ON THE POINTWISE MULTIPLICATION IN BESOV AND LIZORKIN-TRIEBEL SPACES

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Under some sufficient conditions satisfied by F-space of Lizorkin and Triebel and B-space of Besov, we prove some embeddings of types $F \cdot B \hookrightarrow F$, $F \cdot F \hookrightarrow F$, and $B \cdot B \hookrightarrow B$.

1. Introduction and preparations

In Besov spaces and Lizorkin-Triebel spaces, this paper is concerned with proving some embeddings of the form

$$F \cdot B \hookrightarrow F, \qquad F \cdot F \hookrightarrow F, \qquad B \cdot B \hookrightarrow B, \tag{1.1}$$

where F and B, with three indices, will denote the Lizorkin-Triebel space $F^s_{p,q}$ and the Besov space $B^s_{p,q}$, respectively. The different embeddings obtained here are under certain restrictions on the parameters.

In this introduction, we will recall the definition of some spaces and some necessary tools. In Sections 2 and 3, we give the first contribution of this work. The theorems of Section 2 will treat the case $F \cdot B \hookrightarrow F$ where the first theorem is a generalization of the results of Franke [4, Section 3.2, Theorem 1, Section 3.4, Corollary 1] and Marschall [7]. The second theorem is in the sense of Johnsen's works (see [5]). Section 3 will contain a treatment of the embeddings of the types $F \cdot F \hookrightarrow F$ and $B \cdot B \hookrightarrow B$ which presents an improvement of [3].

In the sense of [5, Theorems 6.5, 6.11], some limit cases are considered in Section 4, which constitute the second contribution of this paper. Section 5 is an application of our results to the continuity of pseudodifferential operators on Lizorkin-Triebel spaces.

We will work on the Euclidean space $\mathbb{R}^n$. If $f \in \mathcal{S}$, the Fourier transform is defined by the formula

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx \quad (\xi \in \mathbb{R}^n) \tag{1.2}$$

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and $\mathcal{F}^{-1}f$ denotes the inverse Fourier transform of f; as usual $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended from $\mathcal{S}$ to $\mathcal{S}'$.

Consider a partition of unity

$$\psi(\xi)+\sum_{j=1}^{\infty}\varphi(2^{-j}\xi)=1 \quad (\xi\in\mathbb{R}^n), \tag{1.3}$$

where $\varphi,\psi\in C^\infty$ are positive functions such that $\operatorname{supp}\varphi\subset\{\xi\in\mathbb{R}^n : 1\leq|\xi|\leq 3\}$ and $\operatorname{supp}\psi\subset\{\xi\in\mathbb{R}^n : |\xi|\leq 3\}$. We define the convolution operators Q_j and Δ_k by the following:

$$\begin{aligned} Q_j f &= \mathcal{F}^{-1}(\psi(2^{-j}\cdot\,)) * f \quad (j=1,2,\ldots),\\ \Delta_k f &= \mathcal{F}^{-1}(\varphi(2^{-k}\cdot\,)) * f \quad (k=0,1,\ldots), \end{aligned} \tag{1.4}$$

and we set $Q_0=\Delta_0$. Thus we obtain the Littlewood-Paley decomposition $f=\sum_{j=0}^{\infty}\Delta_j f$ (convergence in $\mathcal{S}'$).

Let us now recall the definitions of $F^s_{p,q}$ and $B^s_{p,q}$, where the general references include [1, 9–13].

Definition 1.1. Let $\gamma>0$, $-\infty<s<\infty$, $0<p<\infty$ (resp., $0<p\leq\infty$), and $0<q\leq\infty$. The space $L^\gamma_p(\ell^s_q)$ (resp., $\ell^s_q(L^\gamma_p)$) is the set of the sequences $\{f_k\}_{k\in\mathbb{N}}\subset S'$ such that $\operatorname{supp}\widehat{f_k}\subset\{\xi\in\mathbb{R}^n : |\xi|<\gamma 2^k\}$ and

$$\begin{gathered} \|\{f_k\}_{k\in\mathbb{N}} \mid L^\gamma_p(\ell^s_q)\| = \|\{2^{ks}f_k\}_{k\in\mathbb{N}} \mid L_p(\ell_q)\|<\infty,\\ (\text{resp., } \|\{f_k\}_{k\in\mathbb{N}} \mid \ell^s_q(L^\gamma_p)\| = \|\{2^{ks}f_k\}_{k\in\mathbb{N}} \mid \ell_q(L_p)\|<\infty). \end{gathered} \tag{1.5}$$

Definition 1.2. (i) Let $0<p<\infty$, $0<q\leq\infty$, and $-\infty<s<\infty$, then

$$F^s_{p,q}=\{f\in\mathcal{S}' : \|\{2^{ks}\Delta_k f\}_{k\in\mathbb{N}} \mid L_p(\ell_q)\|<\infty\}. \tag{1.6}$$

(ii) Let $0<p,q\leq\infty$, and $-\infty<s<\infty$, then

$$B^s_{p,q}=\{f\in\mathcal{S}' : \|\{2^{ks}\Delta_k f\}_{k\in\mathbb{N}} \mid \ell_q(L_p)\|<\infty\}. \tag{1.7}$$

Remark 1.3. We introduce the maximal function

$$\Delta_k^{*,a}f(x)=\sup_{y\in\mathbb{R}^n}\frac{|\Delta_k f(x-y)|}{1+(2^k|y|)^a} \tag{1.8}$$

for all $x\in\mathbb{R}^n$, $f\in\mathcal{S}'$, $a>0$, and $k=0,1,\ldots$. Then, in Definition 1.2(i) (resp., (ii)), we can replace $\Delta_k f$ by $\Delta_k^{*,a}f$ with $a>(n/\min(p,q))$ (resp., $a>n/p$), (cf. see [13, Theorem 2.3.2]).

The product $f \cdot g$ is defined by

$$f \cdot g = \lim_{j \to \infty} Q_j f \cdot Q_j g \quad (\forall f, g \in \mathcal{S}') \tag{1.9}$$

if the limit on the right-hand side exists in $\mathcal{S}'$ (see [10, Section 4.2]), and we have

$$\Delta_k(f \cdot g) = \sum_{j,\ell=0}^{\infty} \Delta_k(\Delta_j g \cdot \Delta_\ell f) = (\Pi_{k,1} + \Pi_{k,2} + \Pi_{k,3})(f,g), \tag{1.10}$$

where

$$\begin{gathered} \Pi_{k,1}(f,g) = \Delta_k(\widetilde{\Delta}_k f \cdot Q_{k+1} g), \qquad \Pi_{k,2}(f,g) = \Delta_k(Q_{k+1} f \cdot \widetilde{\Delta}_k g), \\ \Pi_{k,3}(f,g) = \sum_{j=k}^{\infty} \Delta_k(\Delta_j f \cdot \overline{\Delta}_j g), \end{gathered} \tag{1.11}$$

with $\widetilde{\Delta}_k = \sum_{j=k-2}^{k+4} \Delta_j$ and $\overline{\Delta}_k = \sum_{j=k-1}^{k+1} \Delta_j$.

In the below proofs of the different cases of type (1.1), written as $G_1 \cdot G_2 \hookrightarrow G_3$, to see $f \cdot g$ belongs to G_3, ($f \in G_1$, $g \in G_2$), it suffices to an estimate of terms of the form $\| \{\Pi_{k,i}(f,g)\}_{k \in \mathbb{N}} \mid L_p^\gamma(\ell_q^s) \|$ and $\| \{\Pi_{k,i}(f,g)\}_{k \in \mathbb{N}} \mid \ell_q^s(L_p^\gamma) \|$, $i \in \{1,2,3\}$.

Now we recall some lemmas which are useful for us.

Lemma 1.4. (i) *Let* $-\infty < s_i < \infty$, $0 < p_i < \infty$ *(resp.,* $0 < p_i \le \infty$*), and* $0 < q_i \le \infty$ *(with* $i = 0,1$*). If*

$$s_0 > s_1, \qquad p_0 = p_1, \tag{1.12}$$

or

$$s_0 \ge s_1, \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1} \quad (q_0 \le q_1 \textit{ for Besov space}), \tag{1.13}$$

then it holds

$$F^{s_0}_{p_0,q_0} \hookrightarrow F^{s_1}_{p_1,q_1} \quad (\textit{resp., } B^{s_0}_{p_0,q_0} \hookrightarrow B^{s_1}_{p_1,q_1}). \tag{1.14}$$

(ii) *Let* $-\infty < s$, $s_i < \infty$, $0 < p$, $p_i < \infty$, *and* $0 < q$, $q_i \le \infty$ *(with* $i = 0,1$*) such that* $s_0 - n/p_0 = s - n/p = s_1 - n/p_1$. *If*

$$s_0 > s > s_1, \qquad q_0 \le p \le q_1, \tag{1.15}$$

or

$$s_0 = s = s_1, \qquad q_0 \le \min(p,q), \qquad q_1 \ge \max(p,q), \tag{1.16}$$

then it holds

$$B^{s_0}_{p_0,q_0} \hookrightarrow F^{s}_{p,q} \hookrightarrow B^{s_1}_{p_1,q_1}. \tag{1.17}$$

(iii) *Let* $-\infty < s < \infty$, $0 < p < \infty$ *(resp.,* $0 < p \leq \infty$*), and* $0 < q \leq \infty$. *If*

$$s > \frac{n}{p}, \tag{1.18}$$

or

$$s = \frac{n}{p}, \quad 0 < p \leq 1 \ (\textit{resp.}, \ 0 < q \leq 1), \tag{1.19}$$

then it holds

$$F^s_{p,q} \hookrightarrow L_\infty \quad (\textit{resp.}, \ B^s_{p,q} \hookrightarrow L_\infty). \tag{1.20}$$

Lemma 1.5. *Let* $0 < \gamma < 1$ *and* $0 < q \leq \infty$. *Let* $\{\varepsilon_k\}_{k\in\mathbb{N}}$ *be a sequence of positive real numbers such that* $\|\{\varepsilon_k\}_{k\in\mathbb{N}} \mid \ell_q\| = A < \infty$. *Then the sequences* $\delta_k = \sum_{j=0}^{k} \gamma^{k-j}\varepsilon_j$ *and* $\eta_k = \sum_{j=k}^{\infty} \gamma^{j-k}\varepsilon_j$ *belong to* ℓ_q*, and the estimate*

$$\left\|\{\delta_k\}_{k\in\mathbb{N}} \mid \ell_q\right\| + \left\|\{\eta_k\}_{k\in\mathbb{N}} \mid \ell_q\right\| \leq cA \tag{1.21}$$

holds. The constant c depends only on γ *and q.*

Lemma 1.6. *Let* $0 < p \leq \infty$ *and* $\gamma > 0$. *Let* $\{f_j\}_{j\in\mathbb{N}} \subset L_p$ *be a sequence of functions such that* $\operatorname{supp} \widehat{f}_j \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \gamma 2^j\}$. *Then the estimate*

$$\left\|\Delta_k f_j \mid L_p\right\| \leq c2^{(j-k)\varrho}\left\|f_j \mid L_p\right\| \quad \left(k \leq j < \infty, \ \varrho = \max\left(0, \frac{n}{p} - n\right)\right) \tag{1.22}$$

holds. The constant c depends only on n, p, and γ.

Lemma 1.7. *Let* $0 < p < 1$ *and* $\gamma > 0$. *Let* $\{f_j\}_{j\in\mathbb{N}} \subset L_p$ *be a sequence of functions such that* $\operatorname{supp} \widehat{f}_j \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \gamma 2^j\}$. *Then the estimate*

$$\left\|\sum_{j=0}^{\infty} f_j \mid B^{\varrho}_{p,\infty}\right\| \leq c\left\|\{2^{j\varrho} f_j\}_{j\in\mathbb{N}} \mid L_p(\ell_\infty)\right\| \quad \left(\varrho = \frac{n}{p} - n\right) \tag{1.23}$$

holds. The constant c depends only on n, p, and γ.

Lemma 1.8. *Let* $0 < p \leq q \leq \infty$ *and* $\gamma > 0$. *Then there exists a constant* $c = c(n,p,q) > 0$ *such that for all* $f \in L_p$ *with* $\operatorname{supp} \widehat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \gamma\}$*, one has*

$$\left\|f \mid L_q\right\| \leq c\gamma^{n(1/p-1/q)}\left\|f \mid L_p\right\|. \tag{1.24}$$

For Lemma 1.4, we can see [11, Sections 2.3 and 2.8] and [12, Section 2.7]. Lemma 1.5 follows from Young's inequality in ℓ_q. The proof of Lemma 1.6 is given in [4, Section 2.4, Theorem 1(iii)] and Lemma 1.7 in [7, Lemma 3]. For the proof of Lemma 1.8, we can see [14, Proposition 2.13], $1 \leq p \leq q \leq \infty$, it is the classical inequality of Bernstein.

2. Multiplication of mixed type

The following results give an extension of the sufficient hypotheses used in [5, Theorem 6.1].

THEOREM 2.1. *Let* $0 < p, p_1, p_2 < \infty$, $0 < q$, $q_2 \leq \infty$, $-\infty < s < \infty$, *and* $r > 0$ *be such that*

$$-r + \max\left(0, \frac{n}{p_1} + \frac{n}{p_2} - n\right) < s < \min\left(\frac{n}{p_1}, r\right),$$
$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n}, \quad \frac{1}{q_2} \geq \frac{1}{p_2} - \frac{r}{n}, \quad r < \frac{n}{p_2} \left(\text{resp., } r = \frac{n}{p_2}\right). \tag{2.1}$$

Then it holds

$$F^s_{p_1,q} \cdot B^r_{p_2,q_2} \hookrightarrow F^s_{p,q} \quad \left(\text{resp., } F^s_{p_1,q} \cdot (B^{n/p_2}_{p_2,\infty} \cap L_\infty) \hookrightarrow F^s_{p_1,q}\right). \tag{2.2}$$

COROLLARY 2.2. *Under the hypotheses of Theorem 2.1. If* $r < n/p_2$ (*resp.*, $r = n/p_2$) *then it holds*

$$F^s_{p_1,q} \cdot F^r_{p_2,q_2} \hookrightarrow F^s_{p,q} \quad \left(\text{resp., } F^s_{p_1,q} \cdot F^{n/p_2}_{p_2,q_2} \hookrightarrow F^s_{p_1,q} \text{ for } p_2 \leq 1\right). \tag{2.3}$$

Furthermore, in particular, if $1 < p_1 < \infty$ *and* $r > n/p_1 + n/p_2 - n$, *can be taken* $s = 0$ *in (2.3).*

Proof. Since $F^r_{p_2,q_2} \hookrightarrow B^r_{p_2,t}$ with $t = (1/p_2 - r/n)^{-1}$, we obtain the first embedding. However, the second embedding follows from $F^{n/p_2}_{p_2,q_2} \hookrightarrow B^{n/p_2}_{p_2,\infty} \cap L_\infty$. □

Remark 2.3. In Corollary 2.2, when $r < n/p_2$ (resp., $r = n/p_2$), we obtain [10, Theorems 4.4.3/2(21) and 4.4.4/2(16) (resp., Theorems 4.4.3/2(22) and 4.4.4/2(17))]. The particular case $s = 0$ presents a complement of [10, Theorem 4.4.4/4(i)].

To prove Theorem 2.1, we need the following lemma.

LEMMA 2.4. *Let* $0 < p < \infty$ *and* $a > n/p$. *Then there exists a constant* $c > 0$ *such that*

$$\left\| \{Q^{*,a}_j g\}_{j \in \mathbb{N}} \mid L_p(\ell_\infty) \right\| \leq c \left\| g \mid F^0_{p,2} \right\|, \tag{2.4}$$

for any $g \in F^0_{p,2}$.

Proof. First, we define the maximal function of $Q_j g$, of Hardy-Littelewood type, by the formula

$$MQ_j g(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |Q_j g(y)|\, dy, \tag{2.5}$$

where $B(x,r)$ is the ball centered at x of radius r and $|B(x,r)|$ denotes its measure. Next, let $t > 0$ satisfy $n/a < t < p$. From [13, Theorem 1.3.1], we have

$$Q^{*,a}_j g(x) \leq Q^{*,n/t}_j g(x) \leq c\left(M |Q_j g|^t(x)\right)^{1/t} \quad (\forall x \in \mathbb{R}^n). \tag{2.6}$$

Then we obtain

$$\begin{aligned}\Big\|\sup_{j\in\mathbb{N}} Q_j^{*,a} g \mid L_p\Big\| &\le c\Big\|\Big(\sup_{j\in\mathbb{N}} M|Q_j g|^t\Big)^{1/t} \mid L_p\Big\| = c\Big\|\sup_{j\in\mathbb{N}} M|Q_j g|^t \mid L_{p/t}\Big\|^{1/t}\\ &\le c'\Big\|\sup_{j\in\mathbb{N}} |Q_j g|^t \mid L_{p/t}\Big\|^{1/t}.\end{aligned}\tag{2.7}$$

A proof of the last inequality may be found in [13, Theorem 2.2.2, page 89]. Now, it is easy to see that the last member of (2.7) is bounded by

$$\Big\|\sup_{j\in\mathbb{N}} |Q_j g| \mid L_p\Big\| \le c\|g \mid F^0_{p,2}\|.\tag{2.8}$$

Inequality (2.8) follows from the equality between the local Hardy spaces h_p and $F^0_{p,2}$, (cf. see [12, Section 2.2, page 37, and Theorem 2.5.8/1]). □

Proof of Theorem 2.1
Case 1 ($r < n/p_2$). (i) Estimate of $\{\Pi_{k,1}(f,g)\}_{k\in\mathbb{N}}$. Since

$$\begin{aligned}\big|\Pi_{k,1}(f,g)(x)\big| &= \Big|\int_{\mathbb{R}^n} (\mathcal{F}^{-1}\varphi(y))\,(Q_{k+1}g\cdot\widetilde{\Delta}_k f)\,(x-2^{-k}y)\,dy\Big|\\ &\le cQ^{*,a_1}_{k+1}g(x)\widetilde{\Delta}_k^{*,a_2}f(x)\quad(\forall x\in\mathbb{R}^n),\end{aligned}\tag{2.9}$$

where Q_k^{*,a_1} and $\widetilde{\Delta}_k^{*,a_2}$ are defined as in Remark 1.3, we obtain

$$\|\{2^{ks}\Pi_{k,1}(f,g)\}_{k\in\mathbb{N}} \mid \ell_q\| \le c\sup_{j\in\mathbb{N}}(Q_j^{*,a_1}g)\|\{2^{ks}\widetilde{\Delta}_k^{*,a_2}f\}_{k\in\mathbb{N}} \mid \ell_q\|,\tag{2.10}$$

where a_1 and a_2 are real numbers at our disposal. We set $1/b = 1/p_2 - r/n$. The left-hand side of (2.10), in L_p-norm, is bounded by

$$c\Big\|\sup_{j\in\mathbb{N}} Q_j^{*,a_1}g \mid L_b\Big\|\,\|\{2^{ks}\widetilde{\Delta}_k^{*,a_2}f\}_{k\in\mathbb{N}} \mid L_{p_1}(\ell_q)\|.\tag{2.11}$$

Choose $a_1 > n/b$ and $a_2 > n/\min(p_1,q)$, then both Lemma 2.4 and the embedding $B^r_{p_2,q_2}\hookrightarrow F^0_{b,2}$ yield that (2.11) is estimated as desired.

(ii) Estimate of $\{\Pi_{k,2}(f,g)\}_{k\in\mathbb{N}}$. Let $u\in\mathbb{R}$ such that

$$\max\Big(0,\frac{1}{p_1}-\frac{r}{n}\Big) < \frac{1}{u} < \min\Big(\frac{1}{p_1},\frac{1}{p_1}-\frac{s}{n}\Big).\tag{2.12}$$

We set

$$\frac{1}{v} = \frac{1}{p_2}+\frac{1}{u},\qquad \sigma = s-\frac{n}{p}+\frac{n}{v},\qquad \beta = s-\frac{n}{p_1}+\frac{n}{u}.\tag{2.13}$$

We have

$$\ell_p^{\sigma}(L_v^{\gamma}) \hookrightarrow L_p^{\gamma}(\ell_q^{s}), \qquad F_{p_1,q}^{s} \hookrightarrow B_{u,p_1}^{\beta}. \tag{2.14}$$

For the first embedding of (2.14), we can see [4, Section 2.3, Theorem 3]. On the other hand, the Hölder inequality yields

$$2^{k\sigma}\|\Pi_{k,2}(f,g) \mid L_v\| \le c2^{kr}\|\widetilde{\Delta}_k g \mid L_{p_2}\|\left(2^{k\beta}\sum_{j=0}^{k+1}2^{-j\beta}\cdot 2^{j\beta}\|\Delta_j f \mid L_u\|\right). \tag{2.15}$$

We set $1/\widetilde{q}_2 = 1/p - 1/p_1$. Applying, successively, the Hölder inequality again in ℓ_p-norm and Lemma 1.5, we obtain the bound $c\|g \mid B_{p_2,\widetilde{q}_2}^{r}\|\|f \mid B_{u,p_1}^{\beta}\|$. So (2.14) and $B_{p_2,q_2}^{r} \hookrightarrow B_{p_2,\widetilde{q}_2}^{r}$ give

$$\|\{2^{ks}\Pi_{k,2}(f,g)\}_{k\in\mathbb{N}} \mid L_p(\ell_q)\| \le c\|g \mid B_{p_2,q_2}^{r}\|\|f \mid F_{p_1,q}^{s}\|. \tag{2.16}$$

(iii) Estimate of $\{\Pi_{k,3}(f,g)\}_{k\in\mathbb{N}}$. We first consider $1/p_1 + 1/p_2 \le 1$. Let $u \in \mathbb{R}$ such that

$$\max\left(0, \frac{1}{p_1} - \frac{r}{n}, \frac{1}{p_1} - \frac{r+s}{n}\right) < \frac{1}{u} < \frac{1}{p_1}. \tag{2.17}$$

We use the notations v, σ, and β from (2.13). Lemma 1.6 provides

$$2^{k\sigma}\|\Pi_{k,3}(f,g) \mid L_v\| \le c2^{k(\beta+r)}\sum_{j=k}^{\infty}2^{-j(\beta+r)}\cdot 2^{j(\beta+r)}\|\overline{\Delta}_j g \mid L_{p_2}\|\|\Delta_j f \mid L_u\|. \tag{2.18}$$

A similar argument as above yields

$$\|\{2^{k\sigma}\Pi_{k,3}(f,g)\}_{k\in\mathbb{N}} \mid \ell_p(L_v)\| \le c\|\{2^{j(\beta+r)}\|\overline{\Delta}_j g \mid L_{p_2}\|\|\Delta_j f \mid L_u\|\}_{j\in\mathbb{N}} \mid \ell_p\|. \tag{2.19}$$

We set $1/\widetilde{q}_2 = 1/p - 1/p_1$. By the Hölder inequality in ℓ_p-norm, the right-hand side of (2.19) is bounded by $c\|g \mid B_{p_2,\widetilde{q}_2}^{r}\|\|f \mid B_{u,p_1}^{\beta}\|$. Then we conclude the desired estimate by (2.14).

We now study case $1/p_1 + 1/p_2 > 1$. Let $u \in \mathbb{R}$ such that

$$\max\left(0, 1 - \frac{1}{p_2}, \frac{1}{p_1} - \frac{r}{n}\right) < \frac{1}{u} < \frac{1}{p_1}. \tag{2.20}$$

We employ the notations v and σ from (2.13). By Lemma 1.6, we obtain

$$2^{k\sigma}\|\Pi_{k,3}(f,g) \mid L_v\| \le c2^{k\mu}\sum_{j=k}^{\infty}2^{-j\mu}\cdot 2^{j(r+\varrho)}\|\overline{\Delta}_j g \mid L_{p_2}\|\|\Delta_j f \mid L_u\|, \tag{2.21}$$

where $\varrho = s - n/p_1 + n/u$ and $\mu = s + r - n/p_1 - n/p_2 + n > 0$, therefore,

$$\|\{2^{k\sigma}\Pi_{k,3}(f,g)\}_{k\in\mathbb{N}} \mid \ell_p(L_v)\| \le c\|\{2^{j(r+\varrho)}\|\overline{\Delta}_j g \mid L_{p_2}\|\|\Delta_j f \mid L_u\|\}_{j\in\mathbb{N}} \mid \ell_p\|. \tag{2.22}$$

On the right-hand side of (2.22), we employ the Hölder inequality in ℓ_p-norm (with $1/p = 1/p_1 + 1/\widetilde{q}_2$), $F^s_{p_1,q} \hookrightarrow B^{\varrho}_{u,p_1}$, and $B^r_{p_2,q_2} \hookrightarrow B^r_{p_2,\widetilde{q}_2}$ successively. Since $\sigma > s$ and $\nu < p$, we can finish the proof of this case by applying, in the left-hand side of (2.22), embeddings (2.14).

Case 2 ($r = n/p_2$). We only estimate $\{\Pi_{k,1}(f,g)\}_{k\in\mathbb{N}}$. It is sufficient to see that

$$2^{ks}\left|\Pi_{k,1}(f,g)\right| \le c\|g \mid L_\infty\|\left(2^{ks}\widetilde{\Delta}_k^{*,a} f\right) \tag{2.23}$$

with $a > n/\min(p_1,q)$ and to take the $L_{p_1}(\ell_q)$-norm. □

Theorem 2.5. *Let* $0 < p,\ p_1 < \infty$, $0 < p_2,\ q \le \infty$, $-\infty < s < \infty$, *and* $r > 0$ *be such that*

$$-r + \max\left(0, \frac{n}{p_1} + \frac{n}{p_2} - n\right) < s < r. \tag{2.24}$$

If either of the following assertions is satisfied:

(i) $1/p = 1/p_1 + 1/p_2$,

(ii) $\max(1/p_1, s/n) + \max(0, 1/p_2 - r/n) < 1/p < 1/p_1 + 1/p_2$,

then it holds

$$F^s_{p_1,q} \cdot B^r_{p_2,\infty} \hookrightarrow F^s_{p,q}. \tag{2.25}$$

Corollary 2.6. *Let* p, p_1, q, r, s *be as in Theorem 2.5 and* $0 < p_2 < \infty$. *If (i) or (ii) of Theorem 2.5 is satisfied, then the embedding* $F^s_{p_1,q} \cdot F^r_{p_2,\infty} \hookrightarrow F^s_{p,q}$ *holds.*

The proof of Corollary 2.6 is immediate because $F^r_{p_2,\infty} \hookrightarrow B^r_{p_2,\infty}$.

Remark 2.7. We note that Theorem 2.5(i) when $p_2 = \infty$ is given in [4, Section 3.2, Theorem 1]. Also, we note that Corollary 2.6 is given in both [5, Theorem 6.1 with $r = p$ in formula (6.6)] and [10, Theorems 4.4.3/1(7) and 4.4.4/1(7)].

Proof of Theorem 2.5(i). Noting Remark 2.7, we only need to treat the part $0 < p_2 < \infty$.

(i) Estimate of $\{\Pi_{k,1}(f,g)\}_{k\in\mathbb{N}}$. From (2.9) and Lemma 2.4, we have

$$\left\|\{2^{ks}\Pi_{k,1}(f,g)\}_{k\in\mathbb{N}} \mid L_p(\ell_q)\right\| \le c\|g \mid F^0_{p_2,2}\|\|f \mid F^s_{p_1,q}\|. \tag{2.26}$$

By embeddings $B^r_{p_2,\infty} \hookrightarrow B^0_{p_2,\min(p_2,2)} \hookrightarrow F^0_{p_2,2}$, we obtain that the last term of (2.26) is bounded by the desired quantity.

(ii) Estimate of $\{\Pi_{k,2}(f,g)\}_{k\in\mathbb{N}}$. The Hölder inequality provides

$$\|\Pi_{k,2}(f,g) \mid L_p\| \le c\left(2^{-kr}\sum_{j=0}^{k+1} 2^{-j(s-r)} \cdot 2^{-jr}\right)\|g \mid B^r_{p_2,\infty}\|\|f \mid B^s_{p_1,\infty}\|. \tag{2.27}$$

The hypothesis $s < r$ yields

$$\left\|\{2^{ks}\Pi_{k,2}(f,g)\}_{k\in\mathbb{N}} \mid \ell_{\min(p,q)}(L_p)\right\| \le c\|g \mid B^r_{p_2,\infty}\|\|f \mid B^s_{p_1,\infty}\|. \tag{2.28}$$

Using embeddings

$$\ell^s_{\min(p,q)}(L^\gamma_p) \hookrightarrow L^\gamma_p(\ell^s_q), \qquad F^s_{p_1,q} \hookrightarrow B^s_{p_1,\infty}, \tag{2.29}$$

we obtain the desired result.

(iii) Estimate of $\{\Pi_{k,3}(f,g)\}_{k\in\mathbb{N}}$. We set $\varrho = s+r-\max(0,n/p-n)$. Using Lemma 1.6, we obtain

$$\begin{aligned} 2^{ks}\|\Pi_{k,3}(f,g) \mid L_p\| &\le c2^{-kr}\cdot 2^{k\varrho}\sum_{j=k}^{\infty}2^{-j\varrho}\left(2^{jr}\|\overline{\Delta}_j g \mid L_{p_2}\|\right)\left(2^{js}\|\Delta_j f \mid L_{p_1}\|\right) \\ &\le c2^{-kr}\|g \mid B^r_{p_2,\infty}\|\|f \mid B^s_{p_1,\infty}\|. \end{aligned} \tag{2.30}$$

□

In this inequality, we take $\ell_{\min(p,q)}$-norm and we conclude the desired estimate using (2.29).

Proof of Theorem 2.5(ii). (1) *Estimate of* $\{\Pi_{k,1}(f,g)\}_{k\in\mathbb{N}}$. We set $1/u = 1/p - 1/p_1$. As in (2.26), we have the bound $c\|g \mid F^0_{u,2}\|\|f \mid F^s_{p_1,q}\|$ which, by the embeddings $B^r_{p_2,\infty} \hookrightarrow B^{n/p_2-n/u}_{p_2,u} \hookrightarrow F^0_{u,2}$, is estimated as desired.

(2) *Estimate of* $\{\Pi_{k,2}(f,g)\}_{k\in\mathbb{N}}$. In part, for technical reasons, we prove this in three separate cases:

Case 1 ($s<0$). By Lemma 1.6, the Hölder inequality, and Lemma 1.8, we have

$$\|\Pi_{k,2}(f,g) \mid L_p\| \le c2^{(n/p_1+n/p_2-n/p-r-s)k}\|g \mid B^r_{p_2,\infty}\|\|f \mid B^s_{p_1,\infty}\|. \tag{2.31}$$

Since $n/p_1+n/p_2-n/p-r<0$, we obtain an inequality of type (2.28) and finish the proof of this case using (2.29).

Case 2 ($0 \le s < n/p_1$). We set $1/b = 1/p_2 + 1/p_1 - s/n$. We continue with the following subcases.

Subcase 2.1 ($r \le n/p_2$ and $p \le b$ (or $s \le n/p_2 < r$ and $p \le b$)). As in Case 1, we have

$$\|\Pi_{k,2}(f,g) \mid L_p\| \le c\gamma_k 2^{-kr}\|g \mid B^r_{p_2,\infty}\|\|f \mid B^s_{p_1,\infty}\|, \tag{2.32}$$

where

$$\gamma_k = \begin{cases} k+2 & \text{if } p=b, \\ \left(1-2^{n/b-(n/p)-s}\right)^{-1} & \text{if } p<b. \end{cases} \tag{2.33}$$

Now since $\{2^{k(s-r)}\gamma_k\}_{k\in\mathbb{N}} \in \ell_{\min(p,q)}$, we conclude the desired conclusion using (2.28) and (2.29).

Subcase 2.2 ($r \le n/p_2$ and $p > b$ (or $s \le n/p_2 < r$ and $p > b$)). Let $u>0$ satisfy

$$\max\left(0,\frac{1}{p}-\frac{1}{p_2}\right) < \frac{1}{u} < \frac{1}{p_1}-\frac{s}{n}. \tag{2.34}$$

We employ the notations v, σ, and β from (2.13). We have

$$\|\Pi_{k,2}(f,g) \mid L_v\| \leq c\|g \mid B^r_{p_2,\infty}\|\|f \mid B^\beta_{u,\infty}\|(2^{-k(\beta+r)}). \tag{2.35}$$

Since $\{2^{-k(\beta+r-\sigma)}\}_{k\in\mathbb{N}} \in \ell_p$, we can finish the proof of this case using (2.14).
Subcase 2.3 ($n/p_2 < s < r$). We have only case $p < b$ needs to be verified. As in (2.32), we immediately obtain the result.
Case 3 ($s \geq n/p_1$). We have the following subcases.
Subcase 3.1 ($p < p_2$). We set $1/v = 1/p - 1/p_2$. Observe that

$$\begin{aligned} 2^{ks}\|\Pi_{k,2}(f,g) \mid L_p\| &\leq c2^{kr}\|\widetilde{\Delta}_k g \mid L_{p_2}\|\left(2^{k(s-r)}\sum_{j=0}^{k+1}2^{j(n/p_1-n/v)}\|\Delta_j f \mid L_{p_1}\|\right) \\ &\leq c|g \mid B^r_{p_2,\infty}\|\|f \mid B^{n/p_1}_{p_1,\infty}\|\left(2^{k(s-r)}\sum_{j=0}^{k+1}2^{-jn/v}\right). \end{aligned} \tag{2.36}$$

Then, we calculate $\ell_{\min(p,q)}$-norm and conclude the desired estimate by the fact that

$$\left\{2^{k(s-r)}\sum_{j=0}^{k+1}2^{-jn/v}\right\}_{k\in\mathbb{N}} \in \ell_{\min(p,q)}. \tag{2.37}$$

Subcase 3.2 ($s > n/p_1$ and $p \geq p_2$). It suffices to apply both embedding $B^s_{p_1,\infty} \hookrightarrow B^{n/p_1}_{p_1,1}$ and (2.29) to

$$\begin{aligned} \|\Pi_{k,2}(f,g) \mid L_p\| &\leq c\|\widetilde{\Delta}_k g \mid L_p\|\|Q_{k+1}f \mid L_\infty\| \\ &\leq c2^{k(n/p_2-r-n/p)}\|g \mid B^r_{p_2,\infty}\|\|f \mid B^{n/p_1}_{p_1,1}\|. \end{aligned} \tag{2.38}$$

Subcase 3.3 ($s = n/p_1$ and $p \geq p_2$). We choose $\alpha > 0$ such that $\varepsilon = \alpha - n/p + n/p_1 + n/p_2 - r < 0$, then it suffices to apply (2.29) to

$$2^{kn/p_1}\|\Pi_{k,2}(f,g) \mid L_p\| \leq c2^{k\varepsilon}\|g \mid B^r_{p_2,\infty}\|\|f \mid B^{n/p_1-\alpha}_{p_1,\infty}\|. \tag{2.39}$$

(3) *Estimate of* $\{\Pi_{k,3}(f,g)\}_{k\in\mathbb{N}}$. The proof of this case is obtained similarly to the proof of Theorem 2.1 just by replacing (2.17) and (2.20) with

$$\begin{aligned} &\max\left(0, \frac{1}{p} - \frac{1}{p_2}, \frac{1}{p_1} - \frac{r+s}{n}\right) < \frac{1}{u} < \frac{1}{p_1}, \\ &\max\left(0, 1 - \frac{1}{p_2}, \frac{1}{p} - \frac{1}{p_2}\right) < \frac{1}{u} < \frac{1}{p_1}, \end{aligned} \tag{2.40}$$

respectively. ☐

3. Multiplication of types $F \cdot B$ and $B \cdot B$

The next theorem presents a continuation of [3], [5, Theorem 6.1] ,[6], and [7, Section 5].

THEOREM 3.1. *Let* $0 < p_1 < \infty$ *(resp.,* $0 < p_1 \leq \infty$*),* $1 \leq p_2 \leq \infty$, $0 < q \leq \infty$, *and* $n/p_1 - n < s < \min(n/p_1, n/p_2)$. *Then it holds*

$$F^s_{p_1,q} \cdot \left(B^{n/p_2}_{p_2,\infty} \cap L_\infty\right) \hookrightarrow F^s_{p_1,q} \quad \left(\textit{resp., } B^s_{p_1,q} \cdot \left(B^{n/p_2}_{p_2,\infty} \cap L_\infty\right) \hookrightarrow B^s_{p_1,q}\right). \tag{3.1}$$

Remark 3.2. We note that (3.1), in the F-case, was proved by Franke [4, Section 3.4, Corollary 1] but only in the particular case

$$p_2 = \begin{cases} p_1 & \text{if } 0 < p_1 < 2, \\ p_1 (p_1 - 1)^{-1} & \text{if } 2 \leq p_1 < \infty. \end{cases} \tag{3.2}$$

Also this case yields

$$F^s_{p_1,q} \cdot \left(F^{n/p_2}_{p_2,\infty} \cap L_\infty\right) \hookrightarrow F^s_{p_1,q}. \tag{3.3}$$

Remark 3.3. Theorem 3.1, when $1 \leq p_1 \leq p_2 < \infty$, was proved in [3].

Proof of Theorem 3.1. The estimates of $\{\Pi_{k,1}(f,g)\}_{k\in\mathbb{N}}$ and $\{\Pi_{k,3}(f,g)\}_{k\in\mathbb{N}}$ are similar to Theorem 2.1, see also [3]. For $\{\Pi_{k,2}(f,g)\}_{k\in\mathbb{N}}$ we take, in (2.16), $r = n/p_2$, and $q_2 = \widetilde{q}_2 = \infty$, we obtain (3.1) in the F-case. In the B-case, we will employ the notations u, v, σ, and β from (2.12) and (2.13) with the modifications $r = n/p_2$ and $\sigma = s - n/p_1 + n/v$. One has

$$2^{k\sigma} \| \Pi_{k,2}(f,g) \mid L_v \| \leq c \| g \mid B^{n/p_2}_{p_2,\infty} \| \left(2^{k\beta} \sum_{j=0}^{k+1} 2^{-j\beta} \cdot 2^{j\beta} \| \Delta_j f \mid L_u \| \right). \tag{3.4}$$

Since $\beta < 0$, the last inequality, in the ℓ_q-norm, is bounded by the expression $c\|g \mid B^{n/p_2}_{p_2,\infty}\|$ $\|f \mid B^{\beta}_{u,q}\|$. At the end, it suffices to use

$$\ell^\sigma_q (L^\gamma_v) \hookrightarrow \ell^s_q (L^\gamma_{p_1}), \qquad B^s_{p_1,q} \hookrightarrow B^\beta_{u,q}. \tag{3.5}$$

□

4. Some limit cases

We will prove results of independent interest concerning the limit case for the parameters $s + r$, see [5, Theorems 6.5 and 6.11].

THEOREM 4.1. *Let* $0 < p, q, p_i, q_i \leq \infty$, $(i = 1, 2)$, $-\infty < s < \infty$, *and* $r > 0$ *such that*

$$\frac{1}{q_1} + \frac{1}{q_2} \geq 1, \qquad s + r = \frac{n}{p_1} + \frac{n}{p_2} - n > 0. \tag{4.1}$$

If either of the following assertions is satisfied:

(i)

$$r < \frac{n}{p_2}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n}, \quad s < \min\left(\frac{n}{p_1}, r\right), \\ \frac{1}{q_2} \geq \frac{1}{p_2} - \frac{r}{n}, \quad q = \infty, \; p_2 \neq \infty, \tag{4.2}$$

(ii)

$$\max\left(\frac{1}{p_1}, \frac{s}{n}\right) + \max\left(0, \frac{1}{p_2} - \frac{r}{n}\right) < \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \tag{4.3}$$

and either of the following cases is satisfied:
(1) $s < r$, $q_1 \leq q$,
(2) $s = r$, $\max(q_1, q_2) \leq q$,
then it holds

$$B^s_{p_1,q_1} \cdot B^r_{p_2,q_2} \hookrightarrow B^s_{p,q}. \tag{4.4}$$

Remark 4.2. In Theorem 4.1(i), when $r = n/p_2$, we have

$$B^s_{p_1,q_1} \cdot \left(B^{n/p_2}_{p_2,q_2} \cap L_\infty\right) \hookrightarrow B^s_{p_1,q}. \tag{4.5}$$

THEOREM 4.3. *Let* $0 < p, p_1, p_2 < \infty$, $0 < q \leq \infty$, $-\infty < s < \infty$, *and* $r > 0$ *such that*

$$s + r = \frac{n}{p_1} + \frac{n}{p_2} - n > 0. \tag{4.6}$$

If either of the following assertions is satisfied:

(i)

$$r < \frac{n}{p_2}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{r}{n}, \quad s < \min\left(\frac{n}{p_1}, r\right), \tag{4.7}$$

(ii)

$$\max\left(\frac{1}{p_1}, \frac{s}{n}\right) + \max\left(0, \frac{1}{p_2} - \frac{r}{n}\right) < \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}, \quad s \leq r, \tag{4.8}$$

then it holds

$$F^s_{p_1,q} \cdot F^r_{p_2,\infty} \hookrightarrow B^s_{p,\infty}. \tag{4.9}$$

Remark 4.4. In Theorem 4.3(i), when $r = n/p_2$, we have

$$F^s_{p_1,q} \cdot \left(F^{n/p_2}_{p_2,\infty} \cap L_\infty\right) \hookrightarrow B^s_{p_1,\infty}. \tag{4.10}$$

THEOREM 4.5. *Let* $0 < p, p_1 < \infty$, $0 < p_2, q_1, q_2 \leq \infty$, $-\infty < s < \infty$, *and* $r > 0$ *such that*

$$q_1 \geq p_1, \quad \frac{1}{q_1} + \frac{1}{q_2} \geq 1, \qquad s < r, \quad s + r = \max\left(0, \frac{n}{p_1} + \frac{n}{p_2} - n\right). \tag{4.11}$$

If either of the following assertions is satisfied:

(i)

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \tag{4.12}$$

(ii)

$$\max\left(\frac{1}{p_1}, \frac{s}{n}\right) + \max\left(0, \frac{1}{p_2} - \frac{r}{n}\right) < \frac{1}{p} < \frac{1}{p_1} + \frac{1}{p_2}, \tag{4.13}$$

then it holds

$$F^s_{p_1,q_1} \cdot B^r_{p_2,q_2} \hookrightarrow F^s_{p,q_1}. \tag{4.14}$$

Proof of Theorem 4.1(i). (i) *Estimate of* $\{\Pi_{k,1}(f,g)\}_{k\in\mathbb{N}}$. We set $1/u = 1/p_2 - r/n$. The Hölder inequality and Lemma 2.4 give

$$2^{ks}\|\Pi_{k,1}(f,g) \mid L_p\| \leq c\|g \mid F^0_{u,2}\|(2^{ks}\|\widetilde{\Delta}_k f \mid L_{p_1}\|). \tag{4.15}$$

The embedding $B^r_{p_2,q_2} \hookrightarrow F^0_{u,2}$ together with the ℓ_∞-norm of (4.15) and $\ell_{q_1} \hookrightarrow \ell_\infty$ give the desired estimate.

(ii) *Estimate of* $\{\Pi_{k,2}(f,g)\}_{k\in\mathbb{N}}$. Using the notations u, v, σ, and β from (2.12) and (2.13), we have, as in (2.15),

$$2^{k\sigma}\|\{\Pi_{k,2}(f,g)\}_{k\in\mathbb{N}} \mid \ell_\infty(L_v)\| \leq c\|g \mid B^r_{p_2,\infty}\|\|f \mid B^\beta_{u,\infty}\|, \tag{4.16}$$

and the conclusion is obtained by (3.5).

(iii) *Estimate of* $\{\Pi_{k,3}(f,g)\}_{k\in\mathbb{N}}$. We set $1/b = 1/p_1 + 1/p_2$. By Lemma 1.6 and the Hölder inequality, we obtain

$$\|\Pi_{k,3}(f,g) \mid L_b\| \leq c2^{-k(s+r)} \sum_{j=k}^{\infty} (2^{jr}\|\overline{\Delta}_j g \mid L_{p_2}\|)(2^{js}\|\Delta_j f \mid L_{p_1}\|). \tag{4.17}$$

Using $\ell_d \hookrightarrow \ell_1$ (with $1/d = 1/q_1 + 1/q_2$) we employ the Hölder inequality again to conclude that the last term of (4.17) is bounded by $c\|g \mid B^r_{p_2,q_2}\|\|f \mid B^s_{p_1,q_1}\|$. We finish the proof of this case by applying the embedding $\ell^{s+r}_\infty(L^\gamma_b) \hookrightarrow \ell^s_\infty(L^\gamma_p)$. □

Proof of Theorem 4.1(ii). For $\{\Pi_{k,1}(f,g)\}_{k\in\mathbb{N}}$ and $\{\Pi_{k,2}(f,g)\}_{k\in\mathbb{N}}$, we can use the same methods in Theorem 2.5, see also [10, Sections 4.4.3 and 4.4.4].

Estimate of $\{\Pi_{k,3}(f,g)\}_{k\in\mathbb{N}}$. By Lemmas 1.6, 1.8 and the Hölder inequality, we have

$$2^{ks}\|\Pi_{k,3}(f,g)\mid L_p\| \le c2^{k(n/p_1+n/p_2-n/p-r)}\sum_{j=k}^{\infty}\left(2^{jr}\|\overline{\Delta}_j g\mid L_{p_2}\|\right)\left(2^{js}\|\Delta_j f\mid L_{p_1}\|\right). \tag{4.18}$$

Since $n/p_1+n/p_2-n/p-r<0$, we conclude the desired estimate using $\ell_d \hookrightarrow \ell_1$ (with $1/d=1/q_1+1/q_2$). □

Proof of Theorem 4.3(i). (i) *Estimate of* $\{\Pi_{k,1}(f,g)\}_{k\in\mathbb{N}}$. We set $1/u=1/p_2-r/n$, (i.e., $1/p=1/p_1-1/u$). As in (2.9), the choice of $a_1>n/u$ and $a_2>n/p_1$ leads to

$$\begin{aligned}\|\{2^{ks}\Pi_{k,1}(f,g)\}_{k\in\mathbb{N}}\mid \ell_\infty(L_p)\| &\le c\left\|\sup_{j\in\mathbb{N}} Q_j^{*,a_1}g\mid L_u\right\|\|\{2^{ks}\widetilde{\Delta}_k^{*,a_2}f\}_{k\in\mathbb{N}}\mid \ell_\infty(L_{p_1})\|\\ &\le c\|g\mid F^0_{u,2}\|\,\|f\mid B^s_{p_1,\infty}\|.\end{aligned} \tag{4.19}$$

We conclude the desired estimate by applying both $F^r_{p_2,\infty}\hookrightarrow F^0_{u,2}$ and $F^s_{p_1,\infty}\hookrightarrow B^s_{p_1,\infty}$.

(ii) *Estimate of* $\{\Pi_{k,2}(f,g)\}_{k\in\mathbb{N}}$. Using the notations u, v, σ, and β from (2.12) and (2.13), we have

$$2^{k\sigma}\left|\Pi_{k,2}(f,g)\right| \le c\sup_{\ell\in\mathbb{N}}\left(2^{\ell r}\Delta_\ell^{*,a_1}g\right)\left(2^{k\beta}\sum_{j=0}^{k+1}2^{-j\beta}\left(2^{j\beta}\Delta_j^{*,a_2}f\right)\right). \tag{4.20}$$

Since $\beta<0$, then

$$\|\{2^{k\sigma}\Pi_{k,2}(f,g)\}_{k\in\mathbb{N}}\mid \ell_\infty\| \le c\sup_{\ell\in\mathbb{N}}\left(2^{\ell r}\overline{\Delta}_\ell^{*,a_1}g\right)\|\{2^{j\beta}\Delta_j^{*,a_2}f\}_{j\in\mathbb{N}}\mid \ell_\infty\|. \tag{4.21}$$

We choose $a_1>n/p_2$ and $a_2>n/u$. We obtain the desired result by applying the Hölder inequality, the embeddings $L_v^\gamma(\ell_\infty^\sigma)\hookrightarrow L_p^\gamma(\ell_\infty^s)\hookrightarrow \ell_\infty^s(L_p^\gamma)$ and $F^s_{p_1,q}\hookrightarrow F^\beta_{u,\infty}$.

(iii) *Estimate of* $\{\Pi_{k,3}(f,g)\}_{k\in\mathbb{N}}$. We set $1/u=1/p_2+1/p_1$. We begin by the inequality

$$\left\|\sum_{k=0}^{\infty}\Pi_{k,3}(f,g)\mid B^s_{p,\infty}\right\| \le c\left\|\sum_{j=0}^{\infty}Q_j\left(\overline{\Delta}_j g\cdot\Delta_j f\right)\mid B^{s+r}_{u,\infty}\right\|. \tag{4.22}$$

We can write

$$\left|Q_j\left(\overline{\Delta}_j g\cdot\Delta_j f\right)\right| \le c\sup_{j\in\mathbb{N}}\left(\Delta_j^{*,a_1}g\cdot\Delta_j^{*,a_2}f\right). \tag{4.23}$$

We choose $a_1>n/p_2$ and $a_2>n/\min(p_1,q)$. Then Lemma 1.7 gives the correct bound for (4.22). □

The same method works for the proofs of Theorems 4.3(ii) and 4.5. We omit the details.

Remark 4.6. Theorems 4.1(i) and 4.3(i), when $1 \le p \le \infty$, were proved by Johnsen in [5, Theorems 6.11 and 6.5], respectively.

5. Application

We consider $S^0_{1,0}(E)$ (E a Banach space), the class of symbols $(x,\xi) \to a(x,\xi)$ satisfying

$$\|\partial^\beta_\xi a(\cdot,\xi) \mid E\| \le c_\beta (1+|\xi|)^{-|\beta|} \quad (\forall \beta \in \mathbb{N}^n), \tag{5.1}$$

and we define the pseudodifferential operator by the formula

$$\operatorname{Op}_a f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) \hat{f}(\xi) d\xi \quad (\forall f \in \mathcal{S}, \forall x \in \mathbb{R}^n). \tag{5.2}$$

As mentioned in the introduction, the theorems of this section present an application of the previous results in this paper.

Theorem 5.1. *Let $1 \le p, p_1, p_2, q, q_2 \le \infty$, $-\infty < s < \infty$, and $r > 0$. Under the hypotheses of Theorem 2.1 (with $p_2 \ne \infty$ and $r < n/p_2$) or Theorem 2.5, the operator Op_a is bounded from $F^s_{p_1,q}$ to $F^s_{p,q}$, for all $a \in S^0_{1,0}(B^r_{p_2,q_2})$.*

The proof of Theorem 5.1 is based on the following almost-orthogonality lemma.

Lemma 5.2. *Let $\gamma > 1$ and let p, p_1, p_2, q, q_2, r, s be the same as in Theorem 2.1 (with $p_2 \ne \infty$ and $r < n/p_2$) or Theorem 2.5. For all sequences $\{m_j\}_{j\in\mathbb{N}} \subset B^r_{p_2,q_2}$ and all sequences $\{f_j\}_{j\in\mathbb{N}}$ of functions such that $\operatorname{supp}\hat{f}_j \subset \{\xi \in \mathbb{R}^n : \gamma^{-1}2^j \le |\xi| \le \gamma 2^j\}$, the estimate*

$$\left\| \sum_{j=0}^{\infty} m_j \cdot f_j \mid F^s_{p,q} \right\| \le c \|\{f_j\}_{j\in\mathbb{N}} \mid L^\gamma_{p_1}(\ell^s_q)\| \tag{5.3}$$

holds with $c = c' \sup_{j\ge0} \|m_j \mid B^r_{p_2,q_2}\|$.

Proof. Observe that $\Delta_k f_j \ne 0$ and $Q_{k+1} f_j \ne 0$ if $k-N \le j \le k+N+2$ and $j \le k+N+2$, respectively, where $N = [\log_2 \gamma]$; (here $[x]$ denotes the greatest integer less than or equal to x). Then it suffices to apply Theorem 2.1 (and/or Theorem 2.5) to the following decomposition:

$$\Delta_k\left(\sum_{j=0}^{\infty} m_j \cdot f_j\right) = \sum_{\ell=-N}^{N+2} \Pi_{k,1}(m_{k+\ell}, f_{k+\ell}) + \sum_{j=0}^{k+N+2} \Pi_{k,2}(m_j, f_j) + \sum_{\ell=-N}^{N+2} \tilde{\Pi}_{k,3}(m_{\cdot+\ell}, f_{\cdot+\ell}), \tag{5.4}$$

where $\tilde{\Pi}_{k,3}(m_{\cdot+\ell}, f_{\cdot+\ell}) = \sum_{j=k}^{\infty} \Delta_k(\overline{\Delta}_j m_{j+\ell} \cdot \Delta_j f_{j+\ell})$ (see also (2.7)). □

Proof of Theorem 5.1. We begin by writing

$$a(x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} (1+|u|^2)^{-(n+1)/2} a_u(x,\xi) du + \lambda(x,\xi), \tag{5.5}$$

where $\lambda(x,\xi)=0$ for $|\xi|\geq 3$,

$$\left\|\partial_\xi^\beta\lambda(\cdot,\xi)\mid B^r_{p_2,q_2}\right\|\leq c_\beta(1+|\xi|)^{-|\beta|}\quad(\forall\beta\in\mathbb{N}^n),$$

$$\begin{gathered}a_u(x,\xi)=\sum_{j=0}^\infty m_{j,u}(x)\theta_u(2^{-j}\xi),\\ \sup_{j\in\mathbb{N},\,u\in\mathbb{R}^n}\|m_{j,u}\mid B^r_{p_2,q_2}\|\leq c,\\ \theta_u(\xi)=(2\pi)^{-n}(1+|u|^2)^{(n+1-L)/2}e^{iu\cdot\xi}\theta(\xi),\end{gathered}\tag{5.6}$$

θ is a C^∞ function with $\operatorname{supp}\theta\subset\{\xi\in\mathbb{R}^n:1\leq|\xi|\leq 3\}$, and

$$\|\theta_u^{(\beta)}\mid L_\infty\|\leq c\quad(\forall u\in\mathbb{R}^n,\ |\beta|\leq L-n-1).\tag{5.7}$$

For the decomposition (5.5), we refer the reader to [2] or [8].

Now, by Lemma 5.2, we have

$$\|\mathrm{Op}_{a_u}f\mid F^s_{p,q}\|\leq c\|\{2^{js}f_{u,j}\}_{j\in\mathbb{N}}\mid L_{p_1}(\ell_q)\|\leq c'\|f\mid F^s_{p_1,q}\|,\tag{5.8}$$

where $\mathcal{F}(f_{u,j})(\xi)=\theta_u(2^{-j}\xi)\widehat{f}(\xi)$ and c' is independent of u. Next, we can write

$$\mathrm{Op}_\lambda f(x)=\int_{\mathbb{R}^n}(1+|u|^2)^{-(n+1)/2}b_u(x)f(x+u)du,\tag{5.9}$$

where

$$b_u(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{-iu\cdot\xi}(I-\Delta)^{2n}\lambda(x,\xi)d\xi.\tag{5.10}$$

Theorems 2.1 and 2.5 immediately give

$$\begin{aligned}\|\mathrm{Op}_\lambda f\mid F^s_{p,q}\|&\leq\sup_{u\in\mathbb{R}^n}\|b_u\cdot f(\cdot+u)\mid F^s_{p,q}\|\\&\leq\Big(\sup_{u\in\mathbb{R}^n}\|b_u\mid B^r_{p_2,q_2}\|\Big)\|f\mid F^s_{p_1,q}\|\leq c\|f\mid F^s_{p_1,q}\|.\end{aligned}\tag{5.11}$$

□

Theorem 5.3. *Let* $1\leq p,p_1,q,q_1\leq\infty$, $r>0$, *and*

$$-r+\frac{n}{p}+\frac{n}{p_1}-n<s<\min\left(\frac{n}{p},r\right).\tag{5.12}$$

Suppose that $a\in S^0_{1,0}(B^r_{p_1,q_1})$ *if* $r>n/p_1$ *and* $a\in S^0_{1,0}(L_\infty)\cap S^0_{1,0}(B^{n/p_1}_{p_1,\infty})$ *if* $r=n/p_1$. *Then the operator* Op_a *is bounded on* $F^s_{p,q}$ *and* $B^s_{p,q}$.

For the proof, we apply Theorem 3.1 and proceed as in Theorem 5.1, however, we need an almost-orthogonality estimate of the type in Lemma 5.2, that is, the following lemma.

LEMMA 5.4. *Let* $\gamma > 1$, $0 < p, p_1, q, q_1 \leq \infty$, $r \geq n/p_1$, *and* s *be as in Theorem 5.3. For all sequences of functions* $\{f_j\}_{j\in\mathbb{N}}$ *such that* $\operatorname{supp}\widehat{f_j} \subset \{\xi \in \mathbb{R}^n : \gamma^{-1}2^j \leq |\xi| \leq \gamma 2^j\}$ *and all sequences* $\{m_j\}_{j\in\mathbb{N}} \subset B^r_{p_1,q_1}$ *(or* $\{m_j\}_{j\in\mathbb{N}} \subset B^{n/p_1}_{p_1,q_1} \cap L_\infty$*), the estimates*

$$\begin{aligned} \left\| \sum_{j=0}^{\infty} m_j \cdot f_j \mid F^s_{p,q} \right\| &\leq c \| \{f_j\}_{j\in\mathbb{N}} \mid L^\gamma_p(\ell^s_q) \|, \\ \left\| \sum_{j=0}^{\infty} m_j \cdot f_j \mid B^s_{p,q} \right\| &\leq c' \| \{f_j\}_{j\in\mathbb{N}} \mid \ell^s_q(L^\gamma_p) \|, \end{aligned} \tag{5.13}$$

hold. The constants c *and* c' *are of the form* $c'' \sup_{j\in\mathbb{N}}(\|m_j \mid L_\infty\| + \|m_j \mid B^r_{p_1,q_1}\|)$.

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