EXISTENCE OF BLOWUP SOLUTIONS FOR NONLINEAR PROBLEMS WITH A GRADIENT TERM

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We prove the existence of positive explosive solutions for the equation $\Delta u + \lambda(|x|)|\nabla u(x)| = \varphi(x, u(x))$ in the whole space $\mathbb{R}^N$ $(N \geq 3)$, where $\lambda : [0,\infty) \to [0,\infty)$ is a continuous function and $\varphi : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ is required to satisfy some hypotheses detailed below. More precisely, we will give a necessary and sufficient condition for the existence of a positive solution that blows up at infinity.

1. Introduction and the main result

Semilinear elliptic problems involving gradient term with boundary blowup interested many authors. Namely, Bandle and Giarrusso [1] developed existence and asymptotic behaviour results for large solutions of

$$\Delta u + |\nabla u(x)|^a = f(u) \tag{1.1}$$

in a bounded domain.

In the case $f(u) = p(x)u^\gamma$, $a > 0$, and $\gamma > \max(1, a)$, Lair and Wood [7] dealt with the above equation in bounded domain and the whole space. They proved the existence of entire large solution under the condition $\int_0^\infty r \max_{|x|=r} p(x)dr < \infty$ when the domain is $\mathbb{R}^N$.

Recall that u is a large solution on a bounded domain Ω in $\mathbb{R}^N$, if $u(x) \to +\infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$, and u is called an entire large solution if u is defined on $\mathbb{R}^N$ and $\lim_{|x|\to+\infty} u(x) = +\infty$.

Ghergu et al. [3] considered more general equation

$$\Delta u + q(x)|\nabla u(x)|^a = p(x)f(u), \tag{1.2}$$

where $0 \leq a \leq 2$, p and q are Hölder continuous functions on $(0,\infty)$. We note that the Keller-Osserman condition on f (see [6, 8]) remains the key condition for the existence for their works.

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In the present paper, we are interested in the study of nonlinear elliptic problems with boundary blowup, of the type

$$\begin{gathered}\Delta u+\lambda(|x|)\,|\nabla u(x)| = \varphi(x,u(x)), \quad \text{in } \mathbb{R}^N,\\ u \geq 0, \qquad u \not\equiv 0,\\ \lim_{|x|\to+\infty} u(x) = +\infty,\end{gathered} \tag{P}$$

where $\lambda : [0,\infty) \to [0,\infty)$ is a continuous function and φ satisfies the following hypotheses.

(H_1) $\varphi : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ is measurable, continuous with respect to the second variable.

(H_2) There exist nonnegative functions p, q, and f satisfying for each $x \in \mathbb{R}^N$ and $t \geq 0$,

$$p(|x|)f(t) \leq \varphi(x,t) \leq q(|x|)f(t), \tag{1.3}$$

where f is required to satisfy.

(H_3) $f \in \mathscr{C}^1([0,\infty))$ such that $f' \geq 0$, $f(0) = 0$, $f > 0$ on $(0,\infty)$,

$$\int_1^\infty \frac{1}{f(\zeta)}d\zeta = +\infty, \tag{1.4}$$

and p, q are allowed to verify.

(H_4) $p,q : (0,\infty) \to [0,\infty)$ are continuous functions satisfying

$$\int_0^1 s(1-s)q(s)ds < +\infty. \tag{1.5}$$

Clearly, we see by (1.3) that the function p also satisfies (1.5).

In the sequel, we put

$$h(r) = \int_0^r \frac{1}{K(t)}\left(\int_0^t K(s)q(s)ds\right)dt, \quad \text{for r } \in [0,\infty), \tag{1.6}$$

where $K(t) := t^{N-1}\exp(\int_0^t \lambda(s)ds)$, for each $t > 0$, and we define the function F on $[1,\infty)$ by

$$F(t) = \int_1^t \frac{1}{f(\zeta)}d\zeta. \tag{1.7}$$

From the hypotheses adopted on f, we note that the function F is a bijection from $[1,\infty)$ to $[0,\infty)$.

Our main result is the following.

Theorem 1.1. *Assume that* (H_1)–(H_4) *hold. Moreover, assume that*

$$\int_0^\infty \frac{1}{K(t)}\left(\int_0^t K(s)(q-p)(s) f\circ F^{-1}(2h(s))\,ds\right)dt < +\infty. \tag{1.8}$$

Then problem (P) has a positive entire solution if and only if

$$\int_1^\infty \frac{1}{K(t)}\left(\int_0^t K(s)p(s)\,ds\right)dt = +\infty. \tag{1.9}$$

Example 1.2. Let $\alpha \geq 0$ and $\beta \in [0,1]$. Assume that for $t \geq 0$, $f(t) = (1+t)^\beta \ln(1+t)$ and $p(t) = 1/t^\alpha$. Then the following problem:

$$\begin{gathered} \Delta u + \frac{1}{1+|x|}\,|\nabla u(x)| = \frac{(1+u(x))^\beta}{|x|^\alpha}\ln(1+u(x)), \quad \text{in } \mathbb{R}^N, \\ u \geq 0, \qquad u \not\equiv 0, \\ \lim_{|x|\to+\infty} u(x) = +\infty \end{gathered} \tag{1.10}$$

has an explosive solution if and only if $0 \leq \alpha < 2$.

Motivation for the present contribution stems from the one of Ghergu and Rădulescu [4] who considered the following problem:

$$\begin{gathered} \Delta u + |\nabla u(x)| = p(x) f(u), \quad \text{in } \Omega, \\ u \geq 0, \quad \text{in } \Omega, \end{gathered} \tag{1.11}$$

where Ω is either a smooth bounded domain or the whole space and f is a nondecreasing function satisfying $f \in \mathscr{C}^{0,\alpha}_{\text{loc}}(0,\infty)$, $f(0)=0$, $f>0$ on $(0,\infty)$, and $\Lambda = \sup_{x\geq 1} f(x)/x < \infty$. The authors studied the existence and nonexistence of explosive solutions under the assumption that

$$\int_0^\infty r\left(\max_{|x|=r} p(x) - \min_{|x|=r} p(x)\right)\Psi(r)\,dr < +\infty, \tag{1.12}$$

where $\Psi(r) = \exp(\Lambda/(N-2)\int_0^\infty r \min_{|x|=r} p(x)\,dr)$. More precisely, they showed in the case of $\Omega = \mathbb{R}^N$ that the above problem has positive solution if and only if

$$\int_1^\infty e^{-t} t^{1-N}\left(\int_0^t e^s s^{N-1} \min_{|x|=s} p(x)\,ds\right)dt = +\infty. \tag{1.13}$$

We remark that the condition (1.4) adopted on f includes the sublinear case, $\sup_{x\geq 1} f(x)/x < \infty$, studied by Ghergu and Rădulescu [4].

The outline of the paper is as follows. In Section 2, we will give some auxiliary results. The comparison result obtained in Section 2, Theorem 2.6, is used in Section 3 to prove the main result of this work.

2. Auxiliary results

In this section, we suppose that (A,p) satisfies

(H_5) A is a nonnegative continuous function on $[0,\infty)$, positive and differentiable on $(0,\infty)$, and $p:(0,\infty)\to[0,\infty)$ is continuous function satisfying

$$\int_0^1 A(s)p(s)ds<+\infty,\qquad \int_0^1 \frac{1}{A(t)}\left(\int_0^t A(s)p(s)ds\right)dt<+\infty. \tag{2.1}$$

For any given continuous function ψ on $(0,\infty)$, we put

$$h_\psi(r)=\int_0^r \frac{1}{A(t)}\left(\int_0^t A(s)\psi(s)ds\right)dt,\quad \text{for r } \in[0,\infty). \tag{2.2}$$

We consider the following problem:

$$\begin{gathered}\frac{1}{A}(Au')'=p(t)f(u),\quad \text{in } (0,\infty),\\ Au'(0):=\lim_{t\to 0^+}A(t)u'(t)=0,\qquad u(0)=\alpha\ge 1.\end{gathered} \tag{2.3}$$

We state the following.

Theorem 2.1. *Under the hypotheses (H_3) and (H_5), the problem (2.3) has a positive solution $u\in\mathscr{C}([0,\infty))\cap\mathscr{C}^1((0,\infty))$. Further, on $[0,\infty)$,*

$$\alpha+f(\alpha)h_p(r)\le u(r)\le F^{-1}(F(\alpha)+h_p(r)). \tag{2.4}$$

Proof. Let $(u_k)_{k\ge 0}$ be the sequence of functions defined on $[0,\infty)$ by $u_0(r)=\alpha$ and

$$u_{k+1}(r)=\alpha+\int_0^r\frac{1}{A(t)}\left(\int_0^t A(s)p(s)f(u_k(s))ds\right)dt,\quad \forall k\in\mathbb{N}. \tag{2.5}$$

Clearly, we have for each $k\in\mathbb{N}$, $t\to u_k(t)$ is a nondecreasing function on $[0,+\infty)$.

By induction, we prove that $(u_k)_{k\ge 0}$ is a nondecreasing sequence.

Since the function f is nondecreasing, we obtain by (2.5) that for each $k\ge 0$,

$$u_k'(t)\le f(u_k(t))\frac{1}{A(t)}\int_0^t A(s)p(s)ds,\quad t\ge 0. \tag{2.6}$$

That is,

$$\frac{u_k'(t)}{f(u_k(t))}\le\frac{1}{A(t)}\int_0^t A(s)p(s)ds,\quad t\ge 0. \tag{2.7}$$

Then

$$\int_0^r \frac{u_k'(t)}{f(u_k(t))}dt \le \int_0^r \frac{1}{A(t)}\left(\int_0^t A(s)p(s)ds\right)dt, \quad r \ge 0. \tag{2.8}$$

It follows that for each $r \ge 0$,

$$F(u_k(r)) - F(\alpha) = \int_\alpha^{u_k(r)} \frac{1}{f(\zeta)}d\zeta \le h_p(r). \tag{2.9}$$

So

$$u_k(r) \le F^{-1}(F(\alpha) + h_p(r)), \quad r \ge 0. \tag{2.10}$$

Then the sequence $(u_k)_{k\ge 0}$ converges and the function $u = \sup_{k\in\mathbb{N}} u_k$ is finite and satisfies for each $r \ge 0$,

$$u(r) = \alpha + \int_0^r \frac{1}{A(t)}\left(\int_0^t A(s)p(s)f(u(s))ds\right)dt. \tag{2.11}$$

So, $u \in \mathscr{C}([0,\infty)) \cap \mathscr{C}^1((0,\infty))$. Thus u is a solution of the problem (2.3). Moreover, from the monotonicity of f and (2.10), we obtain (2.4). □

Remark 2.2. The solution of problem (2.3) satisfying (2.4) is bounded if and only if

$$\int_0^{+\infty} \frac{1}{A(t)}\left(\int_0^t A(s)p(s)ds\right)dt < +\infty. \tag{2.12}$$

Example 2.3. Let $A(t) = t^\delta$ for $t \in [0,\infty)$, where $\delta \ge 0$. Assume that for $t > 0$, $p(t) = 1/t^\mu(1+t)^{\nu-\mu}$, with $\mu < \min(2, 1+\delta)$ and $\nu \in \mathbb{R}$. Let $a, b \ge 0$ such that $a + b > 0$, $\beta \ge 0$, and $0 \le \alpha \le 1$, set $f(t) = (at^\alpha + b)\ln(1+t^\beta)$ for $t \in [0,\infty)$, then the problem

$$\begin{gathered} \frac{1}{A}(Au')' = \frac{1}{t^\mu(1+t)^{\nu-\mu}} f(u(t)), \quad \text{in } [0,\infty), \\ Au'(0) = 0, \qquad u(0) = u_0 \ge 1 \end{gathered} \tag{2.13}$$

has a positive solution $u \in \mathscr{C}([0,\infty)) \cap \mathscr{C}^\infty((0,\infty))$. Moreover u is bounded if and only if $\delta > 1$ and $\mu < 2 < \nu$.

Corollary 2.4. *Assume (H_3) and (H_5) hold. Assume moreover that (H_6) holds, for all $c > 0$, there exists $k > 0$ such that for all $x, y \in [0,c]$, $|f(x) - f(y)| \le k|x-y|$. Then the problem (2.3) has a unique positive solution $u \in \mathscr{C}([0,\infty)) \cap \mathscr{C}^1((0,\infty))$ satisfying (2.4).*

Proof. Existence follows from Theorem 2.1.

Now, let us prove the uniqueness. Let u and v be positive solutions of the problem (2.3). Then for each $a > 0$ and $r \in [0,a]$, we have

$$|u(r) - v(r)| \le \int_0^r \frac{1}{A(t)}\left(\int_0^t A(s)p(s)|f(u(s)) - f(v(s))|ds\right)dt. \tag{2.14}$$

Since u and v are continuous, it follows that there exists $c>0$ such that $u(r),v(r)\in[0,c]$ for each $r\in[0,a]$.

So, by hypothesis (H_6) and Fubini theorem, we obtain that

$$|u(r)-v(r)|\leq k\int_0^r\left[A(s)p(s)\left(\int_s^a\frac{1}{A(t)}dt\right)\right]|u(s)-v(s)|\,ds. \tag{2.15}$$

By Gronwall's lemma, we deduce that $u(r)=v(r)$ on $[0,a]$. This completes the proof. □

Corollary 2.5. *Let $\lambda:[0,\infty)\to[0,\infty)$ be a continuous function and suppose that f and (A,p) satisfy, respectively, (H_3) and (H_5). Then the problem*

$$\begin{gathered}\frac{1}{A}(Au')'+\lambda(u)(u')^2=p(t)f(u),\quad in\ (0,\infty),\\ Au'(0)=0,\qquad u(0)=\alpha\geq 1\end{gathered}\tag{2.16}$$

has a positive solution $u\in\mathscr{C}([0,\infty))\cap\mathscr{C}^1((0,\infty))$.

Proof. Let $\rho:[0,\infty)\to[0,\infty)$ be the function defined by $\rho(t)=\int_0^t\exp(\int_0^\xi\lambda(s)ds)d\xi$. It is clear that ρ is a bijection from $[0,\infty)$ to itself. Put $v=\rho(u)$. Then v satisfies the following problem:

$$\begin{gathered}\frac{1}{A}(Av')'=p(t)g(v),\quad \text{in } (0,\infty),\\ Av'(0)=0,\qquad v(0)=\rho(\alpha)\geq 1,\end{gathered}\tag{2.17}$$

where the function g is defined on $[0,\infty)$ by $g\circ\rho=\rho' f$. Clearly, g satisfies (H_3). Hence by Theorem 2.1, the above problem has a solution v belonging to $\mathscr{C}([0,\infty))\cap\mathscr{C}^1((0,\infty))$. Therefore, $u=\rho^{-1}(v)$ is a solution of the problem (2.16). This completes the proof. □

Now, we will give a comparison result. For this aim, we suppose in what follows that

(i) (A,p) and (B,q) satisfy (H_5), $p\leq q$, and B/A is nondecreasing function,

(ii) f and g satisfy (H_3) with $0\leq g\leq f$.

For each $c\geq 1$, we define on $[0,+\infty)$ the function

$$m_c(r):=G^{-1}(G(c)+h_q(r)),\tag{2.18}$$

where h_q is the function defined by (2.2) and G^{-1} is the inverse of the function $G(t)=\int_1^t 1/g(\zeta)d\zeta$.

Theorem 2.6. *Assume that the assumptions (i) and (ii) are satisfied. Then for any $\beta\geq 1$ satisfying*

$$\int_0^\infty\frac{1}{B(t)}\left(\int_0^t B(s)(q-p)(s)g(m_\beta(s))\,ds\right)dt<+\infty,\tag{2.19}$$

there exists $\alpha > \beta$ such that problems

$$\begin{aligned}
&\frac{1}{A}(Av')' = p(t)f(v), \quad \text{in } [0,\infty),\\
&Av'(0) = 0, \qquad v(0) = \alpha > 1,\\
&\frac{1}{B}(Bw')' = q(t)g(w), \quad \text{in } [0,\infty),\\
&Aw'(0) = 0, \qquad w(0) = \beta \geq 1
\end{aligned} \tag{2.20}$$

have positive continuous solutions satisfying

$$v \geq w, \quad \text{in } [0,\infty). \tag{2.21}$$

Proof. By Theorem 2.1, for any $\alpha > \beta \geq 1$, problems (2.20) have positive solutions v and w satisfying the integral equations

$$\begin{aligned}
v(r) &= \alpha + \int_0^r \frac{1}{A(t)}\left(\int_0^t A(s)p(s)f(v(s))\,ds\right)dt, \quad r \geq 0,\\
w(r) &= \beta + \int_0^r \frac{1}{B(t)}\left(\int_0^t B(s)q(s)g(w(s))\,ds\right)dt, \quad r \geq 0.
\end{aligned} \tag{2.22}$$

Let $\alpha > \beta \geq 1$. We intend to show that if the constant α is sufficiently large, more precisely

$$\alpha > \beta + \left(\int_0^\infty \left(\int_0^t \frac{B(s)}{B(t)}(q-p)(s)g(m_\beta(s))\,ds\right)dt\right), \tag{2.23}$$

then we have

$$v(r) \geq w(r), \quad r \geq 0. \tag{2.24}$$

Using (ii) and the fact that B/A and f are nondecreasing functions on $[0,\infty)$, we obtain

$$\begin{aligned}
w(r) &= \beta + \int_0^r \left(\int_0^t \frac{B(s)}{B(t)}q(s)g(w(s))\,ds\right)dt\\
&\leq \beta + \int_0^r \frac{1}{B(t)}\left(\int_0^t B(s)(q-p)(s)g(w(s))\,ds\right)dt\\
&\quad + \int_0^r \frac{1}{A(t)}\left(\int_0^t A(s)p(s)f(w(s))\,ds\right)dt.
\end{aligned} \tag{2.25}$$

On the other hand, by (2.4), we have

$$w(r) \leq G^{-1}\left(G(\beta) + \int_0^r \frac{1}{A(t)}\left(\int_0^t A(s)q(s)\,ds\right)dt\right) = m_\beta(r). \tag{2.26}$$

By (2.19) and (2.23), we obtain

$$\begin{aligned} w(r) &- \int_0^r \left(\int_0^t \frac{A(s)}{A(t)} p(s) f(w(s))\, ds \right) dt \\ &\leq \beta + \int_0^r \left(\int_0^t \frac{B(s)}{B(t)} (q-p)(s) g(m_\beta(s))\, ds \right) dt \\ &< \alpha = v(r) - \int_0^r \frac{1}{A(t)} \left(\int_0^t \frac{A(s)}{A(t)} p(s) f(v(s))\, ds \right) dt. \end{aligned} \tag{2.27}$$

Then using a standard comparison theorem [9, Theorem VI, page 17], we obtain (2.21). □

3. Proof of the main result

Proof of Theorem 1.1. Recall that for each $t > 0$, $K(t) := t^{N-1} \exp(\int_0^t \lambda(s) ds)$.

Necessity. We will proceed by contradiction. Suppose that (1.9) fails and let u be an entire large solution of problem (P). Let

$$v(x) := \int_1^{u(x)+1} \frac{1}{f(\zeta)} d\zeta. \tag{3.1}$$

Define the spherical mean of v by

$$\overline{v}(r) := \frac{1}{w_N r^{N-1}} \int_{|x|=r} v(x) d\sigma_x, \tag{3.2}$$

where w_N denotes the surface of the unit sphere in $\mathbb{R}^N$.

Since u is a positive entire large solution of (P), it follows by (1.4) that v is positive and $\lim_{|x|\to\infty} v(x) = +\infty$.

By [2, Section 1, Proposition 6], we obtain

$$\Delta \overline{v} = \overline{v}'' + \frac{N-1}{r} \overline{v}' = \overline{\Delta v}. \tag{3.3}$$

So

$$\Delta \overline{v} + \lambda(|x|) \nabla \overline{v} \leq \frac{1}{w_N r^{N-1}} \int_{|x|=r} \Delta v(x) + \lambda(|x|) \,|\, \nabla v(x) \,|\, d\sigma_x. \tag{3.4}$$

By computation, we have on the ball

$$\begin{aligned} \Delta v(x) + \lambda(|x|) \,|\, \nabla v(x) \,| &= \frac{1}{f(u(x)+1)} \Delta u(x) + \left(\frac{1}{f} \right)' (u(x)+1) \,|\, \nabla u(x) \,|^2 \\ &\quad + \frac{1}{f(u(x)+1)} \lambda(|x|) \,|\, \nabla u(x) \,|. \end{aligned} \tag{3.5}$$

Using the fact that $f' \geq 0$, we obtain

$$\begin{aligned}
\Delta\overline{v} + \lambda(|x|)\nabla\overline{v} &\leq \frac{1}{w_N r^{N-1}} \int_{|x|=r} \frac{1}{f(u(x)+1)} (\Delta u(x) + \lambda(|x|)\,|\nabla u(x)|)\,d\sigma_x \\
&\leq \frac{1}{w_N r^{N-1}} \int_{|x|=r} \frac{1}{f(u(x)+1)} p(|x|) f(u(x))\,d\sigma_x \leq p(r).
\end{aligned} \tag{3.6}$$

That is,

$$\overline{v}'' + \frac{N-1}{r}\overline{v}' + \lambda(r)\overline{v}' \leq p(r). \tag{3.7}$$

Then

$$\left(r^{N-1}\exp\left(\int_0^r \lambda(s)ds\right)\overline{v}'\right)' \leq r^{N-1}\exp\left(\int_0^r \lambda(s)ds\right)p(r). \tag{3.8}$$

Integrating (3.8) yields for each $r \geq r_0 > 0$, $\overline{v}(r) \leq \overline{v}(r_0) + \int_0^r 1/K(t)(\int_0^t K(s)p(s)ds)dt$. Thus $\overline{v}$ is bounded, contradiction. It follows that (P) has no positive large solution.

Sufficiency. Suppose that (1.9) holds. We will use the comparison result given by Theorem 2.6 for $A(t) = B(t) = K(t) = t^{N-1}\exp(\int_0^t \lambda(s)ds)$, p, q, and f satisfying, respectively, (H_4) and (H_3).

Let $\beta \geq 1$. Put for $r \geq 0$,

$$m_\beta(r) := F^{-1}(F(\beta) + h(r)), \tag{3.9}$$

where h is the function defined by (1.6).

First, we claim that

$$\int_0^\infty \frac{1}{K(t)}\left(\int_0^t K(s)(q-p)(s)f(m_\beta(s))ds\right)dt < +\infty. \tag{3.10}$$

In fact, by (1.3) and (1.9), there exists $0 < r_0 < +\infty$ such that

$$F(\beta) < \int_0^{r_0} \frac{1}{K(t)}\left(\int_0^t K(s)q(s)ds\right)dt = h(r_0). \tag{3.11}$$

Then

$$\begin{aligned}
&\int_0^{r_0} \frac{1}{K(t)}\left(\int_0^t K(s)(q-p)(s)f(m_\beta(s))ds\right)dt \\
&\quad< \int_0^{r_0} \frac{1}{K(t)}\left(\int_0^t K(s)(q-p)(s)f\circ F^{-1}(2h(r_0))ds\right)dt \\
&\quad< f\circ F^{-1}(2h(r_0)) \int_0^{r_0} \frac{1}{K(t)}\left(\int_0^t K(s)q(s)ds\right)dt < +\infty.
\end{aligned} \tag{3.12}$$

On the other hand, by (1.8), we obtain

$$\begin{aligned}
&\int_{r_0}^{\infty}\frac{1}{K(t)}\left(\int_0^t K(s)(q-p)(s)f(m_\beta(s))\,ds\right)dt \\
&\quad<\int_{r_0}^{\infty}\frac{1}{K(t)}\left(\int_0^t K(s)(q-p)(s)f\circ F^{-1}(2h(s))\,ds\right)dt<+\infty.
\end{aligned}\tag{3.13}$$

This yields (3.10).

Thus by Theorem 2.6, there exists $\alpha>\beta$ such that the problems

$$\begin{gathered}
\frac{1}{K}(Kv')'=p(t)f(v),\quad \text{in } [0,\infty),\\
Kv'(0)=0,\qquad v(0)=\alpha>1,\\
\frac{1}{K}(Kw')'=q(t)f(w),\quad \text{in } [0,\infty),\\
Kw'(0)=0,\qquad w(0)=\beta\geq 1
\end{gathered}\tag{3.14}$$

have positive solutions satisfying $v\geq w$ in $[0,\infty)$.

Now, for all $k\geq 0$, we consider the problem

$$\begin{gathered}
\Delta u_k+\lambda(|x|)\,|\nabla u_k(x)|=\varphi(x,u_k(x)),\quad \text{in } B(0,k),\\
u_k(x)=v(k),\quad \text{on } \partial B(0,k).
\end{gathered}\tag{P_k}$$

It is clear that w and v are positive sub- and supersolutions of (P_k). Then the problem (P_k) has at least a positive solution u_k and

$$w(|x|)\leq u_k(x)\leq v(|x|),\quad \text{in } B(0,k),\ \forall k\geq 1. \tag{3.15}$$

Now, by [5, Theorem 14.3], the sequence $(\nabla u_k)_k$ is bounded on every compact set in $\mathbb{R}^N$. Consequently, the sequence $(u_k)_k$ is bounded and equicontinuous on each compact of $\mathbb{R}^N$. Therefore, by Ascoli-Arzèla theorem, the sequence $(u_k)_k$ has a uniformly convergent, subsequence $(u_k^1)_k$ in $\mathscr{C}(\overline{B(0,1)})$. Setting $u^1=\lim_{k\to+\infty}u_k^1$. Then $(\varphi(\cdot,u_k^1))_k$ converges uniformly to $\varphi(\cdot,u^1)$ and so $(\Delta u_k^1+\lambda(|x|)|\nabla u_k^1(x)|)_k$ converges uniformly to $\varphi(\cdot,u^1)$ on $B(0,1)$.

Then, using the fact that $(\Delta+\lambda\nabla)$ is a closed operator, we conclude that u^1 satisfies (P) in $B(0,1)$.

Similarly, the sequence $(u_k^1)_k$ has a uniformly convergent sequence $(u_k^2)_k$ on $B(0,2)$ and let $u^2=\lim_{k\to+\infty}u_k^2$. Using the same arguments as above, we claim that u^2 satisfies (P) in $B(0,2)$. Further, we have $u^2=u^1$ on $B(0,1)$.

Repeating this procedure, we construct a sequence $(u^n)_n$ satisfying (P) in $B(0,n)$ and $u^{n+1}=u^n$ on $B(0,n)$, for all n. The sequence $(u^n)_n$ converges in $L^\infty_{\text{loc}}(\mathbb{R}^N)$ to the function u given by $u(x)=u^n(x)$ on $B(0,n)$.

Using (3.15), we obtain $w \leq u^n \leq v$ in $B(0,n)$, for all $n \geq 1$. Letting n to $+\infty$, it follows that $w \leq u \leq v$ in $\mathbb{R}^N$ and u satisfies the equation

$$\Delta u + \lambda(|x|)\,|\nabla u(x)| = \varphi(x,u(x)), \quad \text{in } \mathbb{R}^N. \tag{3.16}$$

By (1.9) and Remark 2.2, we obtain $\lim_{|x|\to\infty} w(x) = +\infty$.

Consequently, u is a positive entire large solution of problem (P). □

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