C-COMPACTNESS MODULO AN IDEAL

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We investigate the concepts of quasi-H-closed modulo an ideal which generalizes quasi-H-closedness and C-compactness modulo an ideal which simultaneously generalizes C-compactness and compactness modulo an ideal. We obtain a characterization of maximal C-compactness modulo an ideal. Preservation of C-compactness modulo an ideal by functions is also investigated.

1. Introduction

In the present paper, we consider a topological space equipped with an ideal, a theme that has been treated by Vaidyanathaswamy [15] and Kuratowski [6] in their classical texts. An *ideal* $\mathcal{I}$ on a set X is a nonempty subset of $P(X)$, the power set of X, which is closed for subsets and finite unions. An ideal is also called a *dual filter*. $\{\phi\}$ and $P(X)$ are trivial examples of ideals. Some useful ideals are (i) $\mathcal{I}_f$, the ideal of all finite subsets of X, (ii) $\mathcal{I}_c$, the ideal of all countable subsets of X, (iii) $\mathcal{I}_n$, the ideal of all nowhere dense subsets in a topological space (X,τ), *and* (iv) $\mathcal{I}_s$, the set of all scattered sets in (X,τ). For an ideal $\mathcal{I}$ on X and $A \subset X$, we denote the ideal $\{I \cap A : I \in \mathcal{I}\}$ by $\mathcal{I}_A$.

A topological space (X,τ) with an ideal $\mathcal{I}$ on X is denoted by $(X,\tau,\mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I},\tau)$ (called the adherence of A *modulo an ideal* $\mathcal{I}$) or $A^*(\mathcal{I})$ or just A^* is the set $\{x \in X : A \cap U \notin \mathcal{I}$ for every open neighborhood U of $x\}$. $A^*(\mathcal{I},\tau)$ has been called the *local function* of A with respect to $\mathcal{I}$ in [6]. It is easy to see that (i) for the ideal $\{\phi\}$, A^* is the closure of A, (ii) for the ideal $P(X)$, A^* is ϕ, and (iii) for ideal $\mathcal{I}_f$, A^* is the set of all ω-accumulation points of A. For general properties of the operator $*$, we refer the readers to [5, 14].

Observe that the operator $\mathrm{cl}^* : P(X) \to P(X)$ defined by $\mathrm{cl}^*(A)=A \cup A^*$ is a Kuratowski closure operator on X and hence generates a topology $\tau^*(\mathcal{I})$ or just τ^* on X finer than τ. As has already been observed, $\tau^*(\{\phi\}) = \tau$ and $\tau^*(P(X))$ = the discrete topology. A description of open sets in $\tau^*(\mathcal{I})$ as given in Vaidyanathaswamy [15] is given in the following.

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Theorem 1.1. *If τ is a topology and $\mathcal{I}$ is an ideal, both defined on X, then*

$$\beta = \beta(\tau,\mathcal{I}) = \{V - I : V \in \tau,\ I \in \mathcal{I}\} \text{ is a base for the topology } \tau^*(\mathcal{I}) \text{ on } X. \tag{1.1}$$

Ideals have been used frequently in the fields closely related to topology, such as real analysis, measure theory, and lattice theory. Some interesting illustrations of $\tau^*(\mathcal{I})$ are as follows [5].

(1) If τ is the topology generated by the partition $\{\{2n-1,2n\} : n \in \mathbb{N}\}$ on the set $\mathbb{N}$ of natural numbers, then $\tau^*(\mathcal{I}_f)$ is the discrete topology.

(2) If τ is the indiscrete topology on a set X, then $\tau^*(\mathcal{I}_f)$ is the cofinite topology on X, and $\tau^*(\mathcal{I}_c)$ is the co-countable topology on X. If for a fixed point $p \in X$, $\mathcal{I}$ denotes the ideal $\{A \subset X : p \notin A\}$, then $\tau^*(\mathcal{I})$ is the particular point topology on X.

(3) For any topological space (X,τ), $\tau^*(\mathcal{I}_n)$ is the τ^α topology of Njåstad [10].

(4) If τ is the usual topology on the real line $\mathbb{R}$ and $\mathcal{I}$ is the ideal of all subsets of Lebesgue measure zero, then τ^*-Borel sets are precisely the Lebesgue measurable sets of $\mathbb{R}$.

2. Quasi-H-closed modulo an ideal space

The concept of compactness modulo an ideal was introduced by Newcomb [9] and has been studied among others by Rancin [11], and Hamlett and Janković [3]. A space (X,τ) is defined to be *compact modulo an ideal* $\mathcal{I}$ on X or just $(\mathcal{I})$ compact space if for every open cover $\mathcal{U}$ of X, there is a finite subfamily $\{U_1,U_2,\dots,U_n\}$ such that $X - \bigcup_{i=1}^n U_i \in \mathcal{I}$. In this section, we define *quasi-H-closedness* modulo an ideal and study some of its properties. In the process, we get some interesting characterizations of *quasi-H-closed* spaces.

Definition 2.1. Let (X,τ) be a topological space and $\mathcal{I}$ an ideal on X. X is *quasi-H-closed modulo* $\mathcal{I}$ or just $(\mathcal{I})$QHC if for every open cover $\mathcal{U}$ of X, there is a finite subfamily $\{U_1,U_2,\dots,U_n\}$ of $\mathcal{U}$ such that $X - \bigcup_{i=1}^n \mathrm{cl}(U_i) \in \mathcal{I}$. Such a subfamily is said to be *proximate* subcover modulo $\mathcal{I}$ or just $(\mathcal{I})$ proximate subcover.

A subset A of a topological space (X,τ) is said to be *preopen* [8] if $A \subset \mathrm{int}(\mathrm{cl}(A))$. The collection of all preopen sets of a space (X,τ) is denoted by PO(X). An ideal $\mathcal{I}$ of subsets of a topological space (X,τ) is said to be *codense* [1] if the complement of each of its members is dense. Note that an ideal $\mathcal{I}$ is codense if and only if $\mathcal{I} \cap \tau = \{\phi\}$. Codense ideals are called τ-boundary ideals in [9]. An ideal $\mathcal{I}$ of subsets of a topological space (X,τ) is said to be *completely codense* [1] if $\mathcal{I} \cap \mathrm{PO}(X) = \{\phi\}$. Obviously, every completely codense ideal is codense. Note that if $(\mathbb{R},\tau)$ is the set $\mathbb{R}$ of real numbers equipped with the usual topology τ, then $\mathcal{I}_c$ is codense but not completely codense ideal. It is proved in [1] that an ideal $\mathcal{I}$ is completely codense if and only if $\mathcal{I} \subset \mathcal{I}_n$.

From the discussion of Section 1, the proof of the following theorem is immediate.

Theorem 2.2. *For a space (X,τ), the following are equivalent:*

(a) *(X,τ) is quasi-H-closed;*

(b) (X,τ) *is* $(\{\phi\})$ QHC*;*
(c) (X,τ) *is* $(\mathcal{I}_f)$ QHC*;*
(d) (X,τ) *is* $(\mathcal{I}_n)$ QHC*;*
(e) (X,τ) *is* $(\mathcal{I})$ QHC *for every codense ideal* $\mathcal{I}$.

The significance of condition in (e) may be seen by considering the set $\mathbb{R}$ of real numbers equipped with the usual topology τ. If A is a finite subset of $\mathbb{R}$ and $\mathcal{I}$ is the ideal of all subsets of $\mathbb{R}-A$, then $(\mathbb{R},\tau)$ is $(\mathcal{I})$ QHC, but not quasi-H-closed.

A family $\mathcal{F}$ of subsets of X is said to have the *finite-intersection property modulo an ideal* $\mathcal{I}$ *on* X *or just* $(\mathcal{I})$ FIP if the intersection of no finite subfamily of $\mathcal{F}$ is a member of $\mathcal{I}$. Recall that a subset in a space is called *regular open* if it is the interior of its own closure. The complement of a regular open set is called *regular closed*. It is proved in [12] that for completely codense ideal $\mathcal{I}$ on a space (X,τ), the collections of regular open sets of (X,τ) and (X,τ^*) are same. The following theorem contains a number of characterizations of $(\mathcal{I})$ QHC spaces. Since the proof is similar to that of a theorem in the next section, we omit it.

Theorem 2.3. *For a space* (X,τ) *and an ideal* $\mathcal{I}$ *on* X, *the following are equivalent:*
(a) (X,τ) *is* $(\mathcal{I})$ QHC*;*
(b) *for each family* $\mathcal{F}$ *of closed sets having empty intersection, there is a finite subfamily* $\{F_1,F_2,F_3,\dots,F_n\}$ *such that* $\bigcap_{i=1}^{n}\operatorname{int}(F_i)\in\mathcal{I}$*;*
(c) *for each family* $\mathcal{F}$ *of closed sets such that* $\{\operatorname{int}(F):F\in\mathcal{F}\}$ *has* $(\mathcal{I})$ FIP, *one has* $\cap\{F: F\in\mathcal{F}\}\neq\phi$*;*
(d) *every regular open cover has a finite* $(\mathcal{I})$ *proximate subcover;*
(e) *for each family* $\mathcal{F}$ *of nonempty regular closed sets having empty intersection, there is a finite subfamily* $\{F_1,F_2,F_3,\dots,F_n\}$ *such that* $\bigcap_{i=1}^{n}\operatorname{int}(F_i)\in\mathcal{I}$*;*
(f) *for each collection* $\mathcal{F}$ *of nonempty regular closed sets such that* $\{\operatorname{int}(F):F\in\mathcal{F}\}$ *has* $(\mathcal{I})$ FIP, *one has* $\bigcap\{F:F\in\mathcal{F}\}\neq\phi$*;*
(g) *for each open filter base* $\mathcal{B}$ *on* $P(X)-\mathcal{I}$, $\bigcap\{\operatorname{cl}(B):B\in\mathcal{B}\}\neq\phi$*;*
(h) *every open ultrafilter on* $P(X)-\mathcal{I}$ *converges.*

It follows from a result in [13] that τ and $\tau^*(\mathcal{I})$ have the same regular open sets, where $\mathcal{I}$ is a completely codense ideal on (X,τ). In particular, if $U\in\tau^*$, then $\operatorname{cl}(U)=\operatorname{cl}^*(U)$. Using this observation along with the previous theorem, we have the following.

Theorem 2.4. *Let* $\mathcal{I}$ *be a completely codense ideal on a space* (X,τ). *Then* (X,τ) *is* $(\mathcal{I})$ QHC *if and only if* (X,τ^*) *is* $(\mathcal{I})$ QHC.

Combining this result with Theorem 2.2, we have the following.

Corollary 2.5. *Let* (X,τ) *be a space and* $\mathcal{I}$ *a completely codense ideal on* X. *Then the following are equivalent:*
(a) (X,τ) *is quasi-*H*-closed;*
(b) (X,τ^*) *is quasi-*H*-closed;*
(c) (X,τ^α) *is quasi-*H*-closed.*

The last equivalence follows because $\tau^\alpha=\tau^*(\mathcal{I}_n)$, where $\mathcal{I}_n$ is the ideal of nowhere dense sets in X.

3. C-compact modulo an ideal space

In this section, we generalize the concepts of C-compactness of Viglino [16] and compactness modulo an ideal due to Newcomb [9] and Rancin [11]. A space (X,τ) is said to be C-compact if for each closed set A and each τ-open covering $\mathcal{U}$ of A, there exists a finite subfamily $\{U_1,U_2,U_3,\ldots,U_n\}$ such that $A \subset \bigcup_{i=1}^{n} \text{cl}(U_i)$.

Definition 3.1. Let (X,τ) be a topological space and $\mathcal{I}$ an ideal on X. (X,τ) is said to be *C-compact modulo $\mathcal{I}$* or just $C(\mathcal{I})$-compact if for every closed set A and every τ-open cover $\mathcal{U}$ of A, there is a finite subcollection $\{U_1,U_2,U_3,\ldots,U_n\}$ such that $A - \bigcup_{i=1}^{n} \text{cl}(U_i) \in \mathcal{I}$.

It follows from the definition that

$$\begin{array}{ccc} \text{compact} & \text{———} & (\mathcal{I})\ \text{compact} \\ \downarrow & & \downarrow \\ C\text{-compact} & \text{———} & C(\mathcal{I})\text{-compact} \\ \downarrow & & \downarrow \\ \text{quasi-}H\text{-closed} & \text{———} & (\mathcal{I})\ \text{QHC} \end{array} \tag{3.1}$$

Also from the definition in Section 1, we have the following.

Theorem 3.2. *For a space (X,τ), the following are equivalent:*
(a) (X,τ) *is C-compact;*
(b) (X,τ) *is $C(\{\phi\})$-compact;*
(c) (X,τ) *is $C(\mathcal{I}_f)$-compact.*

Example 3.3. For n and m in the set N of positive integers, let Y denote the subset of the plane consisting of all points of the form $(1/n,1/m)$ and the points of the form $(1/n,0)$. Let $X = Y \cup \{\infty\}$. Topologize X as follows: let each point of the form $(1/n,1/m)$ be open. Partition N into infinitely many infinite-equivalence classes, $\{Z_i\}_{i=1}^{\infty}$. Let a neighborhood system for the point $(1/i,0)$ be composed of all sets of the form $G \cup F$, where

$$\begin{aligned} G &= \left\{\left(\frac{1}{i},0\right)\right\} \cup \left\{\left(\frac{1}{i},\frac{1}{m}\right) : m \geq k\right\}, \\ F &= \left\{\left(\frac{1}{n},\frac{1}{m}\right) : m \in Z_i,\ n \geq k\right\} \end{aligned} \tag{3.2}$$

for some $k \in N$. Let a neighborhood system for the point ∞ be composed of sets of the form $X \backslash T$, where

$$T = \left\{\left(\frac{1}{n},0\right) : n \in N\right\} \cup \bigcup_{i=1}^{k} \left\{\left(\frac{1}{i},\frac{1}{m}\right) : m \in N\right\} \cup \left\{\left(\frac{1}{n},\frac{1}{m}\right) : m \in Z_i,\ n \in N\right\} \tag{3.3}$$

for some $k \in N$. It is shown in [16] that X is a C-compact space which is not compact. In view of Theorem 3.2, such a space is $C(\mathcal{I}_f)$-compact, but not $(\mathcal{I}_f)$ compact.

Example 3.4. Let $X = R^+ \cup \{a\} \cup \{b\}$, where R^+ denotes the set of nonnegative real numbers and a, b are two distinct points not in R^+. Let $W(a) = \{V \subset X : V = \{a\} \cup \bigcup_{r=m}^{\infty}(2r, 2r+1)\}$, where m is a nonnegative integer, be a neighborhood system for the point a. Let $W(b) = \{V \subset X : V = \{b\} \cup \bigcup_{r=m}^{\infty}(2r-1, 2r)\}$, where m is a nonnegative integer, be a neighborhood system for the point b. Let R^+, with the usual topology, be imbedded in X. Viglino [16] has shown that the space X is not C-compact. If A is a finite subset of X, then (X,τ) is $C(\mathcal{I})$-compact, where $\mathcal{I}$ is the ideal of all subsets of $X - A$.

In view of Examples 3.3 and 3.4, it is clear that the implications shown after Definition 3.1 are, in general, irreversible.

It is proved in [3] that if (X,τ) is quasi-H-closed and $\mathcal{I}$ is an ideal such that $\mathcal{I}_n \subset \mathcal{I}$, then (X,τ) is $(\mathcal{I})$ compact (and hence $C(\mathcal{I})$-compact).

Next, if $\{U_1, U_2, \dots, U_n\}$ is a finite collection of open subsets such that $X - \bigcup_{i=1}^{n} \mathrm{cl}(U_i) \in \mathcal{I}_n$, then $X - \bigcup_{i=1}^{n} \mathrm{cl}(U_i) = \phi$ because $\tau \cap \mathcal{I}_n = \{\phi\}$. But then $\mathrm{int}(\mathrm{cl}(X - \bigcup_{i=1}^{n} U_i)) = X - \bigcup_{i=1}^{n} \mathrm{cl}(U_i) = \phi$ implies that $X - \bigcup_{i=1}^{n} U_i \in \mathcal{I}_n$. Therefore, a space (X,τ) is $(\mathcal{I}_n)$ compact if and only if it is $C(\mathcal{I}_n)$-compact. In view of this discussion, we have the following.

Theorem 3.5. *For a space (X,τ), the following are equivalent:*
(a) (X,τ) *is quasi-H-closed;*
(b) (X,τ) *is* $(\mathcal{I}_n)$ QHC*;*
(c) (X,τ) *is $C(\mathcal{I}_n)$-compact;*
(d) (X,τ) *is $(\mathcal{I}_n)$ compact.*

A space (X,τ) is said to be *Baire* if the intersection of every countable family of open sets in (X,τ) is dense. It is noted in [5] that a space (X,τ) is Baire if and only if $\tau \cap \mathcal{I}_m = \{\phi\}$, where $\mathcal{I}_m$ is the ideal of meager (first category) subsets of (X,τ). Thus, in view of the above theorem, a Baire space (X,τ) is $C(\mathcal{I}_m)$-compact if and only if it is quasi-H-closed.

We now give some characterizations of $C(\mathcal{I})$-compact spaces.

Theorem 3.6. *Let (X,τ) be a space and let $\mathcal{I}$ be an ideal on X. Then the following are equivalent:*
(a) (X,τ) *is $C(\mathcal{I})$-compact;*
(b) *for each closed subset A of X and each family $\mathcal{F}$ of closed subsets of X such that $\bigcap\{F \bigcap A : F \in \mathcal{F}\} = \phi$, there exists a finite subfamily $\{F_1, F_2, F_3, \dots, F_n\}$ such that $\bigcap(\mathrm{int}(F_i)) \cap A \in \mathcal{I}$;*
(c) *for each closed set A and each family $\mathcal{F}$ of closed sets such that $\{\mathrm{int}(F) \cap A : F \in \mathcal{F}\}$ has* $(\mathcal{I})$ FIP, *one has $\cap\{F \cap A : F \in \mathcal{F}\} \neq \phi$;*
(d) *for each closed set A and each regular open cover $\mathcal{U}$ of A, there exists a finite subcollection $\{U_1, U_2, U_3, \dots, U_n\}$ such that $A - \bigcup_{i=1}^{n} \mathrm{cl}(U_i) \in \mathcal{I}$;*
(e) *for each closed set A and each family $\mathcal{F}$ of regular closed sets such that $\bigcap\{F \cap A : F \in \mathcal{F}\} = \phi$, there is a finite subfamily $\{F_1, F_2, F_3, \dots, F_n\}$ such that $\bigcap_{i=1}^{n}(\mathrm{int}(F_i)) \cap A \in \mathcal{I}$;*
(f) *for each closed set A and each family $\mathcal{F}$ of regular closed sets such that $\{\mathrm{int}(F) \cap A : F \in \mathcal{F}\}$ has* $(\mathcal{I})$ FIP, *one has $\bigcap\{F \cap A : F \in \mathcal{F}\} \neq \phi$;*
(g) *for each closed set A, each open cover $\mathcal{U}$ of $X - A$ and each open neighborhood V of A, there exists a finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ of $\mathcal{U}$ such that $X - (V \cup (\bigcup_{i=1}^{n} \mathrm{cl}(U_i))) \in \mathcal{I}$;*

(h) *for each closed set A and each open filter base $\mathcal{B}$ on X such that $\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{I}$, one has $\bigcap\{\text{cl}(B) : B \in \mathcal{B}\} \cap A \neq \phi$.*

Proof. (a)⇒(b). Let (X,τ) be $C(\mathcal{I})$-compact, A a closed subset, and $\mathcal{F}$ a family of closed subsets with $\cap\{F \cap A : F \in \mathcal{F}\} = \phi$. Then $\{X - F : F \in \mathcal{F}\}$ is an open cover of A and hence admits a finite subfamily $\{X - F_i : i = 1,2,\dots,n\}$ such that $A - \bigcup_{i=1}^n \text{cl}(X - F_i) \in \mathcal{I}$. This set in $\mathcal{I}$ is easily seen to be $\bigcap_{i=1}^n \{\text{int}(F_i) \cap A\}$.

(b)⇒(c). This is easy to be established.

(c)⇒(a). Let A be a closed subset, let $\mathcal{U}$ be an open cover of A with the property that for no finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$ of $\mathcal{U}$, one has $A - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{I}$. Then $\{X - U : U \in \mathcal{U}\}$ is a family of closed sets. Since

$$\bigcap_{i=1}^{n}\{X - \text{cl}(U_i)\} \cap A = \bigcap_{i=1}^{n}\{A - \text{cl}(U_i)\} = A - \bigcup_{i=1}^{n}\text{cl}(U_i), \tag{3.4}$$

the family $\{\text{int}(X - U) \cap A : U \in \mathcal{U}\}$ has $(\mathcal{I})$FIP. By the hypothesis $\bigcap\{(X - U) \cap A : U \in \mathcal{U}\} \neq \phi$. But then $A - \cup\{U : U \in \mathcal{U}\} \neq \phi$, that is, $\mathcal{U}$ is not a cover of A, a contradiction.

(d)⇒(a). Let A be a closed subset of X and $\mathcal{U}$ an open cover of A. Then $\{\text{int}(\text{cl}(U)) : U \in \mathcal{U}\}$ is a regular open cover of A. Let $\{\text{int}(\text{cl}(U_i)) : i = 1,2,\dots,n\}$ be a finite subfamily such that $A - \bigcup_{i=1}^n \text{cl}(\text{int}(\text{cl}(U_i))) \in \mathcal{I}$. Since U_i is open and for each open set U, $\text{cl}(\text{int}(\text{cl}(U))) = \text{cl}(U)$, we have $A - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{I}$, which shows that X is $C(\mathcal{I})$-compact.

(a)⇒(d). This is obvious.

The proofs for (d)⇒(e)⇒(f)⇒(d) are parallel to (a)⇒(b)⇒(c)⇒(a), respectively.

(a)⇒(g). Let A be a closed set, V an open neighborhood of A, and $\mathcal{U}$ an open cover of $X - A$. Since $X - V \subset X - A$, $\mathcal{U}$ is also an open cover of the closed set $X - V$.

Let $\{U_1, U_2, U_3, \dots, U_n\}$ be a finite subcollection of $\mathcal{U}$ such that $(X - V) - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{I}$. However, the last set is $X - (V \cup \{\bigcup_{i=1}^n \text{cl}(U_i)\})$.

(g)⇒(a). Let A be a closed subset of X and $\mathcal{U}$ an open covering of A. If H denotes the union of members of $\mathcal{U}$, then $F = X - H$ is a closed set and $X - A$ is an open neighborhood of F. Also $\mathcal{U}$ is an open cover of $X - F$. By hypothesis, there is a finite subcollection $\{U_1, U_2, U_3, \dots, U_n\}$ of $\mathcal{U}$ such that

$$X\left((X - A) \cup \left\{\bigcup_{i=1}^{n}\text{cl}(U_i)\right\}\right) \in \mathcal{I}. \tag{3.5}$$

However, this set in $\mathcal{I}$ is nothing but $A - \bigcup_{i=1}^n \text{cl}(U_i)$.

(a)⇒(h). Suppose A is a closed set and $\mathcal{B}$ is any open filter base on X with $\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{I}$. Suppose, if possible, $\bigcap\{\text{cl}(B) : B \in \mathcal{B}\} \cap A = \phi$. Then $\{X - \text{cl}(B) : B \in \mathcal{B}\}$ is an open cover of A. By the hypothesis, there exists a finite subfamily $\{X - \text{cl}(B_i) : i = 1,2,3,\dots,n\}$ such that $A - \bigcup_{i=1}^n \text{cl}(X - \text{cl}(B_i))$ is in $\mathcal{I}$. However, this set is $A \cap (\bigcap_{i=1}^n \text{int}(\text{cl}(B_i)))$ and $A \cap (\bigcap_{i=1}^n B_i)$ is a subset of it. Therefore, $A \cap (\bigcap_{i=1}^n B_i) \in \mathcal{I}$. Since $\mathcal{B}$ is a filter base, we have a $B \in \mathcal{B}$ such that $B \subset \bigcap_{i=1}^n B_i$. But then $A \cap B \in \mathcal{I}$ which contradicts the fact that $\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{I}$.

(h)⇒(a). Suppose that (X,τ) is not $C(\mathcal{I})$-compact. Then there exist a closed subset A of X and an open cover $\mathcal{U}$ of A such that for any finite subfamily $\{U_1, U_2, U_3, \dots, U_n\}$

of $\mathcal{U}$, we have $A - \bigcup_{i=1}^{n} \mathrm{cl}(U_i) \notin \mathcal{I}$. We may assume that $\mathcal{U}$ is closed under finite unions. Then the family $\mathcal{B} = \{X - \mathrm{cl}(U) : U \in \mathcal{U}\}$ is an open filter base on X such that $\{B \cap A : B \in \mathcal{B}\} \subset P(A) - \mathcal{I}$. So, by the hypothesis, $\bigcap\{\mathrm{cl}(X - \mathrm{cl}(U)) : U \in \mathcal{U}\} \cap A \neq \phi$. Let x be a point in the intersection. Then $x \in A$ and $x \in \mathrm{cl}(X - \mathrm{cl}(U)) = X - \mathrm{int}(\mathrm{cl}(U)) \subset X - U$ for each $U \in \mathcal{U}$. But this contradicts the fact that $\mathcal{U}$ is a cover of A. Hence (X,τ) is $C(\mathcal{I})$-compact. □

Next we characterize $C(\mathcal{I})$-compact spaces using some weaker forms of filter base convergence.

Definition 3.7. A filter base $\mathcal{B}$ is said to be $(\mathcal{I})$ *adherent convergent* if for every neighborhood G of the adherent set of $\mathcal{B}$, there exists an element $B \in \mathcal{B}$ such that $(X - G) \cap B \in \mathcal{I}$. Clearly, every adherent convergent filter base is $(\mathcal{I})$ adherent convergent and a filter base is adherent convergent if and only if it is $(\{\phi\})$ adherent convergent.

Theorem 3.8. *A space (X,τ) is $C(\mathcal{I})$-compact if and only if every open filter base on $P(X) - \mathcal{I}$ is $(\mathcal{I})$ adherent convergent.*

Proof. Let (X,τ) be $C(\mathcal{I})$-compact and let $\mathcal{B}$ be an open filter base on $P(X) - \mathcal{I}$ with A as its adherent set. Let G be an open neighborhood of A. Then $A = \bigcap\{\mathrm{cl}(B) : B \in \mathcal{B}\}$, $A \subset G$, and $X - G$ is closed. Now $\{X - \mathrm{cl}(B) : B \in \mathcal{B}\}$ is an open cover of $X - G$ and so by the hypothesis, it admits a finite subfamily $\{X - \mathrm{cl}(B_i) : i = 1,2,3,\ldots,n\}$ such that $(X - G) - \bigcup_{i=1}^{n} \mathrm{cl}(X - \mathrm{cl}(B_i)) \in \mathcal{I}$. But this implies $(X - G) \cap (\bigcap_{i=1}^{n} \mathrm{int}(\mathrm{cl}(B_i))) \in \mathcal{I}$. However, $B_i \subset \mathrm{int}(\mathrm{cl}(B_i))$ implies $(X - G) \cap (\bigcap_{i=1}^{n} B_i) \in \mathcal{I}$. Since $\mathcal{B}$ is a filter base and $B_i \in \mathcal{B}$, there is a $B \in \mathcal{B}$ such that $B \subset \bigcap_{i=1}^{n} B_i$. But then $(X - G) \cap B \in \mathcal{I}$ is required.

Conversely, let (X,τ) be not $C(\mathcal{I})$-compact, and let A be a closed set, and $\mathcal{U}$ an open cover of A such that for no finite subfamily $\{U_1, U_2, U_3, \ldots, U_n\}$ of $\mathcal{U}$, one has $A - \bigcup_{i=1}^{n} \mathrm{cl}(U_i) \in \mathcal{I}$. Without loss of generality, we may assume that $\mathcal{U}$ is closed for finite unions. Therefore, $\mathcal{B} = \{X - \mathrm{cl}(U) : U \in \mathcal{U}\}$ becomes an open filter base on $P(X) - \mathcal{I}$. If x is an adherent point of $\mathcal{B}$, that is, if $x \in \bigcap\{\mathrm{cl}(X - \mathrm{cl}(U)) : U \in \mathcal{U}\} = X - \bigcup\{\mathrm{int}(\mathrm{cl}(U)) : U \in \mathcal{U}\}$, then $x \notin A$, because $\mathcal{U}$ is an open cover of A and for $U \in \mathcal{U}$, $U \subset \mathrm{int}(\mathrm{cl}(U))$. Therefore, the adherent set of $\mathcal{B}$ is contained in $X - A$, which is an open set. By the hypothesis, there exists an element $B \in \mathcal{B}$ such that $(X - (X - A)) \cap B \in \mathcal{I}$, that is, $A \cap B \in \mathcal{I}$, that is, $A \cap (X - \mathrm{cl}(U)) \in \mathcal{I}$, that is, $A - \mathrm{cl}(U) \in \mathcal{I}$ for some $U \in \mathcal{U}$. This however contradicts our assumption. This completes the proof. □

Herrington and Long [4] characterized C-compact spaces using r-convergence of filters and nets. We obtain similar results for $C(\mathcal{I})$-compact spaces in the next definition.

Definition 3.9. Let X be a space, $\phi \neq A \subset X$, and let $\mathcal{B}$ be a filter base on A. $\mathcal{B}$ is said to r-converge to $a \in A$ if for each open set V in X containing a, there is $B \in \mathcal{B}$ with $B \subset \mathrm{cl}(V)$. The filter base $\mathcal{B}$ is said to r-accumulate to a, if for each open set V containing a, $\mathrm{cl}(V) \cap B \neq \phi$ for each $B \in \mathcal{B}$.

Similarly, a net $\varphi : D \to A \subset X$ is said to r-converge to $a \in A$ if for each open set V containing a, there is a $b \in D$ such that $\varphi(c) \in \mathrm{cl}(V)$ for all $c \geq b$. φ is said to r-accumulate to a if for each open set V containing a and each $b \in D$, there is $c \in D$ with $c \geq b$ and $\varphi(c) \in \mathrm{cl}(V)$.

It is known [4] that convergence (accumulation) for filter bases and nets implies r-convergence (r-accumulation), but the converse is not true.

Theorem 3.10. *For a space (X,τ) and an ideal $\mathcal{I}$ on X, the following are equivalent:*
(a) *(X,τ) is $C(\mathcal{I})$-compact;*
(b) *for each closed set A, each filter base $\mathcal{B}$ on $P(A)-\mathcal{I}$ r-accumulates to some $a\in A$;*
(c) *for each closed set A, each maximal filter base $\mathcal{M}$ on $P(A)-\mathcal{I}$ r-converges to some $a\in A$;*
(d) *for each closed set A, each net φ on $P(A)-\mathcal{I}$ r-accumulates to some $a\in A$.*

Proof. (a)⇒(b). Suppose there exist a closed set A and a filter base $\mathcal{B}$ on $P(A)-\mathcal{I}$ which does not r-accumulate to any $a\in A$. Then for each $a\in A$, there exists an open set $U(a)$ containing a and a $B(a)\in\mathcal{B}$ such that $B(a)\cap \operatorname{cl}(U(a))=\phi$. Then $\{U(a):a\in A\}$ is an open cover of the closed set A. By (a), there exists a finite subcollection $\{U(a_i):i=1,2,3,\dots,n\}$ such that $A-\bigcup_{i=1}^{n}\operatorname{cl}(U(a_i))\in\mathcal{I}$. If $B\in\mathcal{B}$ is such that $B\subset\bigcap_{i=1}^{n}B(a_i)$, then $B\cap(A-\bigcup_{i=1}^{n}\operatorname{cl}(U(a_i)))\in\mathcal{I}$, that is, $B-\bigcup_{i=1}^{n}\operatorname{cl}(U(a_i))\in\mathcal{I}$. But the later set is just B, because $B\subset B(a_i)$ and $B(a_i)\cap\operatorname{cl}(U(a_i))=\phi$ for each i. However, $B\in\mathcal{I}$ is a contradiction, because $B\in\mathcal{B}$ and $\mathcal{B}\subset P(A)-\mathcal{I}$.

(b)⇔(c). This follows in view of parts (a), (b), and (c) of [4, Theorem 1].

(b)⇒(a). If possible, let X be not $C(\mathcal{I})$-compact. Then by Theorem 3.6(f), there exist a closed set A and a collection $\mathcal{F}$ of regular closed sets with the property that for every finite subcollection $\{F_1,F_2,F_3,\dots,F_n\}$, $\bigcap_{i=1}^{n}\operatorname{int}(F_i)\cap A\notin\mathcal{I}$, but $\bigcap\{F:F\in\mathcal{F}\}\cap A=\phi$. Now the collection of sets of the form $\bigcap_{i=1}^{n}\operatorname{int}(F_i)\cap A$ for all possible finite subfamilies $\{F_1,F_2,F_3,\dots,F_n\}$ of $\mathcal{F}$ forms a filter base on $P(A)-\mathcal{I}$. By (b), this filter base r-accumulates to some $a\in A$, that is, for each open set $U(a)$ containing a and for each $F\in\mathcal{F}$, $\operatorname{cl}(U(a))\cap(\operatorname{int}(F)\cap A)\neq\phi$. However, $a\in A$ and $A\cap\{F:F\in\mathcal{F}\}=\phi$ imply that there is some $F=F(a)\in\mathcal{F}$ such that $a\notin F(a)$. Then $X-F(a)$ is an open set containing a such that $\operatorname{cl}(X-F(a))\cap(\operatorname{int}(F(a))\cap A)=\phi$. This is a contradiction.

(b)⇔(d). This follows using standard arguments about nets and filters. □

If in the above theorem, A is replaced by the whole space X, we get the characterizations of $(\mathcal{I})$QHC spaces. If in addition we consider completely codense ideal $\mathcal{I}$, we get the characterizations of quasi-H-closed spaces.

4. $C(\mathcal{I})$-compact spaces and functions

A function $f:(X,\tau)-(Y,\varsigma)$ is said to be θ-*continuous* [2] at a point $x\in X$ if for every open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(\operatorname{cl}(U))\subseteq\operatorname{cl}(V)$. A function $f:(X,\tau)-(Y,\varsigma)$ is said to be θ-*continuous* if f is θ-*continuous* for every $x\in X$. The concept of θ-continuity is weaker than that of continuity. An important property of C-compact spaces is that a continuous function from a C-compact space to a Hausdorff space is closed. We prove the following more general results.

Theorem 4.1. *Let $f:(X,\tau,\mathcal{I})-(Y,\varsigma,\vartheta)$ be a θ-continuous function, $(X,\tau,\mathcal{I})$ $C(\mathcal{I})$-compact, (Y,ς) Hausdorff, and $f(\mathcal{I})\subseteq\vartheta$. Then $f(A)$ is $\varsigma^*(\vartheta)$-closed for each closed set A of X.*

Proof. Let A be any closed set in X and $a \notin f(A)$. For each $x \in A$, there exists a ς-open set V_y containing $y = f(x)$ such that $a \notin \mathrm{cl}(V_y)$. Now because f is θ-continuous, there exists an open set U_x containing x such that $f(\mathrm{cl}(U_x)) \subseteq \mathrm{cl}(V_y)$. The family $\{U_x : x \in A\}$ is an open cover of A. Therefore, there exists a finite subfamily $\{U_{x_i} : i = 1,2,\dots,n\}$ such that $A - \bigcup_{i=1}^n \mathrm{cl}(U_{x_i}) \in \mathcal{I}$. But then $f(A - \bigcup_{i=1}^n \mathrm{cl}(U_{x_i})) \in f(\mathcal{I}) \subseteq \vartheta$, that is, $f(A) - f(\bigcup_{i=1}^n \mathrm{cl}(U_{x_i})) \in f(\mathcal{I}) \subseteq \vartheta$ because $f(\mathcal{I})$ is also an ideal. Hence $f(A) - (\bigcup_{i=1}^n \mathrm{cl}(V_{y_i})) \in f(\mathcal{I}) \subseteq \vartheta$. Now $a \notin \mathrm{cl}(V_{y_i})$ for any i implies that $a \in Y - \bigcup_{i=1}^n \mathrm{cl}(V_{y_i})$ which is open in (Y,ς) and $(Y - \bigcup_{i=1}^n \mathrm{cl}(V_{y_i})) \cap f(A) = f(A) - \bigcup_{i=1}^n \mathrm{cl}(V_{y_i}) \in f(\mathcal{I}) \subseteq \vartheta$. Hence $a \notin (f(A))^*(\sigma,\vartheta)$. Thus $(f(A))^*(\sigma,\vartheta) \subset f(A)$ and so $f(A)$ is $\varsigma^*(\vartheta)$-closed. □

Corollary 4.2. *Let $f : (X,\tau,\mathcal{I}) - (Y,\varsigma,\vartheta)$ be a continuous function, $(X,\tau,\mathcal{I})$ $C(\mathcal{I})$-compact, (Y,ς) Hausdorff, and $f(\mathcal{I}) \subseteq \vartheta$. Then $f(A)$ is $\varsigma^*(\vartheta)$-closed for each closed set A of X.*

Theorem 4.3. *Let $f : (X,\tau,\mathcal{I}) - (Y,\varsigma,\vartheta)$ be a continuous surjection, $(X,\tau,\mathcal{I})$ $C(\mathcal{I})$-compact, and $f(\mathcal{I}) \subseteq \vartheta$. Then (Y,ς,ϑ) is $C(\vartheta)$-compact.*

Proof. Let A be any closed subset of (Y,ς) and $\{V_\alpha : \alpha \in \Lambda\}$ any open cover of A by open sets in Y. Then $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an open cover of $f^{-1}(A)$ which is closed in X. Hence, by the hypothesis, there exists a finite subcollection $\{f^{-1}(V_{\alpha_i}) : i = 1,2,\dots,n\}$ such that $f^{-1}(A) - \bigcup_{i=1}^n \mathrm{cl}(f^{-1}(V_{\alpha_i})) \in \mathcal{I}$. Since f is continuous, $\mathrm{cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{cl}(B))$ for every subset B of Y. Hence we have $f^{-1}(A) - \bigcup_{i=1}^n f^{-1}(\mathrm{cl}(V_{\alpha_i})) = f^{-1}(A - \bigcup_{i=1}^n \mathrm{cl}(V_{\alpha_i})) \in \mathcal{I}$. Since f is surjective, $A - \bigcup_{i=1}^n \mathrm{cl}(V_{\alpha_i}) \in f(\mathcal{I}) \subset \vartheta$. Hence Y is $C(\vartheta)$-compact. □

Theorem 4.4. *If the product space ΠX_α of nonempty family of topological spaces (X_α,τ_α) is $C(\mathcal{I})$-compact, then each (X_α,τ_α) is $C(p_\alpha(\mathcal{I}))$-compact, where p_α is the projection map and $\mathcal{I}$ is an ideal on ΠX_α.*

Proof. This follows from Theorem 4.3. □

5. $C(\mathcal{I})$-compact spaces and subspaces

In this section, we introduce three types of $C(\mathcal{I})$-compact subsets and use them to obtain new characterizations of $C(\mathcal{I})$-compact spaces and a characterization of maximal $C(\mathcal{I})$-compact spaces.

Definition 5.1. Let (X,τ) be a space and $\mathcal{I}$ an ideal on X. A subset Y of X is said to be $C(\mathcal{I})$-compact if the subspace (Y,τ_Y) is $C(\mathcal{I})$-compact.

Some useful results about such subspaces are contained in the following theorem. The proofs are easy to establish.

Theorem 5.2. *Let (X,τ) be a space and $\mathcal{I}$ an ideal on X. Then*

(a) *a subspace Y is $C(\mathcal{I})$-compact if and only if it is $C(\mathcal{I}_Y)$-compact;*
(b) *a clopen subspace of a $C(\mathcal{I})$-compact space is $C(\mathcal{I})$-compact;*
(c) *if Y is a regular closed subset of a $C(\mathcal{I})$-compact space $(X,\tau,\mathcal{I})$ and $\mathcal{I}$ is codense, then (Y,τ_Y) is quasi-H-closed;*
(d) *a finite union of $C(\mathcal{I})$-compact subspaces of X is $C(\mathcal{I})$-compact.*

Definition 5.3. A subset Y of (X,τ) is said to be *$C(\mathcal{I})$-compact relative to τ* if every τ-open cover of every relatively closed subset A of Y has a finite subfamily whose τ-closures cover A except a set in $\mathcal{I}$.

Some useful properties of such spaces are contained in the following.

Theorem 5.4. *Let (X,τ) be a space and $\mathcal{I}$ an ideal on X. Then the following hold.*

(a) *A closed subspace of a $C(\mathcal{I})$-compact relative to τ subspace of (X,τ) is $C(\mathcal{I})$-compact relative to τ.*

(b) *If (X,τ) is Hausdorff and Y is $C(\mathcal{I})$-compact relative to τ, then Y is $\tau^*(\mathcal{I})$-closed.*

(c) *If Y is a $C(\mathcal{I})$-compact relative to τ subspace of (X,τ) and $f:(X,\tau)-(Z,\varsigma)$ is a continuous bijection, then $f(Y)$ is $C(f(\mathcal{I}))$-compact relative to ς.*

(d) *$C(\mathcal{I})$-compactness relative to τ is contractive.*

The following characterization of $C(\mathcal{I})$-compact spaces is obtained using $C(\mathcal{I})$-compact relative to τ subspaces. The proof is easy.

Theorem 5.5. *A space (X,τ) with an ideal $\mathcal{I}$ is $C(\mathcal{I})$-compact if and only if every proper closed subset of X is $C(\mathcal{I})$-compact relative to τ.*

Definition 5.6. A subset Y of a space (X,τ) is said to be *closure $C(\mathcal{I})$-compact* if for every τ_Y-closed subset K of Y and every τ-open cover $\mathcal{U}$ of $\mathrm{cl}(K)$, there is a finite subcollection $\{U_1,U_2,U_3,\ldots,U_n\}$ of $\mathcal{U}$ such that $K-\bigcup_{i=1}^{n}\mathrm{cl}_Y(U_i\cap Y)\in\mathcal{I}$.

It is easy to see that closure $C(\mathcal{I})$-compactness is contractive.

Example 5.7. Since closed subsets of $C(\mathcal{I})$-compact spaces are not necessarily $(\mathcal{I})$QHC, a space (X,τ) which is $C(\mathcal{I})$-compact relative to τ may fail to be closure $C(\mathcal{I})$-compact. Moreover, $]0,1]$ as a subspace of $[0,1]$ is closure $C(\mathcal{I})$-compact with $\mathcal{I}=\{\phi\}$, but not $C(\mathcal{I})$-compact relative to the usual topology. Thus the concepts of $C(\mathcal{I})$-compact relative to τ and closure $C(\mathcal{I})$-compact are independent concepts.

We now have the following characterization of $C(\mathcal{I})$-compact spaces.

Theorem 5.8. *A space (X,τ) is $C(\mathcal{I})$-compact for an ideal $\mathcal{I}$ on X if and only if every open subset of X is closure $C(\mathcal{I})$-compact.*

Proof. Let (X,τ) be $C(\mathcal{I})$-compact and Y an open subset of X. Let K be a τ_Y-closed subset of Y, and let $\mathcal{U}$ be a τ-open cover of $\mathrm{cl}(K)$. Then there exists a finite subcollection $\{U_1,U_2,U_3,\ldots,U_n\}$ of $\mathcal{U}$ such that $\mathrm{cl}\,K-\bigcup_{i=1}^{n}\mathrm{cl}(U_i)\in\mathcal{I}$. Since Y is open, therefore, $\mathrm{cl}_Y(U\cap Y)=\mathrm{cl}(U)\cap Y$ and so, by hereditary property of $\mathcal{I}$, $K-\bigcup_{i=1}^{n}\mathrm{cl}_Y(U_i\cap Y)\in\mathcal{I}$. Thus Y is closure $C(\mathcal{I})$-compact.

Conversely, let all open subsets of X be closure $C(\mathcal{I})$-compact. Let K be a closed and $\mathcal{U}$ an open cover of K. Choose a $U_0\in\mathcal{U}$. Then $Y=X-\mathrm{cl}(U_0)$ is an open subset of X and $K\cap Y$ is a τ_Y-closed subset of Y. Moreover, $\mathcal{U}-\{U_0\}$ is an open cover of $\mathrm{cl}(K\cap Y)$. By the hypothesis, there exists a finite subcollection $\{U_1,U_2,U_3,\ldots,U_n\}$ of $\mathcal{U}-\{U_0\}$ such that $K\cap Y-\bigcup_{i=1}^{n}\mathrm{cl}_Y(U_i\cap Y)\in\mathcal{I}$. But then $K\cap Y-\bigcup_{i=1}^{n}\mathrm{cl}(U_i)\in\mathcal{I}$ as $\mathrm{cl}_Y(U_i\cap Y)=\mathrm{cl}(U_i)\cap Y$ and $\mathcal{I}$ is hereditary. Therefore, $K-\bigcup_{i=0}^{n}\mathrm{cl}(U_i)\in\mathcal{I}$. Hence (X,τ) is $C(\mathcal{I})$-compact.

Finally, we obtain a characterization of a maximal $C(\mathcal{I})$-compact space. Recall that a space (X,τ) with property P is said to be *maximal P* if there is no topology σ on X which has property P and is strictly finer than τ. For a topological space (X,τ) and a subset A of X, $\tau(A)=\{U\cup(V\cap A): U,V\in\tau\}$ is a topology called *simple extension* [7] of τ by A. $\tau(A)$ is strictly finer than τ if and only if $A\notin\tau$. □

Theorem 5.9. *A topological space (X,τ) is maximal $C(\mathcal{I})$-compact if and only if for every subset A of X such that A is closure $C(\mathcal{I})$-compact and $X-A$ is $C(\mathcal{I})$-compact relative to τ, one has $A\in\tau$.*

Proof. First we assume that (X,τ) is maximal $C(\mathcal{I})$-compact and that A is a subset of X satisfying the given conditions. First, we show that $(X,\tau(A))$ is $C(\mathcal{I})$-compact. Let K be a $\tau(A)$-closed subset of X. Then $K=K_1\cup(K_2\cap(X-A))$, where K_1 and K_2 are τ-closed sets. Let

$$\mathcal{U}=\{U_\alpha\cup(V_\alpha\cap A): U_\alpha,V_\alpha\in\tau,\ \alpha\in\Delta\} \tag{5.1}$$

be a $\tau(A)$-open cover of K. Then $\nu=\{U_\alpha:\alpha\in\Delta\}$ is a τ-open cover of $K\cap(X-A)=(K_1\cup K_2)\cap(X-A)$. Since, by assumption, $X-A$ is $C(\mathcal{I})$-compact relative to τ, we have a finite subcollection $\{U_{\alpha_1},U_{\alpha_2},U_{\alpha_3},\dots,U_{\alpha_n}\}$ of ν such that $K\cap(X-A)-\bigcup_{i=1}^n \mathrm{cl}(U_{\alpha_i})\in\mathcal{I}$. Since $\tau(A)$ is finer than τ, this subcollection is $\tau(A)$-open and $K\cap(X-A)-\bigcup_{i=1}^n \mathrm{cl}_{\tau(A)}(U_{\alpha_i})\in\mathcal{I}$. Next, $\mathcal{W}=\{U_\alpha\cup V_\alpha:\alpha\in\Delta\}$ is a τ-open cover of $\mathrm{cl}(K\cap A)=\mathrm{cl}(K_1\cap A)=\mathrm{cl}_{\tau(A)}(K_1\cap A)$ and therefore by assumption on A, there exists a finite subcollection $\{U_{\beta_i}\cup V_{\beta_i}: i=1,2,\dots,k\}$ of $\mathcal{W}$ such that

$$K_1\cap A-\bigcup_{i=1}^{k}\mathrm{cl}_{\tau_A}\left[(U_{\beta_i}\cup V_{\beta_i})\cap A\right]\in\mathcal{I}. \tag{5.2}$$

However, τ_A, the restriction of τ to A, is nothing but $\tau(A)\mid A$, the restriction of $\tau(A)$ to A. Therefore,

$$K_1\cap A-\bigcup_{i=1}^{k}\mathrm{cl}_{\tau(A)|A}\left[(U_{\beta_i}\cup V_{\beta_i})\cap A\right]\in\mathcal{I}. \tag{5.3}$$

Now $\{U_{\alpha_i}\cup(V_{\alpha_i}\cap A): i=1,2,\dots,n\}\cup\{U_{\beta_i}\cup(V_{\beta_i}\cap A): i=1,2,\dots,k\}$ is a finite $\tau(A)$ $(\mathcal{I})$ proximate cover of K which is a subcover of $\mathcal{U}$. Thus the topology $\tau(A)$ on X is also $C(\mathcal{I})$-compact. However, by the maximality of τ, we have $\tau(A)=\tau$. But then $A\in\tau$ as desired.

Conversely, let (X,τ) be not maximal $C(\mathcal{I})$-compact. Then there is a $C(\mathcal{I})$-compact topology σ on X which is strictly finer than τ. Let $A\in\sigma-\tau$. Then A is σ-closure $C(\mathcal{I})$-compact by Theorem 5.8. Since the property of closure $C(\mathcal{I})$-compact is carried over to coarser topologies, A is τ-closure $C(\mathcal{I})$-compact. Also $X-A$ is $C(\mathcal{I})$-compact relative to σ and hence $C(\mathcal{I})$-compact relative to τ. By the hypothesis, then $A\in\tau$, a contradiction. □

Remark 5.10. The readers can generalize the above concepts in bitopological spaces to unify various types of compactness.

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