_{a,b}(u), \sup_{b\in B}\inf_{a \in A} F_{a,b}(u)\right\}. \end{aligned} \tag{1.5}$$

Remark 1.2. It is clear that ${}_{\delta}F_{A,B}(u) = {}_{\delta}F_{B,A}(u)$, ${}_{D}F_{A,B}(u) = {}_{D}F_{B,A}(u)$, and ${}_{H}F_{A,B}(u) = {}_{H}F_{B,A}(u)$, for all $A, B \in B(X)$ and $u > 0$.

If $A = \{x\}$, we denote ${}_\delta F_{\{x\},B}(u) = {}_\delta F_{x,B}(u)$, ${}_D F_{\{x\},B}(u) = {}_D F_{x,B}(u)$, and ${}_H F_{\{x\},B}(u) = {}_H F_{x,B}(u)$.

Let $(X,\mathcal{F},t)$ be a complete Menger space, and let $T : X \to B(X)$ be a set-valued function and $I : X \to X$ a single-valued function. Then we say that S and I are compatible if

$$\lim_{n\to\infty} {}_H F_{SIx_n,ISx_n}(u) = 1, \tag{1.6}$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} {}_\delta F_{Ix_n,Sx_n}(u) = 1, \quad \forall u > 0. \tag{1.7}$$

Let $\{A_n\}$ be a sequence in $B(X)$. We say that $\{A_n\}$ δ-converges to a set A in X if

$$\lim_{n\to\infty} {}_\delta F_{A_n,A}(u) = 1, \quad \text{for every } u > 0, \tag{1.8}$$

and it is denoted by $A_n \xrightarrow{\delta} A$.

2. Main results

In this paper, we let $\mathbb{R}^+$ denote the set of all nonnegative real numbers, let $\mathbb{N}$ denote the set of all positive integers, and let $(X,\mathcal{F},t)$ be a Menger space with $t(x,y) = \min(x,y)$.

We first prove the following lemmas.

Lemma 2.1. *Let $(X,\mathcal{F},\min)$ be a Menger space. Then for $A,B,C \in B(X)$ and for $u,v > 0$,*

$$ {}_\delta F_{A,C}(u+v) \geq \min\{{}_\delta F_{A,B}(u), {}_\delta F_{B,C}(v)\}. \tag{2.1}$$

Proof. For all $u,v > 0$, we have

$$\min\{{}_\delta F_{A,B}(u), {}_\delta F_{B,C}(v)\} \leq \min\{F_{a,b}(u), F_{b,c}(v)\} \leq F_{a,c}(u+v) \tag{2.2}$$

for each $a \in A$, $b \in B$, and $c \in C$.

This implies that $\min\{{}_\delta F_{A,B}(u), {}_\delta F_{B,C}(v)\} \leq {}_\delta F_{A,C}(u+v)$. □

Lemma 2.2. *Let $(X,\mathcal{F},\min)$ be a Menger space. Then for $A,B \in B(X)$, $c \in X$, and for $u,v > 0$,*

$$ {}_H F_{A,c}(u+v) \geq \min\{{}_H F_{A,B}(u), {}_H F_{B,c}(v)\}. \tag{2.3}$$

Proof. Since for each $a,b,c \in X$ and for all $u,v > 0$,

$$F_{a,c}(u+v) \geq \min\{F_{a,b}(u), F_{b,c}(v)\}. \tag{2.4}$$

By taking $\inf_{c\in C}$, we have

$$\inf_{c\in C} F_{a,c}(u+v) \geq \min\left\{F_{a,b}(u), \inf_{c\in C} F_{b,c}(v)\right\}. \tag{2.5}$$

Hence,

$$\begin{aligned}\sup_{a\in A}\inf_{c\in C} F_{a,c}(u+v) &\geq \sup_{a\in A}\min\left\{F_{a,b}(u),\inf_{c\in C}F_{b,c}(v)\right\}\\ &= \min\left\{\sup_{a\in A}F_{a,b}(u),\inf_{c\in C}F_{b,c}(v)\right\}\\ &\geq \min\left\{\sup_{a\in A}\inf_{b\in B}F_{a,b}(u),\inf_{c\in C}F_{b,c}(v)\right\}.\end{aligned} \tag{2.6}$$

Next, by taking $\sup_{b\in B}$, we have

$$\begin{aligned}\sup_{a\in A}\inf_{c\in C} F_{a,c}(u+v) &\geq \sup_{b\in B}\min\left\{\sup_{a\in A}\inf_{b\in B}F_{a,b}(u),\inf_{c\in C}F_{b,c}(v)\right\}\\ &\geq \min\left\{\sup_{a\in A}\inf_{b\in B}F_{a,b}(u),\sup_{b\in B}\inf_{c\in C}(v)\right\}.\end{aligned} \tag{2.7}$$

Similarly, for each $a,b,c\in X$ and for all $u,v>o$,

$$F_{a,c}(u+v)\geq \min\{F_{a,b}(u),F_{b,c}(v)\}. \tag{2.8}$$

By taking $\inf_{c\in C}$, we have

$$\inf_{a\in A}F_{a,c}(u+v)\geq \min\left\{\inf_{a\in A}F_{a,b}(u),F_{b,c}(v)\right\}. \tag{2.9}$$

Hence,

$$\begin{aligned}\sup_{c\in C}\inf_{a\in A} F_{a,c}(u+v) &\geq \sup_{c\in C}\min\left\{\inf_{a\in A}F_{a,b}(u),F_{b,c}(v)\right\}\\ &= \min\left\{\inf_{a\in A}F_{a,b}(u),\sup_{c\in C}F_{b,c}(v)\right\}\\ &\geq \min\left\{\inf_{a\in A}F_{a,b}(u),\sup_{c\in C}\inf_{b\in B}F_{b,c}(v)\right\}.\end{aligned} \tag{2.10}$$

Next, by taking $\sup_{b\in B}$, we have

$$\begin{aligned}\sup_{c\in C}\inf_{a\in A} F_{a,c}(u+v) &\geq \sup_{b\in B}\min\left\{\inf_{a\in A}F_{a,b}(u),\sup_{c\in C}\inf_{b\in B}F_{b,c}(v)\right\}\\ &\geq \min\left\{\sup_{b\in B}\inf_{a\in A}F_{a,b}(u),\sup_{c\in C}\inf_{b\in B}(v)\right\}.\end{aligned} \tag{2.11}$$

Therefore, we obtain that

$$\begin{aligned} {}_HF_{A,c}(u+v) &= \min\left\{\sup_{c\in C}\inf_{a\in A} F_{a,c}(u+v), \sup_{a\in A}\inf_{c\in C} F_{a,c}(u+v)\right\} \\ &\geq \min\left\{\sup_{b\in B}\inf_{a\in A} F_{a,b}(u), \sup_{c\in C}\inf_{b\in B}(v), \sup_{a\in A}\inf_{b\in B} F_{a,b}(u), \sup_{b\in B}\inf_{c\in C}(v)\right\} \\ &= \min\left\{{}_HF_{A,B}(u), {}_HF_{B,c}(v)\right\}. \end{aligned} \tag{2.12}$$

□

Lemma 2.3. *Let $(X,\mathcal{F},\min)$ be a Menger space. If $A,B\in B(X)$, then* $\lim_{u\to\infty} {}_\delta F_{A,B}(u)=1$.

Proof. For any $x\in A$ and $y\in B$, by Lemma 2.1, we have

$$_\delta F_{A,B}(u) \geq \min\left\{{}_\delta F_{A,x}\left(\frac{u}{3}\right), {}_\delta F_{x,y}\left(\frac{u}{3}\right), {}_\delta F_{y,B}\left(\frac{u}{3}\right)\right\}. \tag{2.13}$$

Letting $u\to\infty$, we have

$$\lim_{u\to\infty} {}_\delta F_{A,B}(u) \geq \min\left\{\lim_{u\to\infty} {}_\delta F_{A,x}\left(\frac{u}{3}\right), \lim_{u\to\infty} {}_\delta F_{x,y}\left(\frac{u}{3}\right), \lim_{u\to\infty} {}_\delta F_{y,B}\left(\frac{u}{3}\right)\right\}. \tag{2.14}$$

Since $x\in A$, $y\in B$, and $A,B\in B(X)$, we have

$$\lim_{u\to\infty} {}_\delta F_{A,x}\left(\frac{u}{3}\right) = 1. \tag{2.15}$$

Similarly, we have

$$\lim_{u\to\infty} {}_\delta F_{y,B}\left(\frac{u}{3}\right) = 1. \tag{2.16}$$

By the definition of the PM-space, we have that $\lim_{u\to\infty} F_{x,y}(u/3)=1$.

Therefore, we conclude that

$$\lim_{u\to\infty} {}_\delta F_{A,B}(u) = 1. \tag{2.17}$$

This completes the proof. □

The following lemma which was introduced by Chang [3], will play an important role for this paper.

Lemma 2.4. *If $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is a strictly increasing, continuous function such that $0<\phi(u)<u$ for all $u>0$, $\lim_{u\to\infty}\phi(u)=\infty$, and if for each $u>0$, $\phi^0(u)=u$ and $\phi^{-n}(u)=\phi^{-1}(\phi^{-n+1}(u))$ for each $n\in\mathbb{N}$ are denoted, then $\lim_{n\to\infty}\phi^{-n}(u)=\infty$.*

In the sequel, we let $\Phi=\{\phi:\mathbb{R}^+\to\mathbb{R}^+ : \phi$ is a strictly increasing, continuous function with $\phi(t)<t$ for all $t>0\}$.

Lemma 2.5. *Let $(X,\mathcal{F},\min)$ be a Menger space and $\{Y_n\}$ a sequence in $B(X)$. If for each $u>0$ and for each $n\in\mathbb{N}$,*

$$_\delta F_{Y_{n+1},Y_{n+2}}(\phi(u)) \geq {}_\delta F_{Y_n,Y_{n+1}}(u), \quad \phi\in\Phi, \tag{2.18}$$

then

$$\lim_{n\to\infty} {}_{\delta}F_{Y_n,Y_{n+1}}(u) = 1. \tag{2.19}$$

Proof. For $u > 0$, by induction, we have

$${}_{\delta}F_{Y_{n+1},Y_{n+2}}(u) \geq {}_{\delta}F_{Y_n,Y_{n+1}}(\phi^{-1}(u)) \geq \cdots \geq {}_{\delta}F_{Y_1,Y_2}(\phi^{-n}(u)), \quad \text{for each } n \in \mathbb{N}. \tag{2.20}$$

By Lemma 2.4, we also have that $\phi^{-n}(u) \to \infty$, as $n \to \infty$.

Next, since Y_n is a bounded set and ${}_{\delta}F_{Y_1,Y_2}(\phi^{-n}(u)) \to 1$ as $n \to \infty$, hence we have

$$\lim_{n\to\infty} {}_{\delta}F_{Y_{n+1},Y_{n+2}}(u) = 1. \tag{2.21}$$

□

Lemma 2.6. *Let $(X,\mathcal{F},\min)$ be a Menger space, and let $A,B \in B(X)$. If*

$${}_{\delta}F_{A,B}(\phi(u)) \geq {}_{\delta}F_{A,B}(u), \quad \textit{for } u > 0, \tag{2.22}$$

then $A = B = a$, for some $a \in X$.

Proof. For $u > 0$, by induction, we have

$${}_{\delta}F_{A,B}(u) \geq {}_{\delta}F_{A,B}(\phi^{-1}(u)) \geq \cdots \geq {}_{\delta}F_{A,B}(\phi^{-n}(u)). \tag{2.23}$$

Since $A,B \in B(X)$, by Lemma 2.3, we have

$$\lim_{n\to\infty} {}_{\delta}F_{A,B}(\phi^{-n}(u)) = 1, \tag{2.24}$$

and by Lemma 2.5, we have ${}_{\delta}F_{A,B}(u) = 1$ for $u > 0$. Thus we conclude that $A = B = \{a\}$ for some $a \in X$. □

The following lemma was introduced by Schweizer and Sklar [8].

Lemma 2.7. *Let $(X,\mathcal{F},\min)$ be a Menger space. If $a_n \to a$ and $b_n \to b$, then for $u > 0$,*

$$\liminf_{n\to\infty} F_{a_n,b_n}(u) = F_{a,b}(u). \tag{2.25}$$

From Lemma 2.7, we conclude the following lemma.

Lemma 2.8. *Let $(X,\mathcal{F},\min)$ be a Menger space. If $A_n \xrightarrow{\delta} a$ and $B_n \xrightarrow{\delta} b$, then for $u > 0$,*

$$\liminf_{n\to\infty} {}_{\delta}F_{A_n,B_n}(u) = F_{a,b}(u). \tag{2.26}$$

Proof. For $u > 0$ and for $\varepsilon > 0$. Since $F_{a,b}(u)$ is left continuous function at u, there exists a positive number k with $0 < 2k < u$ such that $F_{a,b}(u) - F_{a,b}(u-2k) < \varepsilon$.

Since $k > 0$ and $A_n \xrightarrow{\delta} a$, $B_n \xrightarrow{\delta} b$, hence we may take $m \in \mathbb{N}$ such that for $n \geq m$,

$${}_{\delta}F_{A_n,a}(k) \geq F_{a,b}(u-2k), \qquad {}_{\delta}F_{B_n,b}(k) \geq F_{a,b}(u-2k). \tag{2.27}$$

Hence, for $n > m$,

$$\begin{aligned} {}_\delta F_{A_n,B_n}(u) &\geq \min\left\{{}_\delta F_{A_n,b}(u-k), {}_\delta F_{b,B_n}(k)\right\} \\ &\geq \min\left\{{}_\delta F_{A_n,a}(k), {}_\delta F_{a,b}(u-2k), {}_\delta F_{b,B_n}(k)\right\} = F_{a,b}(u-2k), \end{aligned} \tag{2.28}$$

and hence

$$-{}_\delta F_{A_n,B_n}(u) \leq -F_{a,b}(u-2k). \tag{2.29}$$

Therefore, we conclude that

$$F_{a,b}(u) - {}_\delta F_{A_n,B_n}(u) < F_{a,b}(u) - F_{a,b}(u-2k) < \varepsilon. \tag{2.30}$$

Taking $\lim_{n\to\infty}\inf$, we have

$$F_{a,b}(u) - \liminf_{n\to\infty} {}_\delta F_{A_n,B_n}(u) < \varepsilon. \tag{2.31}$$

For any $a_n \in A_n$, $b_n \in B_n$, since $A_n \xrightarrow{\delta} a$ and $B_n \xrightarrow{\delta} b$, we have $a_n \to a$, $b_n \to b$. Thus, for $u > 0$

$${}_\delta F_{A_n,B_n}(u) \leq F_{a_n,b_n}(u). \tag{2.32}$$

Taking $\lim_{n\to\infty}\inf$, we have

$$\liminf_{n\to\infty} {}_\delta F_{A_n,B_n}(u) \leq \liminf_{n\to\infty} F_{a_n,b_n}(u). \tag{2.33}$$

By Lemma 2.7, we have

$$\liminf_{n\to\infty} F_{a_n,b_n}(u) = F_{a,b}(u), \text{ and so } F_{a,b}(u) - \liminf_{n\to\infty} {}_\delta F_{A_n,B_n}(u) \geq 0. \tag{2.34}$$

Therefore, for any $\varepsilon > 0$,

$$\varepsilon > F_{a,b}(u) - \liminf_{n\to\infty} {}_\delta F_{A_n,B_n}(u) \geq 0. \tag{2.35}$$

This implies that

$$\liminf_{n\to\infty} {}_\delta F_{A_n,B_n}(u) = F_{a,b}(u), \quad \text{for } u > 0. \tag{2.36}$$

□

The following two theorems are our main results for this paper.

Theorem 2.9. *Let $(X,\mathcal{F},\min)$ be a complete Menger space. Let $f,g,\eta,\xi : X \to X$ be four single-valued functions, and let $S,T : X \to B(X)$ two set-valued functions. If the following conditions are satisfied:*

(i) $S(X) \subset \xi g(X)$, $T(X) \subset \eta f(X)$,

(ii) $\eta f = f\eta$, $\xi g = g\xi$, $Sf = fS$, $Tg = gT$,

(iii) ηf *or* ξg *is continuous,*

(iv) $(S,\eta f)$ *and* $(T,\xi g)$ *are compatible, and*

(v) *for* $u > 0$,

$$
\begin{aligned}
&{}_{\delta}F_{Sx,Ty}(\phi(u)) \\
&\quad \geq \min\left\{F_{\eta fx,\xi gy}(u), {}_{\delta}F_{\eta fx,Sx}(u), {}_{\delta}F_{\xi gy,Ty}(u), {}_{\delta}F_{\xi gy,Sx}(\beta u), {}_{\delta}F_{\eta fx,Ty}((2-\beta)u)\right\}
\end{aligned}
\tag{2.37}
$$

for all $x,y \in X$, $\beta \in (0,2)$, where $\phi \in \Phi$, then f, g, η, ξ, S, and T have a unique common fixed point z in X.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ recursively as follows:

$$
\xi g x_{2n+1} \in S x_{2n} = Z_{2n}, \qquad \eta f x_{2n+2} \in T x_{2n+1} = Z_{2n+1}. \tag{2.38}
$$

For $n \in \mathbb{N}$ and for all $u > 0$, and $\beta = (1-\alpha)$ with $\alpha \in (0,1)$,

$$
\begin{aligned}
&{}_{\delta}F_{Z_{2n},Z_{2n+1}}(\phi(u)) \\
&\quad = {}_{\delta}F_{Sx_{2n},Tx_{2n+1}}(\phi(u)) \\
&\quad \geq \min\left\{F_{\eta fx_{2n},\xi gx_{2n+1}}(u), {}_{\delta}F_{\eta fx_{2n},Sx_{2n}}(u), {}_{\delta}F_{\xi gx_{2n+1},Tx_{2n+1}}(u), {}_{\delta}F_{\xi gx_{2n+1},Sx_{2n}}((1-\alpha)u),\right. \\
&\qquad\qquad \left. {}_{\delta}F_{\eta fx_{2n},Tx_{2n+1}}((1+\alpha)u)\right\} \\
&\quad \geq \min\left\{{}_{\delta}F_{Z_{2n-1},Z_{2n}}(u), {}_{\delta}F_{Z_{2n-1},Z_{2n}}(u), {}_{\delta}F_{Z_{2n},Z_{2n+1}}(u), {}_{\delta}F_{Z_{2n},Z_{2n}}((1-\alpha)u),\right. \\
&\qquad\qquad \left. {}_{\delta}F_{Z_{2n-1},Z_{2n+1}}((1+\alpha)u)\right\} \\
&\quad \geq \min\left\{{}_{\delta}F_{Z_{2n-1},Z_{2n}}(u), {}_{\delta}F_{Z_{2n-1},Z_{2n}}(u), {}_{\delta}F_{Z_{2n},Z_{2n+1}}(u), 1, {}_{\delta}F_{Z_{2n-1},Z_{2n}}((u), {}_{\delta}F_{Z_{2n},Z_{2n+1}}(\alpha u))\right\} \\
&\quad = \min\left\{{}_{\delta}F_{Z_{2n-1},Z_{2n}}(u), {}_{\delta}F_{Z_{2n},Z_{2n+1}}(u), {}_{\delta}F_{Z_{2n},Z_{2n+1}}(\alpha u)\right\}.
\end{aligned}
\tag{2.39}
$$

As t-norm = min is continuous, letting $\alpha \to 1$, we have

$$
{}_{\delta}F_{Z_{2n},Z_{2n+1}}(\phi(u)) \geq \min\left\{{}_{\delta}F_{Z_{2n-1},Z_{2n}}(u), {}_{\delta}F_{Z_{2n},Z_{2n+1}}(u)\right\}. \tag{2.40}
$$

By Lemma 2.6, we have

$$_{\delta}F_{Z_{2n},Z_{2n+1}}(\phi(u)) \geq _{\delta} F_{Z_{2n-1},Z_{2n}}(u). \tag{2.41}$$

Similarly, we also can prove that for $n \in \mathbb{N}$ and for all $u > 0$,

$$_{\delta}F_{Z_{2n+1},Z_{2n+2}}(\phi(u)) \geq {}_{\delta}F_{Z_{2n},Z_{2n+1}}(u). \tag{2.42}$$

So, we have

$$_{\delta}F_{Z_{n+1},Z_{n+2}}(\phi(u)) \geq {}_{\delta}F_{Z_{n},Z_{n+1}}(u), \quad \forall n \in \mathbb{N},\ u > 0. \tag{2.43}$$

By Lemma 2.5, we conclude that

$$\lim_{n\to\infty} {}_{\delta}F_{Z_n,Z_{n+1}}(u) = 1, \quad \forall u > 0. \tag{$*$}$$

Now, we consider the condition (v) with $\beta = 1$, and then we claim that

$$\text{for } \varepsilon > 0,\ \lambda \in (0,1) \quad \text{there is } M(\varepsilon,\lambda) \in \mathbb{N} \text{ such that } {}_{\delta}F_{Z_n,Z_m}(\varepsilon) \geq 1-\lambda \text{ for } n,m \geq M. \tag{2.44}$$

If it is not the case, then there exists $\varepsilon' > 0$, $\lambda' \in (0,1)$ such that for $k \in \mathbb{N}$, there exist $n_k > m_k \geq k$ such that

(1) n_k is even and m_k is odd,

(2) $_{\delta}F_{Z_{n_k},Z_{m_k}}(\varepsilon') < 1-\lambda'$, and

(3) n_k is the smallest even number such that (1) and (2) hold.

By $(*)$, we may choose $m_1 \in \mathbb{N}$ such that for $n \geq m_1$,

$$_{\delta}F_{Z_n,Z_{n+1}}\left(\min\left\{\frac{\varepsilon'}{2}, \frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right\}\right) > 1-\lambda'. \tag{2.45}$$

So for $k > m_1$, $n_k \geq m_k + 3$, and so for $k > m_1$,

$$\begin{aligned}
1-\lambda' &> {}_{\delta}F_{Z_{n_k},Z_{m_k}}(\varepsilon') = {}_{\delta} F_{Sx_{n_k},Tx_{m_k}}(\varepsilon') \\
&\geq \min\{F_{\eta f_{x_{n_k}},\xi g_{x_{m_k}}}(\phi^{-1}(\varepsilon')), {}_{\delta}F_{\eta f_{x_{n_k}},S_{x_{n_k}}}(\phi^{-1}(\varepsilon')), {}_{\delta}F_{\xi g_{x_{m_k}},T_{x_{m_k}}}(\phi^{-1}(\varepsilon')), \\
&\qquad {}_{\delta}F_{\xi g_{x_{m_k}},S_{x_{n_k}}}(\phi^{-1}(\varepsilon')), {}_{\delta}F_{\eta f_{x_{n_k}},T_{x_{m_k}}}(\phi^{-1}(\varepsilon'))\} \\
&\geq \min\{{}_{\delta}F_{Z_{n_{k-1}},Z_{m_{k-1}}}(\phi^{-1}(\varepsilon')), {}_{\delta}F_{Z_{n_{k-1}},Z_{n_k}}(\phi^{-1}(\varepsilon')), {}_{\delta}F_{Z_{m_{k-1}},Z_{m_k}}(\phi^{-1}(\varepsilon')), \\
&\qquad {}_{\delta}F_{Z_{n_k},Z_{m_{k-1}}}(\phi^{-1}(\varepsilon')), {}_{\delta}F_{Z_{n_{k-1}},Z_{m_k}}(\phi^{-1}(\varepsilon'))\}.
\end{aligned} \tag{2.46}$$

Since

$$
\begin{aligned}
{}_\delta F_{Z_{n_{k-1}},Z_{m_k}}(\phi^{-1}(\varepsilon')) &\geq \min\{{}_\delta F_{Z_{n_{k-1}},Z_{n_{k-2}}}(\phi^{-1}(\varepsilon')-\varepsilon'),{}_\delta F_{Z_{n_{k-2}},Z_{m_k}}(\varepsilon')\},\\
{}_\delta F_{Z_{m_{k-1}},Z_{n_k}}(\phi^{-1}(\varepsilon')) &\geq \min\left\{{}_\delta F_{Z_{m_{k-1}},Z_{n_{k-1}}}\left(\frac{\phi^{-1}(\varepsilon')+\varepsilon'}{2}\right),{}_\delta F_{Z_{n_{k-1}},Z_{n_k}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right)\right\}\\
&\geq \min\left\{{}_\delta F_{Z_{n_{k-1}},Z_{n_{k-2}}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right),{}_\delta F_{Z_{n_{k-2}},Z_{m_{k-1}}}(\varepsilon'),\right.\\
&\qquad\left.{}_\delta F_{Z_{n_{k-1}},Z_{n_k}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right)\right\}\\
&\geq \min\left\{{}_\delta F_{Z_{n_{k-1}},Z_{n_{k-2}}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right),{}_\delta F_{Z_{n_{k-2}},Z_{m_k}}\left(\frac{\varepsilon'}{2}\right),{}_\delta F_{Z_{m_k},Z_{m_{k-1}}}\left(\frac{\varepsilon'}{2}\right),\right.\\
&\qquad\left.{}_\delta F_{Z_{n_{k-1}},Z_{n_k}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right)\right\}\\
&\geq \min\left\{{}_\delta F_{Z_{n_{k-1}},Z_{n_{k-2}}}(\varepsilon'),{}_\delta F_{Z_{n_{k-2}},Z_{m_k}}(\varepsilon'),{}_\delta F_{Z_{m_k},Z_{m_k}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right),\right.\\
&\qquad\left.{}_\delta F_{Z_{m_k},Z_{m_{k-1}}}(\varepsilon'),{}_\delta F_{Z_{n_{k-1}},Z_{n_k}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right)\right\},\\
{}_\delta F_{Z_{n_{k-1}},Z_{m_{k-1}}}(\phi^{-1}(\varepsilon')) &\geq \min\left\{{}_\delta F_{Z_{n_{k-1}},Z_{n_{k-2}}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right),{}_\delta F_{Z_{n_{k-2}},Z_{m_{k-1}}}\left(\frac{\phi^{-1}(\varepsilon')+\varepsilon'}{2}\right)\right\}\\
&\geq \min\left\{{}_\delta F_{Z_{n_{k-1}},Z_{n_{k-2}}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right),{}_\delta F_{Z_{n_{k-2}},Z_{m_k}}(\varepsilon'),\right.\\
&\qquad\left.{}_\delta F_{Z_{m_{k-1}},Z_{m_k}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right)\right\},
\end{aligned}
\tag{2.47}
$$

so for $k>m_1$, we have

$$1-\lambda' > {}_\delta F_{Z_{n_k},Z_{m_k}}(\varepsilon') \geq 1-\lambda', \tag{2.48}$$

which is a contradiction. And, since X is complete, hence for any choice of z_n in Z_n, the sequence $\{z_n\}$ must converge to some point, say, z in X. The point z is independent of the choice of z_n and so we have

$$\eta f x_{2n} \longrightarrow z, \qquad \xi g x_{2n+1} \longrightarrow z, \qquad S x_{2n} \longrightarrow \{z\}, \qquad T x_{2n+1} \longrightarrow \{z\}. \tag{2.49}$$

That is, for $u>0$,

$$F_{\eta f x_{2n},z}(u) \longrightarrow 1, \quad F_{\xi g x_{2n+1},z}(u) \longrightarrow 1, \quad {}_\delta F_{S x_{2n},z}(u) \longrightarrow 1, \quad {}_\delta F_{T x_{2n+1},z}(u) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty. \tag{2.50}$$

Assume that the function ηf is continuous, then for $u > 0$, we have

$$\lim_{n\to\infty} F_{(\eta f)^2 x_{2n},\eta f z}(u) = 1, \qquad \lim_{n\to\infty} {}_{\delta}F_{\eta f S x_{2n},\eta f z}(u) = 1. \tag{2.51}$$

By $\lim_{n\to\infty} F_{\eta f x_{2n},z}(u) = 1$ and $\lim_{n\to\infty} {}_{\delta}F_{Sx_{2n},z}(u) = 1$, we obtain $\lim_{n\to\infty} {}_{\delta}F_{Sx_{2n},\eta f x_{2n}}(u) = 1$. Since S and ηf are compatible, and for $u > 0$, $\lim_{n\to\infty} {}_{\delta}F_{Sx_{2n},\eta f x_{2n}}(u) = 1$, we have $\lim_{n\to\infty} {}_{H}F_{\eta f Sx_{2n},S\eta f x_{2n}}(u) = 1$ and ${}_{H}F_{S\eta f x_{2n},\eta f z}(u) \geq \min\{{}_{H}F_{\eta f Sx_{2n},S\eta f x_{2n}}(u/2), {}_{H}F_{\eta f Sx_{2n},\eta f z}(u/2)\}$. And, since $\lim_{n\to\infty} {}_{H}F_{\eta f Sx_{2n},S\eta f x_{2n}}(u/2) = 1$, $\lim_{n\to\infty} {}_{H}F_{\eta f Sx_{2n},\eta f z}(u/2) = 1$, we have

$$\lim_{n\to\infty} {}_{H}F_{S\eta f x_{2n},\eta f z}(u) = \lim_{n\to\infty} {}_{\delta}F_{S\eta f x_{2n},\eta f z}(u) = 1. \tag{2.52}$$

In order to complete the proof, we will divide it into 5 steps as follows:

Step 1. For $u > 0$ with $\beta = 1$ in the condition (v),

$$\begin{aligned} {}_{\delta}F_{S\eta f x_{2n},Tx_{2n+1}}(\phi(u)) \geq \min\{ & F_{(\eta f)^2 x_{2n},\xi g x_{2n+1}}(u), {}_{\delta}F_{(\eta f)^2 x_{2n},S\eta f x_{2n}}(u), {}_{\delta}F_{\xi g x_{2n+1},Tx_{2n+1}}(u), \\ & {}_{\delta}F_{\xi g x_{2n+1},S\eta f x_{2n}}(u), {}_{\delta}F_{(\eta f)^2 x_{2n},Tx_{2n+1}}(u)\}. \end{aligned} \tag{2.53}$$

Taking $\lim_{n\to\infty}$, by Lemma 2.8,

$$F_{\eta f z,z}(\phi(u)) \geq \min\{F_{\eta f z,z}(u), F_{\eta f z,\eta f z}(u), F_{z,z}(u), F_{\eta f z,z}(u), F_{\eta f z,z}(u)\} = F_{\eta f z,z}(u). \tag{2.54}$$

So we get $\eta f z = z$.

Step 2. For $u > 0$ with $\beta = 1$ in the condition (v),

$$\begin{aligned} {}_{\delta}F_{Sz,z}(\phi(u)) & \\ &= \liminf_{n\to\infty} {}_{\delta}F_{Sz,Tx_{2n+1}}(\phi(u)) \\ &\geq \liminf_{n\to\infty} \min\{F_{\eta f z,\xi g x_{2n+1}}(u), {}_{\delta}F_{\eta f z,Sz}(u), {}_{\delta}F_{\xi g x_{2n+1},Tx_{2n+1}}(u), {}_{\delta}F_{Sz,\xi g x_{2n+1}}(u), {}_{\delta}F_{\eta f z,Tx_{2n+1}}(u)\} \\ &\geq \min\{F_{z,z}(u), {}_{\delta}F_{z,Sz}(u), F_{z,z}(u), {}_{\delta}F_{z,Sz}(u), F_{z,z}(u)\} = {}_{\delta}F_{z,Sz}(u). \end{aligned} \tag{2.55}$$

So we get $Sz = \{z\}$.

Hence, by Steps 1 and 2, we have $Sz = \{z\} = \{\eta f z\}$.

Step 3. By the condition (i), since $SX \subset \xi g X$, there exists $z' \in X$ such that $\{\xi g z'\} = Sz = \{z\}$.

So for any $u > 0$ with $\beta = 1$ in the condition (v)

$$\begin{aligned} & {}_{\delta}F_{Sx_{2n},Tz'}(\phi(u)) \\ & \quad \geq \min\{F_{\eta f x_{2n},\xi g z'}(u), {}_{\delta}F_{\eta f x_{2n},Sx_{2n}}(u), {}_{\delta}F_{\xi g z',Tz'}(u), {}_{\delta}F_{\eta f z',Sx_{2n}}(u), {}_{\delta}F_{\eta f x_{2n},Tz'}(u)\}. \end{aligned} \tag{2.56}$$

Taking $\lim_{n\to\infty}\inf$, by Lemma 2.8,

$${}_{\delta}F_{z,Tz'}(\phi(u)) \geq \min\{F_{z,z}(u), F_{z,z}(u), {}_{\delta}F_{z,Tz'}(u), F_{z,z}(u), {}_{\delta}F_{z,Tz'}(u)\} = {}_{\delta}F_{z,Tz'}(u). \tag{2.57}$$

So we get $Tz' = \{z\}$. Hence, $\{\xi gz'\} = \{z\} = Tz'$.

By Step 2, we may let $\{z\} = \{\eta fz\} = \{Sz\} = \{\xi gz'\} = \{Tz'\}$.

Since S and ηf are compatible and $\{\eta fz\} = Sz$, we get $\eta fSz = S\eta fz$, that is, $\{\eta fz\} = Sz$. Now,

$$\begin{aligned}
{}_\delta F_{Sz,z}(\phi(u)) &= {}_\delta F_{Sz,Tz'}(\phi(u)) \\
&\geq \min\{F_{\eta fz,\xi gz'}(u), {}_\delta F_{\eta fz,Sz}(u), {}_\delta F_{\xi gz',Tz'}(u), {}_\delta F_{\eta fz,Tz'}(u), {}_\delta F_{Sz,\xi gz'}(u)\} \\
&= {}_\delta F_{\eta fz,z}(u) = {}_\delta F_{Sz,z}(u).
\end{aligned} \tag{2.58}$$

This implies $Sz = \{z\} = \{\eta fz\}$.

Choose z' in X such that $\{\xi gz'\} = Sz = \{z\}$, then

$$\begin{aligned}
&{}_\delta F_{z,Tz'}(\phi(u)) \\
&\quad = {}_\delta F_{Sz,Tz'}(\phi(u)) \\
&\quad \geq \min\{F_{\eta fz,\xi gz'}, {}_\delta F_{\eta fz,Sz}(u), {}_\delta F_{\xi gz',Tz'}(u) {}_\delta F_{\eta fz,Tz'}(u), {}_\delta F_{Sz,\xi gz'}(u)\} = {}_\delta F_{z,Tz'}(u).
\end{aligned} \tag{2.59}$$

By Lemma 2.6, we get $Tz' = \{z\}$.

Since T and ξg are compatible and $\{\xi gz'\} = Tz'$, we get $T\xi gz' = \xi gTz'$, that is, $Tz = \{\xi gz\}$.

Now, for $u > 0$,

$$\begin{aligned}
&{}_\delta F_{Sz,Tz}(\phi(u)) \\
&\quad \geq \min\{F_{\eta fz,\xi gz}(u), {}_\delta F_{\eta fz,Sz}(u), {}_\delta F_{\xi gz,Tz}(u), {}_\delta F_{\eta fz,Tz}(u), {}_\delta F_{Sz,\xi gz}(u)\} \\
&\quad = F_{\eta fz,\xi gz}(u) = {}_\delta F_{Sz,Tz}(u).
\end{aligned} \tag{2.60}$$

So we have $Sz = Tz = \{\eta fz\} = \{\xi gz\} = \{z\}$.

Step 4. For $u > 0$ with $\beta = 1$ in the condition (v), we get

$$\begin{aligned}
&{}_\delta F_{Sfz,Tx_{2n+1}}(\phi(u)) \\
&\quad \geq \min\{F_{\eta ffz,\xi gx_{2n+1}}(u), {}_\delta F_{\eta ffz,Sfz}(u), {}_\delta F_{\xi gx_{2n+1},Tx_{2n+1}}(u), {}_\delta F_{\xi gx_{2n+1},Sfz}(u), {}_\delta F_{\eta ffz,Tx_{2n+1}}(u)\}.
\end{aligned} \tag{2.61}$$

By the condition (ii), $\eta f = f\eta$, $Sf = fS$, so we have $\eta f(fz) = f(\eta fz) = fz$ and $S(fz) = \{f(Sz)\} = \{fz\}$. Taking $\lim_{n\to\infty}\inf$, by Lemma 2.8,

$$F_{fz,z}(\phi(u)) \geq \min\{F_{fz,z}(u), F_{fz,fz}(u), F_{z,z}(u), F_{z,fz}(u), F_{fz,z}(u)\} = F_{fz,z}(u). \tag{2.62}$$

So we get $fz = z$.

Hence, by Steps 1 and 4, we have $\eta fz = z$ and $fz = z$, which implies $\eta z = z$. Therefore, $\{z\} = \{fz\} = \{\eta z\} = Sz$.

Step 5. For $u > 0$ with $\beta = 1$ in condition (v), we get

$$
\begin{aligned}
&{}_{\delta}F_{Sx_{2n},Tgz}(\phi(u)) \\
&\quad \geq \min\{F_{\eta f x_{2n},\xi ggz}(u), {}_{\delta}F_{\eta f x_{2n},Sx_{2n}}(u), {}_{\delta}F_{\xi ggz,Tgz}(u), {}_{\delta}F_{\xi ggz,Sx_{2n}}(u), {}_{\delta}F_{\eta f x_{2n},Tgz}(u)\}.
\end{aligned} \tag{2.63}
$$

Since $Tg = gT$ and $\xi g = g\xi$, we have $Tgz = \{gTz\} = \{gz\}$ and $\xi g(gz) = g(\xi gz) = gz$. Taking $\lim_{n\to\infty}\inf$, by Lemma 2.8, we get

$$
F_{z,gz}(\phi(u)) \geq \min\{F_{z,gz}(u), F_{z,z}(u), F_{gz,gz}(u), F_{gz,z}(u), F_{z,gz}(u)\} = F_{z,gz}(u). \tag{2.64}
$$

So we get $gz = z$.

Hence, by Steps 3 and 5, we have $\xi gz = z$ and $gz = z$, which implies $\xi z = z$.

So we have $\{z\} = \{gz\} = \{\xi z\} = Tz$.

Therefore, we have

$$
\{z\} = \{fz\} = \{gz\} = \{\eta z\} = \{\xi z\} = Sz = Tz. \tag{2.65}
$$

Last, we want to prove the uniqueness. Let y be the another commom fixed point of η, f, ξ, g, S, and T. Then for $u > 0$,

$$
\begin{aligned}
F_{z,y}(\phi(u)) &= {}_{\delta}F_{Sz,Ty}(\phi(u)) \\
&\geq \min\{F_{\eta fz,\xi gy}(u), {}_{\delta}F_{\eta fz,Sz}(u), {}_{\delta}F_{\xi gy,Ty}(u), {}_{\delta}F_{\xi gy,Sz}(u), {}_{\delta}F_{\eta fz,Ty}(u)\} \\
&\geq \min\{F_{z,y}(u), F_{z,z}(u), F_{y,y}(u), F_{y,z}(u), F_{ygz}(u)\} = F_{z,y}(u).
\end{aligned} \tag{2.66}
$$

This implies $y = z$. We complete the proof. □

If we take $f = g = I$, the identity map on X in Theorem 2.9, then we immediately have the following corollary.

Corollary 2.10. *Let $(X, \mathcal{F}, \min)$ be a complete Menger space. Let $\eta, \xi : X \to X$ be two single-valued functions, and let $S, T : X \to B(X)$ be two set-valued functions. If the following conditions are satisfied:*

(i) $S(X) \subset \xi(X)$, $T(X) \subset \eta(X)$,

(ii) *η or ξ is continuous,*

(iii) *(S, η) and (T, ξ) are compatible,*

(iv) *for $u > 0$,*

$$
{}_{\delta}F_{Sx,Ty}(\phi(u)) \geq \min\{F_{\eta x,\xi y}(u), {}_{\delta}F_{\eta x,Sx}(u), {}_{\delta}F_{\xi y,Ty}(u), {}_{\delta}F_{\xi y,Sx}(\beta u), {}_{\delta}F_{\eta x,Ty}((2-\beta)u)\} \tag{2.67}
$$

for all $x, y \in X$, $\beta \in (0, 2)$, where $\phi \in \Phi$, then η, ξ, S, and T have a unique common fixed point z in X.

By the same process of the proof of Theorem 2.9, we also get the results of Theorem 2.11.

Theorem 2.11. *Let $(X,\mathcal{F},\min)$ be a complete Menger space. Let $f,g,\eta,\xi : X \to X$ be four single-valued functions, and let $S,T : X \to B(X)$ be two set-valued functions. If the following conditions are satisfied:*

(i) $S(X) \subset \xi g(X)$, $T(X) \subset \eta f(X)$,
(ii) $\eta f = f\eta$, $\xi g = g\xi$, $Sf = fS$, $Tg = gT$,
(iii) ηf *or* ξg *is continuous,*
(iv) $(S,\eta f)$ *and* $(T,\xi g)$ *are compatible,*
(v) *for* $u > 0$,

$$ {}_{\delta}F_{Sx,Ty}(\phi(u)) \geq \min\{F_{\eta fx,\xi gy}(u), {}_{\delta}F_{\eta fx,Sx}(u), {}_{\delta}F_{\xi gy,Ty}(u), {}_{D}F_{\xi gy,Sx}(u) + {}_{D}F_{\eta fx,Ty}(u)\} \tag{2.68} $$

for all $x,y \in X$, where $\phi \in \Phi$, then f, g, η, ξ, S, and T have a unique common fixed point z in X.

If we take $f = g = I$, the identity map on X in Theorem 2.11, then we immediately have the following corollary.

Corollary 2.12. *Let $(X,\mathcal{F},\min)$ be a complete Menger space. Let $\eta,\xi : X \to X$ be two single-valued functions, and let $S,T : X \to B(X)$ be two set-valued functions. If the following conditions are satisfied:*

(i) $S(X) \subset \xi(X)$, $T(X) \subset \eta(X)$,
(ii) η *or* ξ *is continuous,*
(iii) (S,η) *and* (T,ξ) *are compatible,*
(iv) *for* $u > 0$,

$$ {}_{\delta}F_{Sx,Ty}(\phi(u)) \geq \min\{F_{\eta x,\xi y}(u), {}_{\delta}F_{\eta x,Sx}(u), {}_{\delta}F_{\xi y,Ty}(u), {}_{D}F_{\xi y,Sx}(u) +_{D} F_{\eta x,Ty}(u)\} \tag{2.69} $$

for all $x,y \in X$, where $\phi \in \Phi$, then η, ξ, S, and T have a unique common fixed point z in X.

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