SCHRÖDINGER EQUATIONS IN NONCYLINDRICAL DOMAINS: EXACT CONTROLLABILITY

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We consider an open bounded set $\Omega \subset \mathbb{R}^n$ and a family $\{K(t)\}_{t\geq 0}$ of orthogonal matrices of $\mathbb{R}^n$. Set $\Omega_t = \{x \in \mathbb{R}^n;\ x = K(t)y,\ \text{for all}\ y \in \Omega\}$, whose boundary is Γ_t. We denote by $\widehat{Q}$ the noncylindrical domain given by $\widehat{Q} = \bigcup_{0<t<T}\{\Omega_t \times \{t\}\}$, with the regular lateral boundary $\widehat{\sum} = \bigcup_{0<t<T}\{\Gamma_t \times \{t\}\}$. In this paper we investigate the boundary exact controllability for the linear Schrödinger equation $u' - i\Delta u = f$ in $\widehat{Q}$ $(i^2 = -1)$, $u = w$ on $\widehat{\Sigma}$, $u(x,0) = u_0(x)$ in Ω_0, where w is the control.

1. Introduction

We consider the linear Schrödinger equation in a domain whose boundary is moving in time. Let T be a positive real number and let $\{\Omega_t\}_{t\in[0,T]}$ be a family of bounded open sets of $\mathbb{R}^n$, with regular boundary Γ_t, defined as below. We denote by $\widehat{Q}$ the noncylindrical domain of $\mathbb{R}^{n+1}$ defined by

$$\widehat{Q} = \bigcup_{0<t<T}\{\Omega_t \times \{t\}\} \tag{1.1}$$

with regular lateral boundary

$$\widehat{\Sigma} = \bigcup_{0<t<T}\{\Gamma_t \times \{t\}\}. \tag{1.2}$$

Let us consider K a function such that for each $t \in [0,\infty)$, it associates an orthogonal matrix $K(t) = (a_{ij}(t))_{n\times n}$. Note that $K^{-1}(t) = (a_{ji}(t))_{n\times n}$. We denote $(a'_{ji}(t))_{n\times n}$ by $(K^{-1})'(t)$.

Let Ω be a bounded open set of $\mathbb{R}^n$, with regular boundary Γ. We consider the subsets Ω_t of $\mathbb{R}^n$ defined by

$$\Omega_t = \{x \in \mathbb{R}^n;\ x = K(t)y,\ y \in \Omega\}, \quad 0 \leq t \leq T. \tag{1.3}$$

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We develop the article under the following assumptions:
(H1) $K \in C^2[0,T]$;
(H2) there exists a constant $\alpha > 0$ such that

$$((K^{-1})'K)w \cdot w \geq \alpha|w|^2 \quad \forall w \in \mathbb{C}^n. \tag{1.4}$$

The aim of this paper is to obtain the exact controllability of the following mixed problem:

$$\left| \begin{array}{l} u' - i\Delta u = \hat{f} \quad \text{in } \hat{Q}, \\ u = \left| \begin{array}{ll} \hat{\varphi} & \text{on } \hat{\Sigma}_0 \\ 0 & \text{on } \hat{\Sigma}\backslash\hat{\Sigma}_0 \end{array} \right. \quad (i^2 = -1), \\ u(x,0) = u_0(x) \quad \text{in } \Omega_0, \end{array} \right. \tag{1.5}$$

where $\hat{\Sigma}_0$ is a part of $\hat{\Sigma}$ with positive measure.

We can formulate the exact controllability problem for (1.5) as follows: given $T > 0$ large enough, we want to find a Hilbert space H such that, for each initial data u_0 belonging to H, there exists a control $\hat{\varphi}$ belonging to the space of controls, defined on $\hat{\Sigma}_0$, such that a solution $u = u(x,t)$ of (1.5) satisfies the final condition

$$u(x,T) = 0 \quad \text{in } \Omega_T. \tag{1.6}$$

The methodology (cf. Lions [7]) consists of transforming (1.5) into an equivalent problem in the cylinder $Q = \Omega \times (0,T)$ by the diffeomorphism

$$\tau : \hat{Q} \longrightarrow Q \tag{1.7}$$

defined by $\tau(x,t) = (y,t)$, with $y = K^{-1}(t)x$. The inverse

$$\tau^{-1} : Q \longrightarrow \hat{Q} \tag{1.8}$$

is defined by $\tau^{-1}(y,t) = (K(t)y,t)$. Then by the change of variables $u(x,t) = v(y,t)$, where $y = K^{-1}(t)x$, $y \in \Omega$, and $x \in \Omega_t$, we obtain

$$\begin{gathered} u'(x,t) = v'(y,t) + \nabla v \cdot (K^{-1}(t))'K(t)y, \\ \Delta u(x,t) = \Delta v(y,t). \end{gathered} \tag{1.9}$$

Therefore, we transform the problem (1.5) in the noncylindrical domain $\hat{Q}$ into the following problem in the cylinder Q:

$$\left| \begin{array}{l} v' - i\Delta v + \nabla v \cdot (K^{-1})'Ky = f \quad \text{in } Q, \\ v(y,t) = \left| \begin{array}{ll} \varphi & \text{on } \Sigma_0 \\ 0 & \text{on } \Sigma\backslash\Sigma_0 \end{array} \right. \quad (i^2 = -1), \\ v(y,0) = v_0(y) \quad \text{in } \Omega, \end{array} \right. \tag{1.10}$$

where $\Sigma = \Gamma \times (0,T)$ is the lateral boundary of the cylinder Q and Σ_0 is a part of Σ, that will be defined in Section 5.

We investigate the exact controllability for the equivalent problem (1.10) using HUM (hilbert uniqueness method) idealized by Lions [8]. The particular case where $\tau_t(y) = \mu(t)y$, with $\mu(t)$ a real function defined on nonnegative real numbers $[0,\infty)$, has been analyzed by Miranda and Medeiros [10]. We can also find a study in controllability for Schrödinger equation in cylindrical domains in Lebeau [6] and Machtyngier [9]. We include, in the references at the end of this paper, some works relating to noncylindrical mixed problems for others models and related arguments, such as: [2–5].

The plan of this paper is as follows. In Section 2, we study the properties of the weak solution of the homogeneous boundary value problem for the formal adjoint L^* of the operator of (1.10), which is calculated in Section 4. Section 3 is dedicated to prove the direct and inverse inequalities for the weak solutions obtained in Section 2. Section 5 is dedicated to solve the problem of exact controllability for (1.10) and (1.5), respectively. In the appendix, we prove an identity which is a key point for the second estimate in Section 2. Below we state the main result of our paper.

Theorem 1.1. *Let Ω be a regular, bounded open set of $\mathbb{R}^n$, let Ω_t be defined as in (1.3), and suppose that* (H1) *and* (H2) *hold. If $T > 0$, then, for each $u_0 \in H^{-1}(\Omega_t)$, there exists a control $\widehat{w} \in L^2(\widehat{\Sigma})$ such that u solution of problem (1.5) verifies*

$$u(x,T,\widehat{w}) = 0 \quad \forall x \in \Omega_T. \tag{1.11}$$

2. Weak solutions

The formal adjoint or transposed operator of $Lw = w' - i\Delta w + \nabla w \cdot (K^{-1})'Ky$ is $-L^*v = v' - i\Delta v + \nabla v \cdot (K^{-1})'Ky + \operatorname{tr}((K^{-1})K)v$. We represent by $C(t) = (c_{jk}(t))_{1\le j,k\le n}$ the matrix $(K^{-1}(t))'K(t)$ and by $\operatorname{tr}(C(t))$ its trace.

By $H^1(\Omega)$ we represent the Sobolev space defined by

$$\left\{u \in L^2(\Omega),\ \frac{\partial u}{\partial x_i} \in L^2(\Omega) \text{ for } i = 1,\dots,n\right\}. \tag{2.1}$$

Also by $H_0^1(\Omega)$ we represent the Sobolev space of functions $u \in H^1(\Omega)$ such that $u|_\Gamma = 0$.

In order to apply HUM to the mixed problem (1.10) it is fundamental to know the properties of the weak solution of the homogeneous boundary value problem for the formal adjoint L^*, which is studied in this section.

We consider the following problem for the adjoint L^*.

Given $v_0 \in H_0^1(\Omega)$ and $f \in L^2(0,T;H_0^1(\Omega))$, we want to find a function $v : Q \to \mathbb{C}$ solution, in some sense, of the boundary value problem

$$\left|\begin{array}{l} v' - i\Delta v + \nabla v \cdot Cy + \operatorname{tr}(C)v = f \quad \text{in } Q, \\ v = 0 \quad \text{on } \Sigma, \\ v(0) = v_0 \quad \text{in } \Omega. \end{array}\right. \tag{2.2}$$

Theorem 2.1. *Given $v_0 \in H_0^1(\Omega)$ and $f \in L^2(0,T;H_0^1(\Omega))$, there exists one and only one function $v : Q \to \mathbb{C}$, called the weak solution of (2.2), satisfying*

$$v \in L^\infty(0,T;H_0^1(\Omega)),$$
$$v' \in L^2(0,T;H^{-1}(\Omega)),$$
$$-\int_0^T (v,\varphi')dt + i\int_0^T ((v,\varphi))dt + \int_0^T (\nabla v \cdot C(t)y,\varphi)dt + \int_0^T (\operatorname{tr}(C)v,\varphi)dt = \int_0^T (f,\varphi)dt \tag{2.3}$$

for all $\varphi \in L^2(0,T;H_0^1(\Omega))$, such that $\varphi' \in L^2(0,T;L^2(\Omega))$ with $\varphi(0) = \varphi(T) = 0$,

$$v(0) = v_0 \quad \text{in } \Omega. \tag{2.4}$$

Observe that $(\cdot,\cdot)$ and $|\cdot|$, $((\cdot,\cdot))$ and $\|\cdot\|$ represent the inner product and norm, respectively, in $L^2(\Omega)$ and $H_0^1(\Omega)$.

Proof. We employ the Galerkin method. In fact, let us consider the sequence $(w_j)_{j\in\mathbb{N}}$ of the solutions of the eigenvalue problem

$$((w_j,\varphi)) = \lambda_j(w_j,\varphi), \quad j = 1,2,\ldots, \tag{2.5}$$

for each $\varphi \in H_0^1(\Omega)$. Represent by V_m the subspace generated by $\{w_1,w_2,\ldots,w_m\}$ and let us consider the approximate problem

$$\left|\begin{aligned} &\text{find } v_m \in V_m \text{ solution of } (v_m'(t),w_j) + i((v_m(t),w_j)) + (\nabla v_m(t)\cdot C(t)y,w_j) \\ &\qquad\qquad + \operatorname{tr}(C(t))(v_m(t),w_j) = (f(t),w_j) \quad \text{for } j = 1,2,\ldots,m, \\ &v_m(0) = v_{0m} \text{ strongly convergent to } v_0 \text{ in } H_0^1(\Omega). \end{aligned}\right. \tag{2.6}$$

Note that if $v_m(t) \in V_m$, then $v_m(y,t) = \sum_{j=1}^m g_{jm}(t)w_j(y)$. It follows that (2.6) is a system of ordinary differential equations in the unknowns $g_{jm}(t)$, $j = 1,2,\ldots,m$. This system has a local solution on $[0,t_m)$, for some $t_m \in (0,T)$, and each $g_{jm}(t)$ belongs to $H^1(0,t_m)$. The extension to interval $[0,T]$ is a consequence of the following estimate.

First estimate. Multiply both sides of (2.6) by $\overline{g_{jm}(t)}$, adding from $j = 1$ to $j = m$, we obtain

$$\begin{aligned} &(v_m'(t),v_m(t)) + i((v_m(t),v_m(t))) + (\nabla v_m(t)\cdot C(t)y,v_m(t)) \\ &\quad + \operatorname{tr}(C(t))(v_m(t),v_m(t)) = (f(t),v_m(t)). \end{aligned} \tag{2.7}$$

Note that $\overline{z}$ is the complex conjugate of z.

Taking the double of the real parts of the last equality's both sides, we obtain

$$\begin{aligned} &2\operatorname{Re}(v_m'(t),v_m(t)) + 2\operatorname{Re}(\nabla v_m(t)\cdot C(t)y,v_m(t)) + 2\operatorname{tr}(C(t))|v_m(t)|^2 \\ &\quad = 2\operatorname{Re}(f(t),v_m(t)). \end{aligned} \tag{2.8}$$

Observe that $\nabla v_m(t)\cdot y = (\partial v_m(t)/\partial y_j)y_j$ and repeated indexes mean summation.

We analyze the first and second terms of the last equality.
(i) $2\,\mathrm{Re}(v_m'(t), v_m(t))$.
We have

$$2\,\mathrm{Re}\left(v_m'(t), v_m(t)\right) = \frac{d}{dt}\left|v_m(t)\right|^2. \tag{2.9}$$

(ii) $2\,\mathrm{Re}(\nabla v_m(t)\cdot C(t)y, v_m(t))$.
By Gauss' lemma,

$$\int_\Omega \frac{\partial}{\partial y_j}\left(v_m(t)c_{jk}(t)y_k\overline{v_m(t)}\right)dy = 0. \tag{2.10}$$

We observe that

$$\frac{\partial}{\partial y_j}\left(c_{jk}(t)y_k\right) = c_{jj}(t)y_j, \tag{2.11}$$

and therefore

$$\int_\Omega \frac{\partial v_m(t)}{\partial y_j}c_{jk}(t)y_k\overline{v_m(t)}dy + \int_\Omega v_m(t)c_{jj}(t)\overline{v_m(t)}dy + \int_\Omega v_m(t)c_{jk}(t)y_k\overline{\frac{\partial v_m(t)}{\partial y_j}}dy = 0. \tag{2.12}$$

Thus,

$$2\,\mathrm{Re}\left(\nabla v_m(t)\cdot C(t)y, v_m(t)\right) = -\sum_{j=1}^{n}\int_\Omega c_{jj}(t)\left|v_m(t,y)\right|^2dy = -\mathrm{tr}(C)\left|v_m(t)\right|^2. \tag{2.13}$$

Substituting (2.9) and (2.13) in (2.8), it follows that

$$\frac{d}{dt}\left|v_m(t)\right|^2 + \mathrm{tr}(C)\left|v_m(t)\right|^2 \le 2\left|f(t)\right|\left|v_m(t)\right|. \tag{2.14}$$

From assumption (H2) that $\mathrm{tr}(C(t)) \ge 0$, it follows that

$$\frac{d}{dt}\left|v_m(t)\right|^2 \le 2\left|f(t)\right|\left|v_m(t)\right|, \tag{2.15}$$

that is,

$$\frac{d}{dt}\left(\left|v_m(t)\right|^2\right) - \left|v_m(t)\right|^2 \le \left|f(t)\right|^2. \tag{2.16}$$

Multiplying (2.16) by e^{-t} and integrating from 0 to t we obtain

$$\left|v_m(t)\right|^2 \le C_1\left\{\left|v_{0m}\right|^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))}\right\}. \tag{2.17}$$

Since $v_m(0) = v_{0m} \to v_0$ in $H_0^1(\Omega)$, we conclude that

$$\left(v_m(t)\right) \text{ is bounded in } L^\infty\left(0,T;L^2(\Omega)\right). \tag{2.18}$$

Second estimate. Multiply both sides of (2.6) by $\lambda_j\overline{g_{jm}(t)}$, adding from $j=1$ to $j=m$ and taking the double of real parts, we obtain

$$\begin{aligned}&2\operatorname{Re}\left(v_m'(t),-\Delta v_m(t)\right)+2\operatorname{Re}\left(\nabla v_m(t)\cdot C(t)y,-\Delta v_m(t)\right)\\&\qquad+2\operatorname{tr}\left(C(t)\right)\left(v_m(t),-\Delta v_m(t)\right)=2\operatorname{Re}\left(f(t),-\Delta v_m(t)\right).\end{aligned}\tag{2.19}$$

(i) Analysis of $2\operatorname{Re}(v_m'(t),-\Delta v_m(t))$.
We have,

$$2\operatorname{Re}\left(v_m'(t),-\Delta v_m(t)\right)=2\operatorname{Re}\left(\left(v_m'(t),v_m(t)\right)\right)=\frac{d}{dt}\left\|v_m(t)\right\|^2.\tag{2.20}$$

(ii) Analysis of $2\operatorname{Re}(\nabla v_m(t)\cdot C(t)y,-\Delta v_m(t))$.
By Green's formula,

$$\begin{aligned}&\left(\frac{\partial v_m(t)}{\partial y_j}c_{jk}(t)y_k,-\Delta v_m(t)\right)\\&\quad=\left(\frac{\partial}{\partial y_l}\left[\frac{\partial v_m(t)}{\partial y_j}c_{jk}(t)y_k\right],\frac{\partial v_m(t)}{\partial y_l}\right)-\int_\Gamma c_{jk}y_k\cdot\nu_j\left|\frac{\partial v_m}{\partial\nu}\right|^2d\Gamma\\&\quad=\left(\frac{\partial}{\partial y_l}\left(\frac{\partial v_m(t)}{\partial y_j}\right)c_{jk}(t)y_k,\frac{\partial v_m(t)}{\partial y_l}\right)+\left(\frac{\partial v_m(t)}{\partial y_j}c_{jk}(t)\delta_l^k,\frac{\partial v_m(t)}{\partial y_l}\right)\\&\qquad-\int_\Gamma c_{jk}y_k\cdot\nu_j\left|\frac{\partial v_m}{\partial\nu}\right|^2d\Gamma=\int_\Omega\frac{\partial}{\partial y_l}\left(\frac{\partial v_m(t)}{\partial y_j}\right)c_{jk}(t)y_k\overline{\frac{\partial v_m(t)}{\partial y_l}}dy\\&\qquad+\int_\Omega\frac{\partial v_m(t)}{\partial y_j}c_{jl}(t)\overline{\frac{\partial v_m(t)}{\partial y_l}}dy-\int_\Gamma c_{jk}y_k\cdot\nu_j\left|\frac{\partial v_m}{\partial\nu}\right|^2d\Gamma.\end{aligned}\tag{2.21}$$

It follows from (2.21) that

$$\begin{aligned}&2\operatorname{Re}\left(\frac{\partial v_m(t)}{\partial y_j}c_{jk}(t)y_k,-\Delta v_m(t)\right)\\&\quad=2\operatorname{Re}\left(\frac{\partial}{\partial y_l}\left(\frac{\partial v_m(t)}{\partial y_j}\right)c_{jk}(t)y_k,\frac{\partial v_m(t)}{\partial y_l}\right)\\&\qquad+2\operatorname{Re}\left(\frac{\partial v_m(t)}{\partial y_j}c_{jl}(t),\frac{\partial v_m(t)}{\partial y_l}\right)-2\int_\Gamma c_{jk}y_k\cdot\nu_j\left|\frac{\partial v_m}{\partial\nu}\right|^2d\Gamma.\end{aligned}\tag{2.22}$$

Now, by Gauss' lemma,

$$\int_\Omega\frac{\partial}{\partial y_j}\left(\frac{\partial v_m(t)}{\partial y_l}c_{jk}(t)y_k\overline{\frac{\partial v_m(t)}{\partial y_l}}\right)dy=\int_\Gamma\nu_j\cdot\left(\frac{\partial v_m(t)}{\partial y_l}c_{jk}(t)y_k\overline{\frac{\partial v_m(t)}{\partial y_l}}\right)d\Gamma.\tag{2.23}$$

That is,

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial y_j}\left(\frac{\partial v_m(t)}{\partial y_l}\right)c_{jk}(t)y_k\overline{\frac{\partial v_m(t)}{\partial y_l}}dy\\
&\qquad+\int_\Omega \frac{\partial v_m(t)}{\partial y_l}c_{jj}(t)\overline{\frac{\partial v_m(t)}{\partial y_l}}dy+\int_\Omega \frac{\partial v_m(t)}{\partial y_l}c_{jk}(t)y_k\overline{\frac{\partial}{\partial y_j}\left(\frac{\partial v_m(t)}{\partial y_l}\right)}dy\\
&\quad=\int_\Gamma \nu_j\cdot\left(\frac{\partial v_m(t)}{\partial y_l}c_{jk}(t)y_k\overline{\frac{\partial v_m(t)}{\partial y_l}}\right)d\Gamma=\int_\Gamma \nu_j c_{jk}(t)y_k\left|\frac{\partial v_m}{\partial \nu}\right|^2 d\Gamma.
\end{aligned}\tag{2.24}$$

Therefore,

$$2\,\mathrm{Re}\left(\frac{\partial}{\partial y_j}\left(\frac{\partial v_m(t)}{\partial y_l}\right)c_{jk}(t)y_k,\frac{\partial v_m(t)}{\partial y_l}\right)=\int_\Gamma \nu_j c_{jk}(t)y_k\left|\frac{\partial v_m}{\partial \nu}\right|^2 d\Gamma-c_{jj}(t)\int_\Omega\left|\frac{\partial v_m}{\partial y_l}\right|^2 dy.\tag{2.25}$$

Substituting (2.25) into (2.22), we obtain

$$\begin{aligned}
&2\,\mathrm{Re}\left(\frac{\partial v_m(t)}{\partial y_j}c_{jk}(t)y_k,-\Delta v_m(t)\right)\\
&\quad=2\,\mathrm{Re}\left(\frac{\partial v_m(t)}{\partial y_j}c_{jl}(t),\frac{\partial v_m(t)}{\partial y_l}\right)-c_{jj}(t)\int_\Omega\left|\frac{\partial v_m}{\partial y_l}\right|^2 dy-\int_\Gamma \nu_j c_{jk}(t)y_k\left|\frac{\partial v_m}{\partial \nu}\right|^2 d\Gamma.
\end{aligned}\tag{2.26}$$

Substituting the expressions (2.20) and (2.26) into (2.19) we obtain

$$\begin{aligned}
&\frac{d}{dt}\left|v_m(t)\right|^2+2\,\mathrm{Re}\left(\frac{\partial v_m(t)}{\partial y_j}c_{jl}(t),\frac{\partial v_m(t)}{\partial y_l}\right)-c_{jj}(t)\int_\Omega\left|\frac{\partial v_m}{\partial y_l}\right|^2 dy\\
&\qquad-\int_\Gamma \nu_j c_{jk}(t)y_k\left|\frac{\partial v_m}{\partial \nu}\right|^2 d\Gamma+2\,\mathrm{tr}\,(C(t))\|v_m(t)\|^2=2\,\mathrm{Re}\,(f(t),-\Delta v_m(t)).
\end{aligned}\tag{2.27}$$

By the identity of the appendix, we modify (2.27) as follows:

$$\begin{aligned}
&\frac{d}{dt}\|v_m(t)\|^2+\mathrm{tr}\,(C(t))\|v_m(t)\|^2-\frac{d}{dt}\mathrm{Im}\,(v_m(t),C(t)y\cdot\nabla v_m(t))\\
&\qquad+\mathrm{Im}\,(v_m(t),C'(t)y\cdot\nabla v_m(t))-\mathrm{tr}\,(C(t))\,\mathrm{Im}\,(v_m(t),C(t)y\cdot\nabla v_m(t))\\
&\qquad+2\,\mathrm{Im}\,(P_m f(t),C(t)y\cdot\nabla v_m(t))+\mathrm{tr}\,(C(t))\,\mathrm{Im}\,(P_m f(t),v_m(t))\\
&\quad=2\,\mathrm{Re}\,((f(t),v_m(t))).
\end{aligned}\tag{2.28}$$

Setting $\varphi(t)=\mathrm{Im}(v_m(t),C(t)y\cdot\nabla v_m(t))$ and $\psi(t)=\|v_m(t)\|^2-\varphi(t)$, we obtain from (2.28) that

$$\frac{d}{dt}\psi(t)+\mathrm{tr}\,(C(t))\psi(t)=g(t),\tag{2.29}$$

where

$$g(t)=2\operatorname{Re}\left(\left(f(t),v_m(t)\right)\right)-\operatorname{tr}\left(C(t)\right)\operatorname{Im}\left(f(t),v_m(t)\right)\\ -2\operatorname{Im}\left(P_mf(t),C(t)y\cdot\nabla v_m(t)\right)-\operatorname{Im}\left(v_m(t),C'(t)y\cdot\nabla v_m(t)\right). \tag{2.30}$$

Solving the differential equation (2.29) we attain

$$\psi(t)=\psi(0)\exp\left(-\int_0^t\operatorname{tr}\left(C(r)\right)dr\right)+\int_0^t\exp\left(-\int_s^t\operatorname{tr}\left(C(r)\right)dr\right)g(s)ds. \tag{2.31}$$

Since $\psi(0)=\|v_m(0)\|^2-\varphi(0)$, we have from the last equality

$$\left\|v_m(t)\right\|^2=\varphi(t)+\left(\left\|v_m(0)\right\|^2-\varphi(0)\right)\exp\left(-\int_0^t\operatorname{tr}\left(C(r)\right)dr\right)\\ +\int_0^t\exp\left(-\int_s^t\operatorname{tr}\left(C(r)\right)dr\right)g(s)ds. \tag{2.32}$$

Observe that

(i)

$$\begin{aligned}\varphi(t)&\le\left|v_m(t)\right|\left|C(t)y\cdot\nabla v_m(t)\right|\le\left|v_m(t)\right|\sum_{l=1}^n\left(\sum_{k=1}^n\left|c_{lk}(t)\right|\left|y_k\right|\right)\left|\frac{\partial v_m}{\partial y_l}\right|\\ &\le\left|v_m(t)\right|\left[\sum_{l=1}^n\left(\sum_{k=1}^n\left|c_{lk}(t)\right|\left|y_k\right|\right)^2\right]^{1/2}\left(\sum_{l=1}^n\left|\frac{\partial v_m}{\partial y_l}\right|^2\right)^{1/2}\\ &\le\left|v_m(t)\right|\left[\sum_{l=1}^n\left(\sum_{k=1}^n\left|c_{lk}(t)\right|^2\right)\left(\sum_{k=1}^n\left|y_k\right|^2\right)\right]^{1/2}\left(\sum_{l=1}^n\left|\frac{\partial v_m}{\partial y_l}\right|^2\right)^{1/2}\\ &=\left|v_m(t)\right||y|\left[\sum_{l=1}^n\sum_{k=1}^n\left|c_{lk}(t)\right|^2\right]^{1/2}\left|\nabla v_m\right|\\ &\le\left|v_m(t)\right||y|n\left(\max_{l,k=1,\dots,n}\left(\max_{t\in[0,T]}\left|c_{lk}(t)\right|\right)\right)\left|\nabla v_m\right|\\ &\le M(\Omega,n,\mathcal{M}_C)\left|v_m(t)\right|\left|\nabla v_m\right|\le\frac{M(\Omega,n,\mathcal{M}_C)^2}{2}\left|v_m(t)\right|^2+\frac{\left|\nabla v_m\right|^2}{2},\end{aligned} \tag{2.33}$$

where $\mathcal{M}_C=(\max_{l,k=1,\dots,n}(\max_{t\in[0,T]}|c_{lk}(t)|))$ and $M(\Omega,n,\mathcal{M}_C)=(\max_{y\in\Omega}|y|)n\mathcal{M}_C$. Since

(ii)

$$\exp\left(-\int_s^t\operatorname{tr}\left(C(r)\right)dr\right)\le\exp\left(\int_s^t\left|\operatorname{tr}\left(C(r)\right)\right|dr\right)\le\exp\left(n\mathcal{M}_CT\right), \tag{2.34}$$

(iii)

$$\left|g(s)\right|\le M(n,\Omega,\mathcal{M}_C)\left(\left\|f(s)\right\|+\left|v_m(s)\right|\right)\left\|v_m(s)\right\|, \tag{2.35}$$

we have

$$\begin{aligned} &\left| \int_0^t \exp\left(-\int_s^t \operatorname{tr}(C(r))\,dr \right) g(s)\,ds \right| \\ &\quad \le M(n,\Omega,\mathcal{M}_C,T) \int_0^t (\|f(s)\| + |v_m(s)|)\|v_m(s)\|\,ds, \end{aligned} \tag{2.36}$$

where

$$M(n,\Omega,\mathcal{M}_C,T) = \exp(n\mathcal{M}_C T) M(n,\Omega,\mathcal{M}_C) \tag{2.37}$$

and therefore

$$\begin{aligned} \|v_m(t)\|^2 \le{}& \frac{1}{2} M(n,\Omega,\mathcal{M}_C) |v_m(t)|^2 + \frac{1}{2}\|v_m(t)\|^2 + M(n,\Omega,\mathcal{M}_C,T)\|v_{0m}\|^2 \\ &+ M(n,\Omega,\mathcal{M}_C,T) \int_0^t (\|f(s)\| + |v_m(s)|)\|v_m(s)\|\,ds. \end{aligned} \tag{2.38}$$

From the first estimate, it follows that

$$|v_m(t)|^2 \le \alpha_0(T). \tag{2.39}$$

Then (2.39) implies

$$\frac{\|v_m(t)\|^2}{2} \le M(n,\Omega,\mathcal{M}_C,T,\|v_{0m}\|) + M(n,\Omega,\mathcal{M}_C,T) \int_0^t (\|f(s)\| + |v_m(s)|)\|v_m(s)\|\,ds, \tag{2.40}$$

where

$$M(n,\Omega,\mathcal{M}_C,T,\|v_{0m}\|) = \frac{1}{2} M(n,\Omega,\mathcal{M}_C)\alpha_0^2 + M(n,\Omega,\mathcal{M}_C,T)\|v_{0m}\|^2. \tag{2.41}$$

Then, by an inequality similar to Gronwall-Bellman's one, see Brézis [1, page 157], we obtain

$$\|v_m(t)\|^2 \le \alpha_1(T). \tag{2.42}$$

Then

$$(v_m) \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)). \tag{2.43}$$

Then, from (2.43), we can extract a subsequence v_μ of v_m such that

$$v_\mu \text{ converges to } v \text{ weak star in } L^\infty(0,T;H_0^1(\Omega)). \tag{2.44}$$

Applying the approximate equation (2.6) to $\theta \in \mathcal{D}(0,T)$, we get

$$\begin{aligned} &\int_0^T (v_\mu,w_j)\theta'\,dt + \int_0^T i((v_\mu,w_j))\theta\,dt + \int_0^T (Cy\cdot\nabla v_\mu,w_j)\theta\,dt \\ &\quad + \int_0^T \operatorname{tr}(C(t))(v_\mu,w_j)\theta\,dt = \int_0^T (f,w_j)\theta\,dt. \end{aligned} \tag{2.45}$$

Taking the limit when $\mu \to \infty$, j fixed, we obtain that v is a solution in the sense of Theorem 2.1.

We observe that

$$\frac{d}{dt}(v(t),w)+i((v(t),w))+(Cy\cdot\nabla v(t),w)+\operatorname{tr}(C(t))(v(t),w)=(f(t),w), \tag{2.46}$$

in the sense of $\mathcal{D}'(0,T)$, for each $w \in H_0^1(\Omega)$.

Since $G = i\Delta v - Cy\cdot\nabla v - \operatorname{tr}(C)v + f \in L^2(0,T;H^{-1}(\Omega))$, it follows from (2.46) (cf. Temam [11]) that

$$v' \in L^2(0,T;H^{-1}(\Omega)), \quad v' = G \quad \text{in } L^2(0,T;H^{-1}(\Omega)). \tag{2.47}$$

As we have seen $v \in L^2(0,T;H_0^1(\Omega))$, then, identifying $L^2(\Omega)$ to its dual $(L^2(\Omega))'$, we obtain

$$v \in C^0([0,T];L^2(\Omega)). \tag{2.48}$$

The equality stated in Theorem 2.1 follows from the one in (2.47), as the uniqueness of solution does too. We must observe that the solution v also satisfies $v(0) = v_0$.

To prove uniqueness, suppose v_1, v_2 are solutions and $w = v_1 - v_2$. Then, by the equality in (2.47), we obtain

$$w' - i\Delta w + Cy\cdot\nabla w + \operatorname{tr}(C)w = 0 \quad \text{in } L^2(0,T;H^{-1}(\Omega)). \tag{2.49}$$

Thus, multiplying by w and taking the double of the real part of it, we have

$$2\operatorname{Re}\langle w'(t),w(t)\rangle + 2\operatorname{Re}(\nabla w(t)\cdot C(t)y,w(t)) + 2\operatorname{tr}(C(t))|w(t)|^2 = 0. \tag{2.50}$$

Analyzing the first and second terms of the last equality as in the first estimate, we attain

$$\frac{d}{dt}|w(t)|^2 + \operatorname{tr}(C(t))|w(t)|^2 \le 0. \tag{2.51}$$

From assumption (H2) that $\operatorname{tr}(C(t)) \ge 0$ it follows that

$$\frac{d}{dt}|w(t)|^2 \le 0, \tag{2.52}$$

and since $w(0) = 0$, we conclude that $w \equiv 0$ which means the solution v is unique. □

Remark 2.2. If we consider the boundary value problem (2.2) with initial data

$$v_0 \in H_0^1(\Omega)\cap H^2(\Omega) \quad f \in L^2(0,T;H_0^1(\Omega)) \quad \text{such that } f' \in L^1(0,T;L^2(\Omega)), \tag{2.53}$$

by the same argument used to prove Theorem 2.1, we prove the following theorem.

Theorem 2.3. *Given v_0 and f satisfying (2.53), there exists only one function $v : Q \to \mathbb{C}$, satisfying the following conditions:*

$$\left|\begin{array}{l} v \in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \\ v' - i\Delta v + Cy\cdot\nabla v + \operatorname{tr}(C)v = f \quad \text{a.e. in } Q, \\ v(0) = v_0 \quad \text{in } \Omega. \end{array}\right. \tag{2.54}$$

The solution v of Theorem 2.3 is called the strong solution of problem (2.2).

3. Inequalities

HUM is based on two inequalities: one is called direct, Lemma 3.2; another inverse, Lemma 3.3. This section is dedicated to prove these inequalities, for the solution of the boundary value problem (2.2), which are the key points in order to solve the exact controllability problem for the Schrödinger equation, see (5.19).

We begin proving the following identity.

Lemma 3.1. *Let $q = (q_l)$, $1 \le l \le n$, be a vector field, with $q_l \in C^2(\overline{\Omega})$ for all l. Then, for all strong solution v of the adjoint boundary value problem (2.2), there exists the identity*

$$\begin{aligned} &\int_0^T\int_\Gamma q\cdot\nu\left|\frac{\partial v}{\partial\nu}\right|^2 d\Gamma dt \\ &\quad= \operatorname{Im}\left(v, q_l\frac{\partial v}{\partial y_l}\right)\Big|_0^T + \int_0^T \operatorname{Re}\left(\frac{\partial v}{\partial y_j}, \frac{\partial^2 q_l}{\partial y_j\partial y_l}v\right)dt \\ &\qquad+ 2\operatorname{Re}\left(\frac{\partial v}{\partial y_j}, \frac{\partial q_l}{\partial y_j}\frac{\partial v}{\partial y_l}\right)dt + \int_0^T \operatorname{Im}\left(c_{jk}y_k\frac{\partial v}{\partial y_j}, \frac{\partial q_l}{\partial y_l}v\right)dt \\ &\qquad+ \int_0^T 2\operatorname{Im}\left(c_{jk}y_k\frac{\partial v}{\partial y_j}, q_l\frac{\partial v}{\partial y_l}\right)dt + \int_0^T 2\operatorname{Im}\operatorname{tr}(C(t))\left(v, q_l\frac{\partial v}{\partial y_l}\right)dt \\ &\qquad- \int_0^T \operatorname{Im}\left(f, \frac{\partial q_l}{\partial y_l}v\right)dt - \int_0^T 2\operatorname{Im}\left(f, q_l\frac{\partial v}{\partial y_l}\right)dt. \end{aligned} \tag{3.1}$$

Proof. Multiplying both sides of $(2.2)_1$, Section 2, by $q_l(\partial v/\partial y_l)$, integrating on Q, and taking the double of the imaginary parts of the resulting equality's both sides, we have

$$\begin{aligned} &\int_0^T 2\operatorname{Im}\left(v', q_l\frac{\partial v}{\partial y_l}\right)dt + \int_0^T 2\operatorname{Re}\left(-\Delta v, q_l\frac{\partial v}{\partial y_l}\right)dt + \int_0^T 2\operatorname{Im}\left(c_{jk}\cdot y_k\frac{\partial v}{\partial y_j}, q_l\frac{\partial v}{\partial y_l}\right)dt \\ &\quad+ \int_0^T 2\operatorname{Im}\operatorname{tr}(C(t))\left(v, q_l\frac{\partial v}{\partial y_l}\right)dt = \int_0^T 2\operatorname{Im}\left(f, q_l\frac{\partial v}{\partial y_l}\right)dt. \end{aligned} \tag{3.2}$$

We have

$$\frac{d}{dt}\left(v, q_l\frac{\partial v}{\partial y_l}\right) = \left(v', q_l\frac{\partial v}{\partial y_l}\right) + \left(v, q_l\frac{\partial v'}{\partial y_l}\right). \tag{3.3}$$

By Gauss' lemma, we obtain

$$\int_\Omega \frac{\partial}{\partial y_l}(vq_l\overline{v}')\,dy = 0, \quad \text{because } v = 0 \text{ on } \Sigma. \tag{3.4}$$

It follows that

$$\left(v, q_l\frac{\partial v'}{\partial y_l}\right) + \left(q_l\frac{\partial v}{\partial y_l}, v'\right) + \left(\frac{\partial q_l}{\partial y_l}v, v'\right) = 0. \tag{3.5}$$

Substituting (3.5) in (3.3) and integrating on $(0,T)$, we obtain

$$\int_0^T \left(v', q_l\frac{\partial v}{\partial y_l}\right)dt - \int_0^T \left(q_l\frac{\partial v}{\partial y_l}, v'\right)dt = \left(v, q_l\frac{\partial v}{\partial y_l}\right)\Big|_0^T + \int_0^T \left(\frac{\partial q_l}{\partial y_l}v, v'\right)dt. \tag{3.6}$$

Since $z - \overline{z} = 2i\,\mathrm{Im}\,z$, the last identity implies

$$2\,\mathrm{Im}\int_0^T \left(v', q_l\frac{\partial v}{\partial y_l}\right)dt = -i\left(v, q_l\frac{\partial v}{\partial y_l}\right)\Big|_0^T - i\int_0^T \left(\frac{\partial q_l}{\partial y_l}v, v'\right)dt. \tag{3.7}$$

Taking the real parts of both sides of the above identity, we get

$$2\,\mathrm{Im}\int_0^T \left(v', q_l\frac{\partial v}{\partial y_l}\right)dt = \mathrm{Im}\left(v, q_l\frac{\partial v}{\partial y_l}\right)\Big|_0^T - \mathrm{Im}\int_0^T \left(v', \frac{\partial q_l}{\partial y_l}v\right)dt. \tag{3.8}$$

(i) Analysis of $\mathrm{Im}\int_0^T (v', (\partial q_l/\partial y_l)v)dt$.

Multiplying $(2.2)_1$ by $(\partial q_l/\partial y_l)v$, integrating on Ω, and taking the imaginary parts of the resulting equality's both sides, we obtain

$$\begin{aligned}\mathrm{Im}\left(v', \frac{\partial q_l}{\partial y_l}v\right) + \mathrm{Re}\left(-\Delta v, \frac{\partial q_l}{\partial y_l}v\right) + \mathrm{Im}\left(c_{jk}\cdot y_k\frac{\partial v}{\partial y_j}, \frac{\partial q_l}{\partial y_l}v\right)&\\ + \mathrm{Im}\left(\mathrm{tr}\,(C(t))v, \frac{\partial q_l}{\partial y_l}v\right) = \mathrm{Im}\left(f, \frac{\partial q_l}{\partial y_l}v\right)&.\end{aligned} \tag{3.9}$$

By Green's formula, we obtain

$$\mathrm{Re}\left(-\Delta v, \frac{\partial q_l}{\partial y_l}v\right) = \mathrm{Re}\left(\frac{\partial v}{\partial y_j}, \frac{\partial^2 q_l}{\partial y_j\partial y_l}v\right) + \mathrm{Re}\left(\frac{\partial v}{\partial y_j}, \frac{\partial q_l}{\partial y_l}\frac{\partial v}{\partial y_j}\right), \tag{3.10}$$

because $v = 0$ on Γ.

By (3.8), (3.9), and (3.10) we have

$$\begin{aligned}
&2\,\mathrm{Im}\int_0^T\left(v',q_l\frac{\partial v}{\partial y_l}\right)dt\\
&\quad=\mathrm{Im}\left(v,q_l\frac{\partial v}{\partial y_l}\right)\Bigg|_0^T-\int_0^T\mathrm{Im}\left(f,\frac{\partial q_l}{\partial y_l}v\right)dt+\int_0^T\mathrm{Re}\left(\frac{\partial v}{\partial y_j},\frac{\partial^2 q_l}{\partial y_j\partial y_l}v\right)dt\\
&\qquad+\int_0^T\left(\frac{\partial v}{\partial y_j},\frac{\partial q_l}{\partial y_l}\frac{\partial v}{\partial y_j}\right)dt+\int_0^T\mathrm{Im}\left(c_{jk}\cdot y_k\frac{\partial v}{\partial y_j},\frac{\partial q_l}{\partial y_l}v\right)dt.
\end{aligned}\tag{3.11}$$

(ii) Analysis of $\int_0^T 2\,\mathrm{Re}(-\Delta v,q_l(\partial v/\partial y_l))dt$.
By Gauss' lemma,

$$\int_\Omega\frac{\partial}{\partial y_j}\left(\frac{\partial v}{\partial y_j}q_l\frac{\partial\overline{v}}{\partial y_j}\right)dy=\int_\Gamma q_l\cdot\nu_l\frac{\partial v}{\partial y_j}\frac{\partial\overline{v}}{\partial y_j}d\Gamma.\tag{3.12}$$

Since $\partial v/\partial y_j=\nu_j(\partial v/\partial\nu)$, we have from the last identity that

$$2\,\mathrm{Re}\left(\frac{\partial v}{\partial y_j},q_l\frac{\partial}{\partial y_j}\left(\frac{\partial v}{\partial y_j}\right)\right)=-\left(\frac{\partial v}{\partial y_j},\frac{\partial q_l}{\partial y_j}\frac{\partial v}{\partial y_j}\right)+\int_\Gamma q\cdot\nu\left|\frac{\partial v}{\partial\nu}\right|^2d\Gamma.\tag{3.13}$$

By Green's formula, we have

$$\left(-\Delta v,q_l\frac{\partial v}{\partial y_l}\right)=\left(\frac{\partial v}{\partial y_j},\frac{\partial q_l}{\partial y_j}\frac{\partial v}{\partial y_l}\right)+\left(\frac{\partial v}{\partial y_j},q_l\frac{\partial}{\partial y_j}\left(\frac{\partial v}{\partial y_j}\right)\right)-\int_\Gamma q_l\cdot\nu_l\left|\frac{\partial v}{\partial\nu}\right|^2d\Gamma.\tag{3.14}$$

From (3.13), we modify (3.14) obtaining

$$\begin{aligned}
&\int_0^T 2\,\mathrm{Re}\left(-\Delta v,q_l\frac{\partial v}{\partial y_l}\right)dt\\
&\quad=\int_0^T 2\,\mathrm{Re}\left(\frac{\partial v}{\partial y_j},\frac{\partial q_l}{\partial y_j}\frac{\partial v}{\partial y_l}\right)dt-\int_0^T 2\,\mathrm{Re}\left(\frac{\partial v}{\partial y_j},\frac{\partial q_l}{\partial y_l}\frac{\partial v}{\partial yj}\right)dt-\int_0^T\int_\Gamma q\cdot\nu\left|\frac{\partial v}{\partial\nu}\right|^2d\Gamma\,dt.
\end{aligned}\tag{3.15}$$

Substituting (3.11) and (3.15) in the identity (3.2), we obtain the identity (3.1). □

Before stating the direct inequality, the following consideration will be made. Let $(W,\|\cdot\|_W)$ be the Banach space of the weak solutions of (2.2), where $\|v\|_W=\max_{0\le t\le T}\|v(t)\|$. Considering by $\mathcal{F}=\{f\in L^2(0,T;H_0^1(\Omega));f'\in L^1(0,T;L^2(\Omega))\}$ and denoting by S the vector space of the strong solutions of (2.2) relating to $\{v_0,f\}\in(H_0^1(\Omega)\cap H^2(\Omega))\times\mathcal{F}$, it follows that

$$S\ \text{is dense in}\ W.\tag{3.16}$$

Lemma 3.2 (direct inequality). *If v is the weak solution of the boundary value problem (2.2), then there exists the inequality*

$$\int_0^T \int_\Gamma \left|\frac{\partial v}{\partial \nu}\right|^2 d\Gamma\, dt \leq \alpha\left[\|v_0\|^2 + \int_0^T \|f(s)\|^2 ds\right]. \tag{3.17}$$

Proof. If v is the strong solution of (2.2), it satisfies the identity of Lemma 3.1. If we choose the vector field $q = (q_l)_{1\leq l\leq n}$, with $q \in [C^2(\overline{\Omega})]^n$, such that $q = \nu$ on Γ, where ν is the unit exterior normal vector to Γ, then $q \cdot \nu = 1$ on Γ and the left-hand side of the identity (3.1) is reduced to

$$\int_0^T \int_\Gamma \left|\frac{\partial v}{\partial \nu}\right|^2 d\Gamma\, dt. \tag{3.18}$$

The right-hand side of the identity (3.1), for $q = \nu$ on Γ, is bounded by

$$\alpha\left(\max_{0\leq t\leq T}\|v(t)\|^2 + \int_0^T \|f(s)\|^2 ds\right). \tag{3.19}$$

From the second estimate, we know that

$$\|v\|^2 \leq \alpha_1\|v_0\|^2. \tag{3.20}$$

Therefore, we obtain

$$\int_0^T \int_\Gamma \left|\frac{\partial v}{\partial \nu}\right|^2 d\Gamma\, dt \leq \alpha\left[\|v_0\|^2 + \int_0^T \|f(s)\|^2 ds\right]. \tag{3.21}$$

Let us consider the operator $\gamma : S \to L^2(\Sigma)$ such that $\gamma(v) = \partial v/\partial \nu$ and the vector space S is defined previously. From (3.1), we have

$$|\gamma(v)|^2_{L^2(\Sigma)} = \int_0^T \int_\Gamma \left|\frac{\partial v}{\partial \nu}\right|^2 d\Gamma\, dt \leq c\max_{0\leq t\leq T}\|v(t)\|^2 = c\|v\|^2_W. \tag{3.22}$$

Thus, γ is linear and continuous on S that is dense in W. Therefore, γ admits a linear and continuous extension to $\overline{S} = W$. Let v be the weak solution relating to v_0 and f. Then, there exists a sequence of strong solutions (v_n) such that $v_n \to v$ in W.

For each strong solution v_n, the inequality (3.21) is true, and therefore taking the limit we obtain

$$\int_0^T \int_\Gamma \left|\frac{\partial v}{\partial \nu}\right|^2 d\Gamma\, dt \leq \alpha\left[\|v_0\|^2 + \int_0^T \|f(s)\|^2 ds\right] \tag{3.23}$$

now for the weak solution v. □

The next inequality is true on a part $\Gamma(y_0)$ which we will define. In fact, let us consider a point $y_0 \in \mathbb{R}^n$ and represent by $m(y)$ the vector $y - y_0$. We consider the following decomposition of Γ:

$$\Gamma(y_0) = \{y \in \Gamma; Cm(y)\cdot\nu \geq 0\}, \qquad \Gamma_*(y_0) = \{y \in \Gamma; Cm(y)\cdot\nu < 0\}, \tag{3.24}$$

where C is the matrix $(K^{-1}(t))'K(t)$. We also define

$$\sum(y_0) = \Gamma(y_0)\times(0,T), \qquad \sum_{*}(y_0) = \Gamma_*(y_0)\times(0,T). \tag{3.25}$$

Lemma 3.3 (inverse inequality). *If v is a weak solution of the boundary value problem (2.2), with $f = 0$, there exists the inequality*

$$\alpha\|v_0\|^2 \le \int_0^T\int_{\Gamma(y_0)}\left|\frac{\partial v}{\partial \nu}\right|^2 d\Gamma\, dt, \tag{3.26}$$

where the constant α is independent of v and depends only on T, $\|y_0\|$, and Ω.

Proof. Let v be the weak solution of (2.2) corresponding to $v_0 \in H_0^1(\Omega)$. It follows from the identity (3.1), for $\vec{q} = C(y-y_0)$ and $f \equiv 0$, that

$$\begin{aligned}\int_0^T\int_{\Gamma}(Cm(y)\cdot\nu)\left|\frac{\partial v}{\partial \nu}\right|^2 d\Gamma\, dt \\ = (v, Cm(y)\cdot\nabla v)\big|_0^T + 2\int_0^T \operatorname{Re}(C\cdot\nabla v,\nabla v)dt + \int_0^T H(t)dt,\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}H(t) = {}&\operatorname{Im}(Cy\cdot\nabla v,(\operatorname{tr}C)v) + 2\operatorname{Im}(Cy\cdot\nabla v, Cm(y)\cdot\nabla v) \\ &+ 2\operatorname{Im}((\operatorname{tr}C)v, Cm(y)\cdot\nabla v).\end{aligned} \tag{3.28}$$

We have

$$\begin{aligned}H(t) = {}&\operatorname{Im}((\operatorname{tr}C)(Cy\cdot\nabla v),v) + 2\operatorname{Im}(Cy\cdot\nabla v, Cy\cdot\nabla v) - 2\operatorname{Im}(Cy\cdot\nabla v, Cy_0\cdot\nabla v) \\ &+ 2\operatorname{Im}((\operatorname{tr}C)v, Cy\cdot\nabla v) - 2\operatorname{Im}((\operatorname{tr}C)v, Cy_0\cdot\nabla v).\end{aligned} \tag{3.29}$$

Observe that if z is a complex number $-2\operatorname{Im}\overline{z} - \operatorname{Im}z = \operatorname{Im}z$, then

$$2\operatorname{Im}(v,(\operatorname{tr}C)(Cy\cdot\nabla v)) + \operatorname{Im}((\operatorname{tr}C)(Cy\cdot\nabla v),v) = -\operatorname{Im}((\operatorname{tr}C)(Cy\cdot\nabla v),v). \tag{3.30}$$

Using (3.30) we modify H obtaining

$$\begin{aligned}H(t) = {}&-\operatorname{Im}((\operatorname{tr}C)(Cy\cdot\nabla v),v) + 2\operatorname{Im}|Cy\cdot\nabla v|^2 - 2\operatorname{Im}(Cy\cdot\nabla v, Cy_0\cdot\nabla v) \\ &- 2\operatorname{Im}((\operatorname{tr}C)v, Cy_0\cdot\nabla v) = -\operatorname{Im}((\operatorname{tr}C)(Cy\cdot\nabla v),v) \\ &- 2\operatorname{Im}(Cy\cdot\nabla v, Cy_0\cdot\nabla v) - 2\operatorname{Im}((\operatorname{tr}C)v, Cy_0\cdot\nabla v).\end{aligned} \tag{3.31}$$

Substituting (3.31) in (3.27) we find

$$\begin{aligned}\int_0^T\int_\Gamma (Cm(y)\cdot\nu)\left|\frac{\partial v}{\partial\nu}\right|^2 d\Gamma\, dt\\ &= (v, Cm(y)\cdot\nabla v)\big|_0^T + 2\int_0^T \operatorname{Re}(C\cdot\nabla v,\nabla v)dt - \int_0^T \operatorname{Im}(Cy\cdot\nabla v, \operatorname{tr}(C)v)\,dt\\ &\quad - 2\int_0^T \operatorname{Im}(Cy\cdot\nabla v, Cy_0\cdot\nabla v)\,dt - 2\int_0^T \operatorname{Im}(\operatorname{tr}(C)v, Cy_0\cdot\nabla v)\,dt.\end{aligned} \tag{3.32}$$

Now we prove the inverse inequality (3.26). It will be done by steps.

Step 1. We prove that the weak solution of (2.2) satisfies

$$\alpha_1\left|v_0\right|^2 + \int_0^T\int_{\Gamma(y_0)} (Cm(y)\cdot\nu)\left|\frac{\partial v}{\partial\nu}\right|^2 d\Gamma\, dt \geq \alpha_2\|v_0\|^2. \tag{3.33}$$

We note that from (H2) we have

$$2\int_0^T \operatorname{Re}(C\cdot\nabla v,\nabla v)dt \geq 2\alpha\int_0^T \|v\|^2 dt. \tag{3.34}$$

Now, from the second estimate we have

$$\begin{aligned}\|v\|^2 &= \|v_0\|^2\exp\left(-\int_0^t \operatorname{tr}(C)dr\right) - \operatorname{Im}(v_0, Cy\cdot\nabla v_0)\exp\left(-\int_0^t \operatorname{tr}(C)dr\right)\\ &\quad + \operatorname{Im}(v, Cy\cdot\nabla v) - \int_0^t \operatorname{Im}(v, C'y\cdot\nabla v)\exp\left(-\int_0^t \operatorname{tr}(C)dr\right)ds,\end{aligned} \tag{3.35}$$

therefore,

$$\begin{aligned}2\int_0^T \operatorname{Re}(C\cdot\nabla v,\nabla v)dt &\geq 2\alpha\int_0^T \|v\|^2 dt\\ &= 2\alpha\|v_0\|^2\int_0^T \exp\left(-\int_0^t \operatorname{tr}(C)dr\right)dt - 2\alpha\int_0^T \operatorname{Im}(v_0, Cy\cdot\nabla v_0)\exp\left(-\int_0^t \operatorname{tr}(C)dr\right)dt\\ &\quad + 2\alpha\int_0^T \operatorname{Im}(v, Cy\cdot\nabla v)dt - 2\alpha\int_0^T\int_0^t \operatorname{Im}(v, C'y\cdot\nabla v)\exp\left(-\int_0^t \operatorname{tr}(C)dr\right)ds\,dt.\end{aligned} \tag{3.36}$$

Observe that for each $e_{jk} = (\delta_j^k)$, $k = 1,\dots,n$, where

$$\delta_j^k = \begin{cases}1 & \text{if } j = k,\\ 0 & \text{if } j \neq k,\end{cases} \tag{3.37}$$

we have, from (H2), that $C(t)e_{jk}\cdot e_{jk} \geq \alpha|e_{jk}|^2 = \alpha$ and $C(t)e_{jk}\cdot e_{jk} = C_{kk}(t)$, for all $k = 1,\dots,n$ for all $t\in[0,T]$, then $C_{kk}(t)\geq\alpha>0$ and, therefore,

$$\operatorname{tr}(C) \geq n\alpha > 0. \tag{3.38}$$

From (3.38), it follows that, for all $s,t \in [0,T]$, $s \le t$,

$$0 \le \int_s^t \operatorname{tr}(C)dr \le \int_0^T \operatorname{tr}(C)dr, \tag{3.39}$$

and then

$$\exp\left(-\int_0^T \operatorname{tr}(C)dr\right) \le \exp\left(-\int_s^t \operatorname{tr}(C)dr\right) \le 1. \tag{3.40}$$

Therefore,

$$\begin{aligned} &2\alpha\|v_0\|^2\int_0^T \exp\left(-\int_0^t \operatorname{tr}(C)dr\right)dt \\ &\quad\ge 2\alpha\|v_0\|^2\int_0^T \exp\left(-\int_0^T \operatorname{tr}(C)dr\right)dt \ge 2\alpha T C_1(T)\|v_0\|^2. \end{aligned} \tag{3.41}$$

We also have

$$\begin{aligned} &-2\alpha\int_0^T \operatorname{Im}(v_0, Cy\cdot\nabla v_0)\exp\left(-\int_0^t \operatorname{tr}(C)dr\right)dt \\ &\quad\ge -2\alpha|v_0|\|v_0\|\int_0^T |C(t)y|\,dt. \end{aligned} \tag{3.42}$$

Observe that $|C(t)y| = |y|$ for all $t \in [0,T]$, because $\{C(t)\}_{t\ge 0}$ is a family of orthogonal matrices of $\mathbb{R}^n$. Therefore,

$$\int_0^T |C(t)y|\,dt \le M(T,\Omega). \tag{3.43}$$

Substituting (3.43) in (3.42) we obtain

$$-2\alpha\int_0^T \operatorname{Im}(v_0, Cy\cdot\nabla v_0)\exp\left(-\int_0^t \operatorname{tr}(C)dr\right)dt \ge -2\alpha M(T,\Omega)|v_0|\|v_0\|. \tag{3.44}$$

From Young's inequality we get

$$-2\alpha\int_0^T \operatorname{Im}(v_0, Cy\cdot\nabla v_0)\exp\left(-\int_0^t \operatorname{tr}(C)dr\right)dt \ge \frac{-(\alpha M(T,\Omega))^2}{\epsilon}|v_0|^2 - \epsilon\|v_0\|^2, \tag{3.45}$$

where $\epsilon > 0$.

(i) Analysis of $2\alpha\int_0^T \operatorname{Im}(v, Cy\cdot\nabla v)dt$.

We observe that from the second estimate there exists a constant $\alpha_1 > 0$ such that

$$\|v(t)\| \le K_1\|v_0\|. \tag{3.46}$$

Therefore, from Young's inequality, it follows that

$$
\begin{aligned}
2\alpha\int_0^T \operatorname{Im}(v, Cy\cdot\nabla v)dt &\geq -2\alpha_2\int_0^T |v_0|\,|Cy|\,||v_0||dt \\
&\geq \frac{-M(\Omega,T)^2}{\epsilon}|v_0|^2 - \epsilon||v_0||^2.
\end{aligned}
\tag{3.47}
$$

(ii) Analysis of $-2\alpha\int_0^T\int_0^t \operatorname{Im}(v, C'y\cdot\nabla v)\exp(-\int_0^t \operatorname{tr}(C)dr)ds\,dt$.

$$
\begin{aligned}
&-2\alpha\int_0^T\int_0^t \operatorname{Im}(v, C'y\cdot\nabla v)\exp\Big(-\int_0^t \operatorname{tr}(C)dr\Big)ds\,dt \\
&\quad\geq -2\alpha\int_0^T\int_0^t |v|\,|C'y|\,\|v\|ds\,dt \geq -2\alpha_3\int_0^T\int_0^t |v_0|\,|C'y|\,||v_0||ds\,dt \\
&\quad\geq \frac{(M(\Omega,T,n))}{\epsilon}|v_0|^2 - \epsilon||v_0||^2.
\end{aligned}
\tag{3.48}
$$

Substituting (3.42), (3.45), (3.47), and (3.48) in (3.36) we obtain

$$
2\int_0^T \operatorname{Re}(C\cdot\nabla v,\nabla v)dt \geq \big(2M(\alpha,T)-3\epsilon\big)||v_0||^2 - \frac{(M(\alpha,T,n,\Omega))^2}{\epsilon}|v_0|^2.
\tag{3.49}
$$

Choosing $\epsilon > 0$ such that $0<\epsilon<2M(\alpha,T)/3$, then we have

$$
2\int_0^T \operatorname{Re}(C\cdot\nabla v,\nabla v)dt \geq \alpha_4(T)||v_0||^2 - \alpha_5(T)|v_0|^2,
\tag{3.50}
$$

where $\alpha_4(T)$ and $\alpha_5(T)$ are positive constants.

(iii) Analysis of $-2\int_0^T \operatorname{Im}(Cy\cdot\nabla v, Cy_0\cdot\nabla v)dt$.

$$
\begin{aligned}
-2\int_0^T \operatorname{Im}(Cy\cdot\nabla v, Cy_0\cdot\nabla v)dt &\geq -2\int_0^T |Cy|\,\|v\|\,|Cy_0|\,\|v\|dt \\
&\geq -2M(\Omega)\int_0^T \|v\|^2 dt.
\end{aligned}
\tag{3.51}
$$

From (3.35) and (3.40), it follows that

$$
\begin{aligned}
\int_0^T \|v\|^2 dt &\leq ||v_0||^2 T + \int_0^T |(v_0, Cy\cdot\nabla v_0)|dt \\
&\quad + \int_0^T |(v, Cy\cdot\nabla v)|dt + T\int_0^T |(v, C'y\cdot\nabla v)|dt \\
&\leq ||v_0||^2 T \leq \frac{M(\Omega,T,n)}{2\epsilon}|v_0|^2 + \frac{3\epsilon}{2}||v_0||^2.
\end{aligned}
\tag{3.52}
$$

Substituting (3.52) in (3.51), we get

$$
\begin{aligned}
&-2\int_0^T \operatorname{Im}(Cy\cdot\nabla v, Cy_0\cdot\nabla v)dt \\
&\quad\geq -2M(\Omega,T)||v_0||^2 - \frac{M(\Omega,T,n)}{\epsilon}|v_0|^2 - \epsilon M(\Omega)||v_0||^2.
\end{aligned}
\tag{3.53}
$$

(iv) Analysis of $-2\int_0^T \operatorname{Im}(\operatorname{tr}(C)v, Cy_0\cdot\nabla v)dt$.

$$\begin{aligned}-2\int_0^T \operatorname{Im}\left(\operatorname{tr}(C)v, Cy_0\cdot\nabla v\right)dt &\geq -2\int_0^T |\operatorname{tr}(C)v|\,|Cy_0|\,|\nabla v|\,dt\\ &\geq -\frac{(M(T,\Omega))^2}{\epsilon}|v_0|^2-\epsilon\|v_0\|^2.\end{aligned}\tag{3.54}$$

(v) Analysis of $-\int_0^T \operatorname{Im}(Cy\cdot\nabla v, \operatorname{tr}(C)v)dt$.

$$\begin{aligned}&-\int_0^T \operatorname{Im}\left(Cy\cdot\nabla v,(\operatorname{tr}C)v\right)dt\\ &\quad\geq -\int_0^T |Cy|\,\|v\|\,|(\operatorname{tr}C)v|\,dt \geq -\int_0^T M(\Omega,T)\|v(t)\|\,|v(t)|\,dt\\ &\quad\geq -M(\Omega,T,\alpha_1)\|v_0\|\,|v_0| \geq -\frac{(M(\Omega,T,\alpha_1))^2}{2\epsilon}|v_0|^2-\frac{\epsilon}{2}\|v_0\|^2.\end{aligned}\tag{3.55}$$

We also have

$$(v, Cm(y)\cdot\nabla v)\big|_0^T=(v(T), Cm(y)\cdot\nabla v(T))-(v_0, Cm(y)\cdot\nabla v_0),\tag{3.56}$$

therefore, from Young's inequality, it follows that

$$(v, Cm(y)\cdot\nabla v)\big|_0^T \geq \frac{(M(\Omega,\alpha_1))^2}{2\epsilon}|v_0|^2-\frac{\epsilon}{2}\|v_0\|^2.\tag{3.57}$$

Substituting (3.50), (3.53), (3.54), (3.55), and (3.57) in (3.32) we obtain, after some computations, that the weak solution of (2.2) satisfies (3.33).

Step 2. We prove in this step that if v is a weak solution of (2.2) then there exists a constant $\lambda>0$ such that

$$\lambda|v_0|^2 \leq \int_0^T\int_{\Gamma(y_0)}(Cm(y)\cdot\nu)\left|\frac{\partial v}{\partial\nu}\right|^2 d\Gamma\,dt.\tag{3.58}$$

In fact, we argue by contradiction. Suppose (3.58) is false. Considering $v_0\in H_0^1(\Omega)$, there exists a sequence (v_μ) of strong solutions of (2.2), $v_\mu(0)=v_{0\mu}$, such that

$$\int_0^T\int_{\Gamma(y_0)}(Cm(y)\cdot\nu)\left|\frac{\partial v}{\partial\nu}\right|^2 d\Gamma\,dt \leq \frac{1}{\mu}|v_{0\mu}|^2,\tag{3.59}$$

and we can suppose $|v_{0\mu}|=1$. Then, from (3.59), we obtain

$$\lim_{\mu\to\infty}\int_0^T\int_{\Gamma(y_0)}(Cm(y)\cdot\nu)\left|\frac{\partial v}{\partial\nu}\right|^2 d\Gamma\,dt=0\tag{3.60}$$

strongly.

In Step 1 we proved that

$$C_1 \left| v_{0\mu} \right|^2 + \int_0^T \int_{\Gamma(y_0)} (Cm(y) \cdot \nu) \left| \frac{\partial v_\mu}{\partial \nu} \right|^2 d\Gamma\, dt \geq C_2 \|v_{0\mu}\|^2. \tag{3.61}$$

The left-hand side of (3.61) is bounded, then it follows that $(v_{0\mu})$ is bounded in $H_0^1(\Omega)$. Since the embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact, it follows that we can extract a subsequence, still represented by $(v_{0\mu})$, such that

$$\lim_{\mu \to \infty} v_{0\mu} = v_0 \quad \text{strongly in } L^2(\Omega). \tag{3.62}$$

Since $|v_{0\mu}| = 1$, it follows that

$$\left| v_0 \right| = 1. \tag{3.63}$$

We also have

$$\left| v_\mu(t) - v_\eta(t) \right| \leq C(T) \left| v_{0\mu} - v_{0\eta} \right|, \tag{3.64}$$

which implies that

$$\lim_{\mu \to \infty} v_\mu = v \quad \text{in } C^0([0,T]; L^2(\Omega)). \tag{3.65}$$

Integrating by parts for $\xi \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega))$ and $\xi' \in L^\infty(0,T; L^2(\Omega))$, with $\xi(0) = \xi(T) = 0$, we obtain

$$\int_Q v_\mu \left(-\overline{\xi'} - i\Delta\overline{\xi} - (Cy \cdot \nabla\overline{\xi}) \right) dy\, dt = 0. \tag{3.66}$$

When $\mu \to \infty$ in (3.66) we have

$$\left| \begin{aligned} &\int_Q v\left(\overline{\xi'} + i\Delta\overline{\xi} + (Cy \cdot \nabla\overline{\xi})\right) dy\, dt = 0, \\ &v(0g) = v_0. \end{aligned} \right. \tag{3.67}$$

We transform (3.67) into a noncylindrical problem on $\widehat{Q}$ by the mapping $y = K^{-1}(t)x$, $x \in \widehat{Q}$. Consider $\theta(x,t) = v(K^{-1}(t)x, t)$, where v is the weak solution, then $\theta \in C^0([0,T]; H_0^1(\Omega))$.

Represent by $\widehat{G}$ a bounded convex set of $\mathbb{R}^n$ such that its closure contains $\widehat{Q}$ and $\widehat{\Sigma}(y_0)$. Let $\mathbb{O}$ be $\widehat{G} \cap \widehat{Q} \neq \varnothing$ and let $\widetilde{\theta}(x,t)$ equal to $\theta(x,t)$ on $\widehat{Q}$ and zero outside. We prove that

$$\widetilde{\theta}' - i\Delta\widetilde{\theta} = 0 \quad \text{in the sense of } \mathcal{D}'(\widehat{G}). \tag{3.68}$$

By definition, $\widetilde{\theta} = 0$ on $\widehat{G} \setminus \widehat{Q}$, then $\widetilde{\theta} = 0$ on $\widehat{G}$ by Holmgren's theorem. Therefore, $\theta = \widetilde{\theta} = 0$ on $\widehat{Q}$ and then $v = 0$. This is a contradiction because $|v_0| = 1$. $\square$

4. Ultra weak solutions or solutions by transposition

Let us consider the following nonhomogeneous mixed problem:

$$\left| \begin{array}{ll} Lv = 0 & \text{on } Q, \\ v = w & \text{on } \Sigma, \\ v(0) = v_0 & \text{in } \Omega, \end{array} \right. \tag{4.1}$$

where $Lv = v' - i\Delta v + Cy \cdot \nabla v$ was defined in Section 2.

We are looking for a concept of a solution for (4.1). Let us suppose $w \in L^2(\Sigma)$, $v_0 \in H^{-1}(\Omega)$ and consider a function $\theta = \theta(y,t)$ such that $\theta = 0$ on Σ and $\theta(T) = \theta(y,T) = 0$ for $y \in \Omega$. Multiplying both sides of $(4.1)_1$ by $\overline{\theta}$ and integrating on Q, we obtain

$$\int_0^T \int_\Omega (v' - i\Delta v + Cy \cdot \nabla v)\overline{\theta}\, dy\, dt = 0. \tag{4.2}$$

We have the following.

(i)

$$\int_0^T \int_\Omega v'\overline{\theta} dy\, dt = \int_0^T (v', \theta) dt = (v(T), \theta(T)) - (v(0), \theta(0)) - \int_0^T (v, \theta') dt \tag{4.3}$$

or

$$\int_0^T \int_\Omega v'\overline{\theta} dy\, dt = -(v(0), \theta(0)) - \int_0^T \int_\Omega v\overline{\theta'} dy\, dt. \tag{4.4}$$

(ii) $-\int_0^T \int_\Omega i\Delta v\overline{\theta}\, dy\, dt$.

By Green's formula,

$$\begin{aligned} -\int_\Omega \Delta v\overline{\theta} dy &= \int_\Omega \nabla v \nabla\overline{\theta}\, dy - \int_\Gamma \frac{\partial v}{\partial \nu}\overline{\theta}\, d\Gamma, \\ -\int_\Omega v\Delta\overline{\theta} dy &= \int_\Omega \nabla v \nabla\overline{\theta}\, dy - \int_\Gamma v\frac{\partial \overline{\theta}}{\partial \nu} d\Gamma. \end{aligned} \tag{4.5}$$

Observe that by hypothesis $\theta = 0$ on Σ. Then,

$$-i\int_0^T \int_\Omega \Delta v\overline{\theta}\, dy\, dt = -i\int_0^T \int_\Omega v\Delta\overline{\theta}\, dy\, dt + i\int_0^T \int_\Gamma v\frac{\partial\overline{\theta}}{\partial \nu} d\Gamma\, dt. \tag{4.6}$$

(iii) $\int_0^T \int_\Omega Cy \cdot \nabla v\overline{\theta}\, dy\, dt$.

By Gauss' lemma we have

$$\int_\Omega \frac{\partial}{\partial y_l}(v c_{lk} y_k \overline{\theta})\, dy = 0, \tag{4.7}$$

because $\theta = 0$ on Σ.

From (4.7), it follows that

$$\int_\Omega \frac{\partial v}{\partial y_l} c_{lk} y_k \overline{\theta}\, dy + \int_\Omega v c_{lk} \delta_l^k \overline{\theta}\, dy + \int_\Omega v c_{lk} y_k \frac{\partial \overline{\theta}}{\partial y_l} dy = 0 \tag{4.8}$$

or

$$\int_0^T \int_\Omega Cy \cdot \nabla v \overline{\theta}\, dy\, dt = -\int_0^T \int_\Omega \operatorname{tr}(C(t)) v \overline{\theta}\, dy\, dt - \int_0^T \int_\Omega v Cy \cdot \nabla \overline{\theta}\, dy\, dt. \tag{4.9}$$

Adding (4.4), (4.6), and (4.9) and observing (4.2), we get

$$-(v_0, \theta(0)) + \int_0^T \int_\Gamma v \frac{\partial \overline{\theta}}{\partial \nu} d\Gamma dt + \int_Q v\left(-\overline{\theta'} - i\Delta\overline{\theta} - \operatorname{tr}(C(t))\overline{\theta} - Cy \cdot \nabla\overline{\theta}\right) dy\, dt = 0. \tag{4.10}$$

Hence,

$$-\int_Q v \overline{(\theta' - i\Delta\theta + \operatorname{tr}(C)\theta + Cy \cdot \nabla\theta)}\, dy\, dt = (v_0, \theta(0)) - i\int_0^T \int_\Gamma v \frac{\partial \overline{\theta}}{\partial \nu} d\Gamma\, dt. \tag{4.11}$$

Now, we formulate the concept of solution by transposition or ultra weak solution for the problem (4.1).

Given $f \in L^1(0,T;H_0^1(\Omega))$, let θ be the weak solution of the following backward mixed problem

$$\left|\begin{array}{ll} L^*\theta = -f & \text{in } Q, \\ \theta = 0 & \text{on } \Sigma, \\ \theta(T) = 0 & \text{on } \Omega, \end{array}\right. \tag{4.12}$$

where $L^*\theta = \theta' - i\Delta\theta + \operatorname{tr}(C)\theta + Cy \cdot \nabla\theta$ according to Section 2.

Reversing time in (4.12) and defining $\hat{\theta}(y,t) = \theta(y,T-t)$, $\hat{f}(y,t) = f(y,T-t)$, and $\hat{C}(t) = -C(T-t)$, we obtain

$$\hat{\theta}' + i\Delta\hat{\theta} + \operatorname{tr}(\hat{C})\hat{\theta} + \hat{C}y \cdot \nabla\hat{\theta} = \hat{f}. \tag{4.13}$$

Then, we obtain from (4.12) the equivalent problem

$$\left|\begin{array}{ll} \overline{\hat{\theta}'} - i\Delta\overline{\hat{\theta}} + \operatorname{tr}(\hat{C})\overline{\hat{\theta}} + \hat{C}y \cdot \nabla\overline{\hat{\theta}} = \overline{\hat{f}} & \text{in } Q, \\ \overline{\hat{\theta}} = 0 & \text{on } \Sigma, \\ \overline{\hat{\theta}}(0) = 0 & \text{on } \Omega, \end{array}\right. \tag{4.14}$$

which, by Section 2, Theorem 2.1, has only one weak solution $\overline{\hat{\theta}}$. Therefore, we have the following definition.

Definition 4.1. For $v_0 \in H^{-1}(\Omega)$ and $w \in L^2(\Sigma)$, the solution by transposition of the problem (4.1) is called the unique function $v \in L^\infty(0,T;H^{-1}(\Omega))$ such that

$$\int_0^T \langle v(t), f(t)\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} dt = \langle v_0, \theta(0)\rangle - \int_\Sigma wi\frac{\partial\overline{\theta}}{\partial\nu} d\Gamma\, dt \tag{4.15}$$

for all $f \in L^1(0,T;H_0^1(\Omega))$, where θ is the weak solution of (4.12).

Theorem 4.2. *Given $v_0 \in H^{-1}(\Omega)$ and $w \in L^2(\Sigma)$, there exists only one ultra weak solution $v \in L^\infty(0,T;H^{-1}(\Omega))$ of the nonhomogeneous boundary value problem (4.1).*

Proof. The existence is a consequence of Riesz representation theorem for continuous linear functional on $L^1(0,T;H_0^1(\Omega))$.

For the uniqueness, suppose we have two ultra weak solutions v and $\hat{v}$ corresponding to v_0 and f. Then, by definition of ultra weak solution, we have

$$\langle v - \hat{v}, f\rangle = 0, \tag{4.16}$$

where $\langle\cdot,\cdot\rangle$ is the duality between $L^\infty(0,T;H^{-1}(\Omega))$ and $L^1(0,T;H_0^1(\Omega))$. Then, by Hahn-Banach theorem, we have $v - \hat{v} = 0$. □

5. Exact controllability

In this section, at first, we solve the problem of the exact controllability for the nonhomogeneous boundary value problem on the cylinder Q:

$$\left|\begin{array}{l} v' - i\Delta v + Cy\cdot\nabla v = 0 \quad \text{on } Q, \\ v = w \quad \text{on } \Sigma, \\ v(0) = v_0 \quad \text{in } \Omega. \end{array}\right. \tag{5.1}$$

Later, we obtain the exact controllability result for the nonhomogeneous mixed problem (1.5) on the noncylindrical domain $\widehat{Q}$.

The problem of exact controllability for (5.1) can be formulated as follows: given $T > 0$, find a Hilbert space H such that for each initial data $v_0 \in H$, there exists a control w belonging to a space of controls on Σ such that the corresponding solution $v(y,t,w)$ of (5.1) verifies the condition

$$v(y,T,w) = 0 \quad \forall y \in \Omega. \tag{5.2}$$

Theorem 5.1. *Let Ω be a regular, bounded open set of $\mathbb{R}^n$ and suppose that* (H1) *and* (H2) *hold. If $T > 0$, then, for each $v_0 \in H^{-1}(\Omega)$, there exists a control $w \in L^2(\Sigma)$ such that v, solution of problem (5.1), satisfies the condition*

$$v(y,T,w) = 0 \quad \forall y \in \Omega. \tag{5.3}$$

Proof. In order to prove the exact controllability for (5.1), we employ HUM (hilbert uniqueness method) idealized by Lions [8]. We describe the method by steps.

Step 1. Given $\phi_0 \in \mathcal{D}(\Omega)$, let us consider the adjoint problem

$$\left| \begin{array}{l} \phi' - i\Delta\phi + Cy \cdot \nabla\phi + \operatorname{tr}(C)\phi = 0 \quad \text{in } Q, \\ \phi = 0 \quad \text{on } \Sigma, \\ \phi(0) = \phi_0 \quad \text{in } \Omega. \end{array} \right. \tag{5.4}$$

We know by Section 1 that (5.4) has a strong solution. By the direct inequality, we obtain

$$\frac{\partial \phi}{\partial \nu} \in L^2(\Sigma). \tag{5.5}$$

Step 2. We solve the nonhomogeneous backward problem

$$\left| \begin{array}{l} \psi' - i\Delta\psi + Cy \cdot \nabla\psi = 0 \quad \text{in } Q, \\ \psi = \begin{cases} -i\dfrac{\partial \phi}{\partial \nu} & \text{on } \Sigma(y_0), \\ 0 & \text{in } \Sigma \setminus \Sigma(y_0), \end{cases} \\ \psi(T) = 0. \end{array} \right. \tag{5.6}$$

Note that (5.6) is a nonhomogeneous backward problem of the type studied in Section 4. To obtain, from (5.6), the system (4.1) of Section 4, it is sufficient to consider the change of variable $T - t$ in place of t.

The operator Λ. From the solution ψ of (5.6), we define the application

$$\Lambda\{\phi_0\} = \psi(0). \tag{5.7}$$

Observe that, from $\phi_0 \in \mathcal{D}(\Omega)$, we obtain the solution ϕ of (5.4) with regularity (5.5) for the normal derivative. Then, the problem (5.6) is well posed, from which we define Λ.

Step 3. Multiplying both sides of (5.6) by ϕ, the solution of (5.4), and integrating on Q we obtain

$$0 = \langle L\psi, \phi \rangle = \langle \psi, L^*\phi \rangle + \langle \psi(T), \phi(T) \rangle - \langle \psi(0), \phi(0) \rangle + i \int_0^T \int_\Gamma \psi \frac{\partial \overline{\phi}}{\partial \nu} d\Gamma\, dt. \tag{5.8}$$

That is,

$$0 = -\langle \psi(0), \phi_0 \rangle + \int_0^T \int_\Gamma \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma\, dt. \tag{5.9}$$

Therefore,

$$\langle \psi(0), \phi_0 \rangle = \int_0^T \int_\Gamma \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma\, dt. \tag{5.10}$$

Substituting (5.7) in (5.10), we obtain

$$\langle \Lambda\{\phi_0\}, \phi_0 \rangle = \int_0^T \int_\Gamma \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma\, dt. \tag{5.11}$$

If we consider ϕ_0, $\zeta_0 \in \mathcal{D}(\Omega)$ and we represent by ϕ and ζ the corresponding weak solution of (5.4), we obtain, by the same argument used to obtain (5.10), that

$$\langle \Lambda\{\phi_0\}, \zeta_0 \rangle = \int_0^T \int_\Gamma \frac{\partial \phi}{\partial \nu} \frac{\partial \overline{\zeta}}{\partial \nu} d\Gamma\, dt. \tag{5.12}$$

We define in $\mathcal{D}(\Omega)$ the quadratic form

$$||\phi_0||_F^2 = \int_0^T \int_\Gamma \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma\, dt. \tag{5.13}$$

Remark 5.2. It follows from the direct inequality that the quadratic form defined above is a norm in $\mathcal{D}(\Omega)$, induced by the inner product

$$(\phi, \zeta)_F = \int_0^T \int_\Gamma \frac{\partial \phi}{\partial \nu} \frac{\partial \overline{\zeta}}{\partial \nu} d\Gamma\, dt \tag{5.14}$$

defined in $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$.

Let us represent by F the completion of $\mathcal{D}(\Omega)$ with the Hilbertian norm (5.13).

It follows from the remark above and (5.11) that

$$\langle \Lambda\{\phi_0\}, \zeta_0 \rangle = (\phi_0, \zeta_0). \tag{5.15}$$

Then, $b(\phi_0, \zeta_0) = \langle \Lambda\{\phi_0\}, \zeta_0 \rangle$ is a sesquilinear form, Hermitian, and strictly positive in $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$. Its continuity follows from Schwarz inequality. In fact,

$$| b(\phi_0, \zeta_0) | = | \langle \Lambda\{\phi_0\}, \zeta_0 \rangle | \leq ||\phi_0||_F ||\zeta_0||_F. \tag{5.16}$$

It follows that $b(\phi_0, \zeta_0)$ has a unique extension by closure, to the completion F of $\mathcal{D}(\Omega)$. Let us still represent by $b(\phi_0, \zeta_0)$ this extension. It is sesquilinear, Hermitian, and strictly positive in F. Then, by Lax-Milgram's lemma, given $v_0 \in F'$, dual of F, there exists a unique $\phi_0 \in F$ such that

$$\langle \Lambda\{\phi_0\}, \zeta_0 \rangle = b(\phi_0, \zeta_0) = \langle v_0, \zeta_0 \rangle \quad \forall \zeta_0 \in F. \tag{5.17}$$

This means that, given $v_0 \in F'$, there exists a unique $\phi_0 \in F$ such that

$$\Lambda\{\phi_0\} = v_0 \quad \text{in } F'. \tag{5.18}$$

But, by (5.7), we have $\Lambda\{\phi_0\} = \psi(0)$, which implies that the ultra weak solution ψ of the backward problem (5.6) satisfies the initial condition $\psi(0) = v_0$. It follows, by uniqueness, that $\psi = v$ and then $v(T) = 0$, which is the condition (5.2).

We need only to characterize, in terms of space of functions on Ω, the completion F of $\mathcal{D}(\Omega)$ with respect to the norm $\| \cdot \|_F$ given by (5.13).

In fact, from inverse and direct inequalities, we have

$$C_0 ||\phi_0||^2 \leq \int_0^T \int_\Gamma \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma\, dt \leq C_1 ||\phi_0||^2, \tag{5.19}$$

which implies that the norm $\|\cdot\|_F$ in $\mathfrak{D}(\Omega)$, defined in (5.13), is equivalent to the norm $H_0^1(\Omega)$. Therefore, $F = H_0^1(\Omega)$ and $F' = H^{-1}(\Omega)$. □

The proof of Theorem 1.1 follows from the previous theorem and the diffeomorphism τ defined in Section 1.

Appendix

The objective of this appendix is to prove an identity in order to modify the surface integral

$$\int_\Gamma \nu_j \cdot c_{jk}(t) y_k \left| \frac{\partial v_m}{\partial \nu} \right|^2 d\Gamma \tag{A.1}$$

that appears in the second estimate.

In fact, let us consider the approximate equation

$$(v_m'(t), w_j) + i((v_m(t), w_j)) + (C(t)y \cdot \nabla v_m(t), w_j) + \operatorname{tr}(C(t))(v_m(t), w_j) = (f(t), w_j). \tag{A.2}$$

Let us consider $L^2(\Omega) = V_m \oplus V_m^\perp$ and P_m the orthogonal projection from $L^2(\Omega)$ on V_m. We know that

(i) P_m is bounded, self-adjoint;

(ii) $P_m w = w$ for all $w \in V_m$;

(iii) $P_m w = \sum_{j=1}^m (w, w_j) w_j$ for each $w \in L^2(\Omega)$.

Multiplying both sides of the approximate equation (A.2) by w_j and adding for $1 \le j \le m$, we obtain that

$$v_m' - i\Delta v_m + P_m[C(t)y \cdot \nabla v_m] + \operatorname{tr}(C(t)) v_m = P_m f. \tag{A.3}$$

Taking the inner product of both sides of (A.3) with $C(t)y \cdot \nabla v_m$, we obtain

$$\begin{aligned}(v_m', C(t)y \cdot \nabla v_m) - i(\Delta v_m, C(t)y \cdot \nabla v_m) + (P_m[C(t)y \cdot \nabla v_m], C(t)y \cdot \nabla v_m) \\ + \operatorname{tr}(C(t))(v_m(t), C(t)y \cdot \nabla v_m) = (P_m f, C(t)y \cdot \nabla v_m).\end{aligned} \tag{A.4}$$

Note that $P_m^2 = P_m$, then $(P_m[C(t)y \cdot \nabla v_m], C(t)y \cdot \nabla v_m) = |P_m[C(t)y \cdot \nabla v_m]|^2$ is real. Taking the double of the imaginary parts of both sides of (A.5), we have

$$\begin{aligned}2\operatorname{Im}(v_m', C(t)y \cdot \nabla v_m) - 2\operatorname{Im} i(\Delta v_m, C(t)y \cdot \nabla v_m) \\ + 2\operatorname{tr}(C(t))\operatorname{Im}(v_m(t), C(t)y \cdot \nabla v_m) = 2\operatorname{Im}(P_m f, C(t)y \cdot \nabla v_m).\end{aligned} \tag{A.5}$$

Observing that $\operatorname{Im}(iz) = \operatorname{Re} z$, we modify (A.5), obtaining

$$\begin{aligned}2\operatorname{Im}(v_m', C(t)y \cdot \nabla v_m) - 2\operatorname{Re}(\Delta v_m, C(t)y \cdot \nabla v_m) \\ + 2\operatorname{tr}(C(t))\operatorname{Im}(v_m(t), C(t)y \cdot \nabla v_m) = 2\operatorname{Im}(P_m f, C(t)y \cdot \nabla v_m).\end{aligned} \tag{A.6}$$

(i) Analysis of $2\operatorname{Im}(v_m', C(t)y \cdot \nabla v_m)$.

We have

$$\frac{d}{dt}(v_m, C(t)y\cdot\nabla v_m) = (v_m', C(t)y\cdot\nabla v_m) + (v_m, C'(t)y\cdot\nabla v_m) + (v_m, C(t)y\cdot\nabla v_m'). \tag{A.7}$$

By Gauss' lemma,

$$\int_\Omega \frac{\partial}{\partial y_l}(v_m C_{lk} y_k \overline{v_m'})\,dy = 0, \tag{A.8}$$

that is,

$$\int_\Omega \frac{\partial v_m}{\partial y_l} C_{lk} y_k \overline{v_m'}\,dy + \int_\Omega v_m C_{lk}\delta_l^k \overline{v_m'}\,dy + \int_\Omega v_m C_{lk} y_k \frac{\partial \overline{v_m'}}{\partial y_l}\,dy = 0. \tag{A.9}$$

Therefore, from (A.9), we get

$$-(v_m, C(t)y\cdot\nabla v_m') = \mathrm{tr}(C)(v_m, v_m') + (C(t)y\cdot\nabla v_m, v_m'). \tag{A.10}$$

Substituting (A.10) in (A.7), we obtain

$$\begin{aligned}(v_m', C(t)y\cdot\nabla v_m) &- (C(t)y\cdot\nabla v_m, v_m')\\ &= \frac{d}{dt}(v_m, C(t)y\cdot\nabla v_m) - (v_m, C'(t)y\cdot\nabla v_m) + \mathrm{tr}(C)(v_m, v_m').\end{aligned} \tag{A.11}$$

Note that $z-\overline{z} = 2i\,\mathrm{Im}\,z$ and $-i(z-\overline{z}) = 2\,\mathrm{Im}\,z$, which implies

$$2\,\mathrm{Im}\,(v_m', C(t)y\cdot\nabla v_m) = -i\frac{d}{dt}(v_m, C(t)y\cdot\nabla v_m) + i(v_m, C'(t)y\cdot\nabla v_m) - i\,\mathrm{tr}(C)(v_m, v_m'). \tag{A.12}$$

Taking the real parts of both sides in the last equality, we obtain

$$\begin{aligned}2\,\mathrm{Im}\,(v_m', C(t)y\cdot\nabla v_m) = \mathrm{Im}\,&\frac{d}{dt}(v_m, C(t)y\cdot\nabla v_m)\\ &- \mathrm{Im}\,(v_m, C'(t)y\cdot\nabla v_m) + \mathrm{Im}\,\mathrm{tr}(C)(v_m, v_m').\end{aligned} \tag{A.13}$$

From the projection (A.3), we have

$$-v_m' = -i\Delta v_m + P_m[C(t)y\cdot\nabla v_m] + \mathrm{tr}\,(C(t))v_m - P_m f. \tag{A.14}$$

Taking the inner product of both sides of (A.14) with v_m and taking the imaginary parts of both sides, we obtain

$$\mathrm{Im}\,(-v_m', v_m) = -\mathrm{Im}\,i(\Delta v_m, v_m) + \mathrm{Im}\,(P_m[C(t)y\cdot\nabla v_m, v_m]) - \mathrm{Im}\,(P_m f, v_m). \tag{A.15}$$

Observe that $(P_m[C(t)y\cdot\nabla v_m], v_m) = (C(t)y\cdot\nabla v_m, v_m)$, because P_m is self-adjoint and $P_m v_m = v_m$.

Therefore, from (A.15), it follows that

$$\operatorname{Im}(-v'_m, v_m) = -\operatorname{Re}(\Delta v_m, v_m) + \operatorname{Im}(C(t)y \cdot \nabla v_m, v_m) - \operatorname{Im}(P_m f, v_m). \tag{A.16}$$

From (A.16), we obtain

$$\begin{aligned} \operatorname{tr}(C(t)) \operatorname{Im}(v_m, v'_m) = {} & \operatorname{tr}(C(t)) \|v_m\|^2 + \operatorname{tr}(C(t)) \operatorname{Im}(C(t)y \cdot \nabla v_m, v_m) \\ & - \operatorname{tr}(C(t)) \operatorname{Im}(P_m f, v_m). \end{aligned} \tag{A.17}$$

We modify (A.13), by means of (A.17) and obtain

$$\begin{aligned} 2\operatorname{Im}(v'_m, C(t)y \cdot \nabla v_m) = {} & \operatorname{Im} \frac{d}{dt}(v_m, C(t)y \cdot \nabla v_m) - \operatorname{Im}(v_m, C'(t)y \cdot \nabla v_m) \\ & + \operatorname{tr}(C(t)) \|v_m\|^2 + \operatorname{tr}(C(t)) \operatorname{Im}((t)y \cdot \nabla v_m, v_m) \\ & - \operatorname{tr}(C(t)) \operatorname{Im}(P_m f, v_m). \end{aligned} \tag{A.18}$$

From the second estimate, we have

$$\begin{aligned} 2\operatorname{Re}(\nabla v_m(t) \cdot C(t)y, -\Delta v_m(t)) = {} & 2\operatorname{Re}(\nabla v_m(t) \cdot C(t)y, \nabla v_m(t)) \\ & - \operatorname{tr}(C(t)) \|v_m(t)\|^2 - \int_\Gamma C(t)y \cdot \nu \left| \frac{\partial v_m}{\partial \nu} \right|^2 d\Gamma. \end{aligned} \tag{A.19}$$

Substituting (A.18) and (A.19) in (A.6), it follows that

$$\begin{aligned} & -\int_\Gamma C(t)y \cdot \nu \left| \frac{\partial v_m}{\partial \nu} \right|^2 d\Gamma + 2\operatorname{Re}(\nabla v_m(t) \cdot C(t)y, \nabla v_m(t)) - \operatorname{tr}(C(t)) \|v_m(t)\|^2 \\ & \quad = 2\operatorname{Im}(P_m f, \nabla v_m(t) \cdot C(t)y) + \operatorname{tr}(C(t)) \operatorname{Im}(P_m f, v_m) \\ & \qquad - \operatorname{tr}(C(t)) \operatorname{Im}(v_m(t), C(t)y \cdot \nabla v_m) - \operatorname{tr}(C(t)) \|v_m\|^2 \\ & \qquad - \frac{d}{dt} \operatorname{Im}(v_m, C(t)y \cdot \nabla v_m) + \operatorname{Im}(v_m, C'(t)y \cdot \nabla v_m). \end{aligned} \tag{A.20}$$

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