GENERAL VARIATIONAL INCLUSIONS IN L^p SPACES

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It is well known that the variational inclusions are equivalent to the fixed point problems. We use this equivalent alternative formulation to suggest and analyze some iterative methods for solving variational inclusions in L^p spaces. We also consider the convergence analysis of these new iterative methods under some suitable conditions.

1. Introduction

Variational inequalities are being used as a mathematical programming tool in modeling a wide class of problems arising in different branches of pure and applied sciences, see [1–26]. Variational inequalities have been extended and generalized in various directions using novel and innovative techniques. A useful and important generalization is called the general variational inclusion involving the sum of two nonlinear operators T and A. Moudafi and Noor [10] studied the sensitivity analysis of variational inclusions by using the technique of the resolvent equations. Recently, much attention has been given to develop iterative algorithms for solving the variational inclusions. It is known that such algorithms require an evaluation of the resolvent operator of the type $(I+\rho(T+A))^{-1}$. The main difficulty with such problems is that the resolvent operator may be hard to invert. This difficulty has been overcome by using the resolvent operators $(I+\rho T)^{-1}$ and $(I+\rho A)^{-1}$ separately rather than $(I+\rho(T+A))^{-1}$. Such a technique is called the splitting method. Such type of methods for solving variational inclusions has been studied extensively, see [4, 5, 7, 9–11, 17, 19, 23] and the references therein.

In the context of the mixed variational inequalities (variational inclusions), Noor [15–18] has used the resolvent operator and resolvent equations techniques to suggest and analyze a number of resolvent-type iterative methods. A useful feature of these splitting methods is that the resolvent step involves the subdifferential of the proper, convex, and lower semicontinuous function only and the other part facilitates the problem decomposition.

Equally important is the concept of the resolvent equations, which has been introduced by Noor [16, 17] recently in connection with the mixed variational inequalities.

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It has been shown that the general variational inclusions and the resolvent equations are equivalent. This alternate equivalent formulation has played an important part in suggesting many iterative methods for solving variational inclusions and for studying the sensitivity analysis. Resolvent equations technique has been used to suggest and analyze a number of iterative methods for solving general variational inclusions. In spite of these activities, very little attention has been given to study such type of variational inclusions in L^p spaces. This fact has motivated us to study the variational inclusion in the setting of L^p spaces.

In Section 2, we formulate the problems and review the basic definitions and concepts. In Section 3, we establish the equivalence between the general variational inclusions and the resolvent equations. This equivalence is used to suggest a number of new iterative methods for variational inclusions. We study the convergence of some algorithms for strongly monotone and Lipschitz continuous operators in the L^p-spaces.

2. Formulations and basic facts

Let X be a real Banach space with norm $\|\cdot\|$ and dual X^*. We denote the pairing between X^* and X by $\langle\cdot\rangle$. Let K be a nonempty closed convex set in X.

For given nonlinear operators $T,g : X \to X^*$ and a maximal monotone operator $A : X \to X^*$, consider the problem of finding $u \in X$ such that

$$0 \in Tu + A(g(u)), \tag{2.1}$$

which is called the general variational inclusion. Variational inclusions have been studied and considered by many authors including Moudafi and Noor [10], M. A. Noor and K. I. Noor [20], and Uko [26].

For $g = I$, the identity operator, problem (2.1) is equivalent to finding $u \in X$ such that

$$0 \in T(u) + A(u), \tag{2.2}$$

which is known as the variational inclusion problem. Problems (2.1) and (2.2) are also known as finding the zeros of the sum of two maximal monotone operators. It has been shown that a wide class of linear and nonlinear problems arising in several branches of pure and applied sciences can be studied via the variational inclusions (2.1) and (2.2), see [4, 5, 9, 20, 23, 26].

If $A(\cdot) = \partial\varphi(\cdot)$, where $\partial\varphi(\cdot)$ is the subdifferential of a proper, convex, and lower semicontinuous function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, then problem (2.1) reduces to finding $u \in X$ such that

$$0 \in Tu + \partial\varphi(g(u)) \tag{2.3}$$

or equivalently, finding $u \in X : g(u) \in X$ such that

$$\langle Tu, g(v) - g(u)\rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \quad \forall g(v) \in X. \tag{2.4}$$

The inequality of the type (2.4) is called the general mixed variational inequality or the general variational inequality of the second kind [17, 18]. It can be shown that a wide class of linear and nonlinear problems arising in pure and applied sciences can be studied via the general mixed variational inequalities (2.4). We remark that if $g \equiv I$, the identity operator, then the problem (2.4) is equivalent to finding $u \in X$ such that

$$\langle Tu, v-u\rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in X, \tag{2.5}$$

which are called the mixed variational inequalities. For the applications and numerical formulations, see [1, 2, 6, 8, 17, 18] and the references therein.

We note that if φ is the indicator function of a closed convex set K in X, that is,

$$\varphi(u) \equiv I_K(u) = \begin{cases} 0, & \text{if } u \in K \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.6}$$

then the general mixed variational inequality (2.4) is equivalent to finding $u \in X$, $g(u) \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \quad \forall v \in X : g(v) \in K. \tag{2.7}$$

The inequality of the type (2.7) is known as the general variational inequality which was first introduced and studied by Noor [12]. The general variational inequalities are also called the Noor variational inequalities. It turned out that the odd order and nonsymmetric free, unilateral, obstacle, and equilibrium problems can be studied by the general variational inequality (2.1), see [16, 17, 21].

If $K^* = \{u \in X : \langle u, v\rangle \geq 0, \text{ for all } v \in K\}$ is a polar cone of a convex cone K in X, then problem (2.7) is equivalent to finding $u \in X$ such that

$$g(u) \in K, \quad Tu \in K^*, \quad \langle Tu, g(u)\rangle = 0, \tag{2.8}$$

which is called the general complementarity problem. For $g = I$, problem (2.8) is called the generalized complementarity problem. For the theory, applications, and numerical methods of complementarity problems, see [13, 15–17, 21] and the references therein.

For $g \equiv I$, the identity operator, the general variational inequality (2.7) collapses to finding $u \in K$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall u \in K, \tag{2.9}$$

which is called the standard variational inequality introduced and studied by Stampacchia [25]. For the recent state of the art, see [1–26].

We need the following concepts and results.

Definition 2.1. An operator $T : K \to X$ with $D(T)$ and range $R(T)$ in X is said to be strongly accretive if there exists a real number $\alpha > 0$ such that

$$\langle Tu - Tv, j\rangle \geq \alpha\|u-v\|^2, \quad \forall u, v \in K, \tag{2.10}$$

for some $j \in J(u-v)$. Here

$$J(u) = \{f \in X^* : \|f\|^2 = \langle u, f\rangle = \|u\|^2\} \tag{2.11}$$

is called the normalized duality mapping of X. The normalized duality mapping has the following properties.

(1) $J(u) \neq 0$, for all $u \in X$, a Banach space.

(2) $J(\alpha u) = \alpha J(u)$, $\alpha \in [0,\infty)$.

(3) $J(-u) = -J(u)$.

(4) J is bounded, that is, for any bounded subset $A \subset X$, $J(A)$ is a bounded subset in X^*.

(5) X is uniformly smooth if and only if J is single valued and uniformly continuous on any bounded subset of X.

Lemma 2.2 [3, 27]. *Let* $X = L^p$ *(or* $\updownarrow_{\surd}$*),* $2 \leq p < \infty$. *For any* $u, v \in X$,

$$\|u+v\|^2 \leq (p-1)\|u\|^2 + \|v\|^2 + 2\langle u, j(v)\rangle, \quad \forall j \in J(u+v). \tag{2.12}$$

Definition 2.3 [2]. If A is a maximal accretive operator on X, then for a constant $\rho > 0$, the resolvent operator associated with A is defined by

$$J_A(u) = (I+\rho A)^{-1}(u), \quad \forall u \in X, \tag{2.13}$$

where I is the identity operator. It is well known that a monotone operator is maximal accretive if and only if its resolvent operator is defined everywhere. In addition, the resolvent operator is single valued and nonexpansive, that is,

$$\|J_A(u) - J_A(v)\| \leq \|u-v\|, \quad \forall u, v \in X. \tag{2.14}$$

Remark 2.4. It is well known that the subdifferential $\partial\varphi$ of a proper, convex, and lower semicontinuous function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is a maximal monotone operator. We denote by

$$J_\varphi(u) = (I+\rho\partial\varphi)^{-1}(u), \quad \forall u \in X, \tag{2.15}$$

the resolvent operator associated with $\partial\varphi$, which is defined everywhere on X.

We now introduce the general resolvent equations. To be more precise, let $R_A \equiv I - J_A$, where I is the identity operator and $J_A = (I+\rho A)^{-1}$ is the resolvent operator. Let $g : X \to X^*$ be an invertible operator. For given nonlinear operators $T, g : X \to X^*$, consider the problem of finding $z \in X$ such that

$$Tg^{-1}J_A z + \rho^{-1}R_A z = 0, \tag{2.16}$$

where $\rho > 0$ is a constant. Equations of the type (2.16) are called the general resolvent equations. If $g = I$, then general resolvent equations (2.16) collapse to finding $z \in X$ such that

$$TJ_A z + \rho^{-1}R_A z = 0, \tag{2.17}$$

which are known as the resolvent equations. The resolvent equations have been studied by Moudafi and Théra [11], Noor [13]. Using the general duality principle, it has been shown that the resolvent equations (2.17) are equivalent to the variational inclusions (2.2).

If $A(\cdot) \equiv \partial\varphi(\cdot)$, where $\partial\varphi$ is the subdifferential of a proper, convex, and lower semicontinuous function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, then general resolvent equations (2.16) are equivalent to finding $z \in X$ such that

$$Tg^{-1}J_{\varphi}z + \rho^{-1}R_{\varphi}z = 0, \tag{2.18}$$

which are also called the general resolvent equations introduced and studied by Noor [18] in relation with the general mixed variational inequalities (2.3). Using these resolvent equations, Noor [18] has suggested and analyzed a number of iterative methods for solving general mixed variational inequalities. If $g \equiv I$, the identity operator, then the problem (2.18) reduces to finding $z \in X$ such that

$$TJ_{\varphi}z + \rho^{-1}R_{\varphi}z = 0, \tag{2.19}$$

which are called the resolvent equations. For the applications, formulations, and numerical methods of the resolvent equations, see [16, 18, 19].

We remark that if φ is the indicator function of a closed convex set K, then $J_{\varphi} \equiv P_K$, the projection of H onto K. Consequently, problem (2.18) is equivalent to finding $z \in X$ such that

$$Tg^{-1}P_K z + \rho^{-1}Q_K z = 0. \tag{2.20}$$

Equations of the type (2.20) are known as the general Wiener-Hopf equations, which are mainly due to Noor [13]. For $g = I$, we obtain the Wiener-Hopf (normal) equations introduced and studied by Shi [24] in connection with the classical variational inequalities. We would like to mention that the Wiener-Hopf equations technique is being used to develop some implementable and efficient iterative algorithms for solving variational inequalities and related fields, see [19].

3. Main results

In this section, we suggest and analyze some new iterative methods for solving the general variational inclusions (2.1). First of all, we prove that problem (2.1) is equivalent to the fixed point problem by using the definition of the resolvent operator.

Lemma 3.1. *The function $u \in X$ is a solution of the variational inclusion (2.1) if and only if $u \in X$ satisfies the relation*

$$g(u) = J_A[g(u) - \rho Tu], \tag{3.1}$$

where $J_A = (I + \rho A)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

Proof. Let $u \in H$ be a solution of (2.1). Then for a constant $\rho > 0$, inclusion (2.1) can be written as

$$0 \in -g(u) + \rho Tu + (I + \rho A)g(u), \tag{3.2}$$

which is equivalent to finding $u \in H$ such that

$$g(u) = (I+\rho A)^{-1}[g(u)-\rho Tu] = J_A[g(u)-\rho Tu], \tag{3.3}$$

the required result. □

Lemma 3.1 implies that the variational inclusion (2.1) is equivalent to the fixed point problem. This alternate equivalent formulation is very useful from the numerical point of view. This fixed point formulation can be used to suggest and analyze the iterative algorithm for solving variational inclusion (2.1).

Algorithm 3.2. For a given $u_0 \in X$, compute the approximate solution u_{n+1} by the iterative scheme

$$g(u_{n+1}) = J_A[g(u_n)-\rho Tu_n], \quad n = 0,1,2,\dots. \tag{3.4}$$

We now prove that the variational inclusion (2.1) is equivalent to the resolvent equations (2.16) by invoking Lemma 3.1 and this is the prime motivation of our next result.

Theorem 3.3. *The variational inclusion (2.1) has a solution $u \in X$ if and only if the resolvent equation (2.16) has a solution $z \in X$, where*

$$g(u) = J_A z, \tag{3.5}$$

$$z = g(u)-\rho Tu, \tag{3.6}$$

where J_A is the resolvent operator and $\rho > 0$ is a constant.

Proof. Let $u \in X$ be a solution of (2.1). Then, by Lemma 3.1, we have

$$g(u) = J_A[g(u)-\rho Tu]. \tag{3.7}$$

Let $z = g(u)-\rho Tu$ in (3.7). Then

$$g(u) = J_A z, \qquad z = J_A z - \rho T g^{-1} J_A z, \tag{3.8}$$

which implies that

$$T g^{-1} J_A z + \rho^{-1} R_A z = 0, \tag{3.9}$$

the required result. □

From Theorem 3.3, we conclude that the variational inclusion (2.1) and the resolvent equations (2.16) are equivalent. This alternative formulation plays an important and crucial part in suggesting and analyzing various iterative methods for solving variational inclusions and related optimization problems. In this paper, by suitable and appropriate rearrangement, we suggest a number of new iterative methods for solving variational inclusions (2.1).

(I) The equations (2.16) can be written as

$$R_A z = -\rho T g^{-1} J_A z, \tag{3.10}$$

which implies that, using (3.5),

$$z = J_A z - \rho T g^{-1} J_A z = g(u) - \rho T u. \tag{3.11}$$

This fixed point formulation enables us to suggest the following iterative method for solving the variational inclusion (2.1).

Algorithm 3.4. For a given $z_0 \in X$, compute u_{n+1} by the iterative schemes

$$g(u_n) = J_A z_n, \tag{3.12}$$

$$z_{n+1} = g(u_n) - \rho T u_n, \quad n = 0, 1, 2, \ldots. \tag{3.13}$$

(II) The equations (2.16) may be written as

$$\begin{aligned} z &= J_A z - \rho T g^{-1} J_A z + (1 - \rho^{-1}) R_A z \\ &= u - T u + (1 - \rho^{-1}) R_A z, \quad \text{using (3.5)}. \end{aligned} \tag{3.14}$$

Using this fixed point formulation, we suggest the following iterative method.

Algorithm 3.5. For a given $z_0 \in X$, compute u_{n+1} by the iterative schemes

$$\begin{aligned} g(u_n) &= J_A z_n, \\ z_{n+1} &= u_n - T u_n + (1 - \rho^{-1}) R_A z_n, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{3.15}$$

(III) If the operator T is linear and T^{-1} exists, then the resolvent equation (2.16) can be written as

$$z = (I - \rho^{-1} g T^{-1}) R_A z, \tag{3.16}$$

which allows us to suggest the following iterative method.

Algorithm 3.6. For a given $z_0 \in X$, compute z_{n+1} by the iterative scheme

$$z_{n+1} = (I - \rho^{-1} g T^{-1}) R_A z_n, \quad n = 0, 1, 2 \ldots. \tag{3.17}$$

We remark that if $g \equiv I$, the identity operator, then Algorithms 3.4, 3.5, and 3.6 reduce to the following algorithms for solving variational inclusions (2.2).

Algorithm 3.7. For a given $z_0 \in X$, compute z_{n+1} by the iterative schemes

$$\begin{aligned} u_n &= J_A z_n, \\ z_{n+1} &= u_n - \rho T u_n, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{3.18}$$

Convergence analysis of Algorithm 3.7 has been studied by Noor [16] for strongly monotone and co-strongly monotone operators T.

Algorithm 3.8. For a given $z_0 \in X$, compute z_{n+1} by the iterative schemes

$$\begin{aligned} u_n &= J_A z_n, \\ z_{n+1} &= u_n - Tu_n + (1-\rho^{-1}) R_A z_n, \quad n = 0,1,2\ldots. \end{aligned} \tag{3.19}$$

Algorithm 3.9. For a given $z_0 \in X$, compute z_{n+1} by the iterative scheme

$$z_{n+1} = (I - \rho^{-1} T^{-1}) R_A z_n, \quad n = 0,1,2,\ldots. \tag{3.20}$$

We note that Algorithms 3.8 and 3.9 are new even for the variational inclusions (2.2).

We now study the convergence analysis of Algorithm 3.4. One can study the convergence analysis of Algorithms 3.5–3.9 in a similar way.

Theorem 3.10. *Let T, g be both strongly accretive with constants $\alpha > 0$, $\sigma > 0$, and Lipschitz continuous with constants $\beta > 0$, $\delta > 0$, respectively. If*

$$\begin{gathered} \left| \rho - \frac{\alpha}{\beta^2(p-1)} \right| < \frac{\sqrt{\alpha^2 - \beta^2(p-1)k(2-k)}}{\beta^2(p-1)}, \\ \alpha > \beta\sqrt{k(2-k)(p-1)}, \quad k < 1, \end{gathered} \tag{3.21}$$

where

$$k = 2\sqrt{1 - 2\sigma + \delta^2(p-1)}, \tag{3.22}$$

then there exists $z \in X$ satisfying the resolvent equation (2.16) and the sequence $\{z_n\}$ generated by Algorithm 3.4 convergences to z in X strongly.

Proof. Let $z \in X$ be a solution of the resolvent equation (2.16). Then, from (3.6) and (3.13), we have

$$\begin{aligned} ||z_{n+1} - z|| &= ||g(u_n) - g(u) - \rho(Tu_n - Tu)|| \\ &\le ||u_n - u - (g(u_n) - g(u))|| \\ &\quad + ||u_n - u - \rho(Tu_n - Tu)||. \end{aligned} \tag{3.23}$$

Since T is strongly accretive with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, so using Lemma 2.2, we have

$$\begin{aligned} ||u_n - u - \rho(Tu_n - Tu)||^2 &= ||u_n - u||^2 - 2\rho\langle Tu_n - Tu, j(u_n - u)\rangle \\ &\quad + \rho^2(p-1)||Tu_n - Tu||^2 \\ &\le (1 - 2\rho\alpha + \rho^2\beta^2(p-1))||u_n - u||^2, \end{aligned} \tag{3.24}$$

and similarly,

$$||u_n - u - (g(u_n) - g(u))||^2 \le (1 - 2\sigma + \delta^2(p-1))||u_n - u||^2, \tag{3.25}$$

where $\sigma > 0$ and $\delta > 0$ are the strongly accretivity and Lipschitz continuity constants of the operator g, respectively.

Combining (3.22), (3.23), (3.24), and (3.25), we have

$$||z_{n+1} - z|| \leq \left\{\frac{k}{2} + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2(p-1)}\right\}||u_n - u||. \tag{3.26}$$

From (3.5), (3.12), and (3.25), we obtain

$$||u_n - u|| \leq |u_n - u - (g(u_n) - g(u))|| + ||J_A z_n - J_A z|| \leq \frac{k}{2}||u_n - u|| + ||z_n - z||, \tag{3.27}$$

which implies that

$$||u_n - u|| \leq \left(\frac{1}{1 - k/2}\right)||z_n - z||. \tag{3.28}$$

Combining (3.26) and (3.28), we have

$$||z_{n+1} - z|| \leq \frac{k/2 + t(\rho)}{1 - k/2}||z_n - z|| = \theta||z_n - z||, \tag{3.29}$$

where

$$\begin{aligned} t(\rho) &= \sqrt{1 - 2\rho\alpha + \rho^2\beta^2(p-1)}, \\ \theta &= \left(\frac{k}{2} + t(\rho)\right) \Big/ \left(1 - \frac{k}{2}\right). \end{aligned} \tag{3.30}$$

From (3.21), it follows that $\theta < 1$ and consequently, the map defined by (3.6) is a contraction map and has a fixed point $z \in X$ satisfying the resolvent equation (2.16). Furthermore, it follows that the sequence $\{z_n\}$ generated by Algorithm 3.4 converges to z strongly in X, the required result. □

4. Conclusion

In this paper, we have shown that the variational inclusions can be considered in L^p spaces. We have also proved the equivalence between the variational inclusions and the resolvent equations. This equivalence is used to suggest several iterative methods for solving variational inclusions. The convergence criterion of these methods is also analyzed under certain conditions.

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References

[1] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities*, A Wiley-Interscience Publication, John Wiley & Sons, New York, 1984.

[2] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[3] C. E. Chidume, *An iterative process for nonlinear Lipschitzian strongly accretive mappings in L_p spaces*, Journal of Mathematical Analysis and Applications **151** (1990), no. 2, 453–461.

[4] J. Eckstein and D. P. Bertsekas, *On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators*, Mathematical Programming, Series A **55** (1992), no. 3, 293–318.

[5] M. Fukushima, *The primal Douglas-Rachford splitting algorithm for a class of monotone mappings with application to the traffic equilibrium problem*, Mathematical Programming, Series A **72** (1996), no. 1, 1–15.

[6] F. Giannessi and A. Maugeri (eds.), *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York, 1995.

[7] R. Glowinski and P. Le Tallec, *Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics*, SIAM Studies in Applied Mathematics, vol. 9, SIAM, Pennsylvania, 1989.

[8] R. Glowinski, J.-L. Lions, and R. Trémolières, *Numerical Analysis of Variational Inequalities*, Studies in Mathematics and Its Applications, vol. 8, North-Holland, Amsterdam, 1981.

[9] P.-L. Lions and B. Mercier, *Splitting algorithms for the sum of two nonlinear operators*, SIAM Journal on Numerical Analysis **16** (1979), no. 6, 964–979.

[10] A. Moudafi and M. A. Noor, *Sensitivity analysis for variational inclusions by Wiener-Hopf equation techniques*, Journal of Applied Mathematics and Stochastic Analysis **12** (1999), no. 3, 223–232.

[11] A. Moudafi and M. Théra, *Finding a zero of the sum of two maximal monotone operators*, Journal of Optimization Theory and Applications **94** (1997), no. 2, 425–448.

[12] M. A. Noor, *General variational inequalities*, Applied Mathematics Letters **1** (1988), no. 2, 119–122.

[13] ______, *Wiener-Hopf equations and variational inequalities*, Journal of Optimization Theory and Applications **79** (1993), no. 1, 197–206.

[14] ______, *Some recent advances in variational inequalities. I. Basic concepts*, New Zealand Journal of Mathematics **26** (1997), no. 1, 53–80.

[15] ______, *Some recent advances in variational inequalities. II. Other concepts*, New Zealand Journal of Mathematics **26** (1997), no. 2, 229–255.

[16] ______, *Generalized set-valued variational inclusions and resolvent equations*, Journal of Mathematical Analysis and Applications **228** (1998), no. 1, 206–220.

[17] ______, *Some algorithms for general monotone mixed variational inequalities*, Mathematical and Computer Modelling **29** (1999), no. 7, 1–9.

[18] ______, *Fundamentals of mixed quasi variational inequalities*, International Journal of Pure and Applied Mathematics **15** (2004), no. 2, 137–258.

[19] ______, *Some developments in general variational inequalities*, Applied Mathematics and Computation **152** (2004), no. 1, 199–277.

[20] M. A. Noor and K. I. Noor, *Sensitivity analysis for quasi-variational inclusions*, Journal of Mathematical Analysis and Applications **236** (1999), no. 2, 290–299.

[21] M. A. Noor, K. I. Noor, and T. M. Rassias, *Some aspects of variational inequalities*, Journal of Computational and Applied Mathematics **47** (1993), no. 3, 285–312.

[22] M. Patriksson, *Nonlinear Programming and Variational Inequality Problems. A Unified Approach*, Applied Optimization, vol. 23, Kluwer Academic, Dordrecht, 1999.

[23] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM Journal on Control and Optimization **14** (1976), no. 5, 877–898.

[24] P. Shi, *Equivalence of variational inequalities with Wiener-Hopf equations*, Proceedings of the American Mathematical Society **111** (1991), no. 2, 339–346.

[25] G. Stampacchia, *Formes bilinéaires coercitives sur les ensembles convexes*, Comptes Rendus de l'Académie des Sciences. Paris **258** (1964), 4413–4416.
[26] L. U. Uko, *Strongly nonlinear generalized equations*, Journal of Mathematical Analysis and Applications **220** (1998), no. 1, 65–76.
[27] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Analysis **16** (1991), no. 12, 1127–1138.

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